Homework 6 (due: Fr, Nov. 3)

(1) The last price of Wells Fargo (Ticker: WFC) on Thursday, 10/26/17, was $55.62. Options with expiration 11/17/17 had following last prices:

\[
\begin{array}{ccc}
\text{call} & \text{strike} & \text{put} \\
1.11 & 55 & 0.78 \\
0.85 & 55.5 & 1.04 \\
0.63 & 56 & 1.33 \\
\end{array}
\]

The spreads of the options were fairly low in the 0.04–0.06 range. The indicated price for the options is the average of bid and ask, which gives a good approximation for a realistic price. The call-put parity (neglecting interest rates) requires the (approximate) relation \( K + c = S_0 + p \) between the current stock price \( S_0 \), the strike price \( K \), the price \( c \) of the call, and the price \( p \) of the put.

(a) Compute \( S_0 + p - K - c \) for each of the three strike prices.

(b) The computations in (a) show that the call-put parity seems to be violated. Explain the reason for this!

(2) (a) A call ratio spread is implemented by buying a call \( O_1 \) and selling a larger number of calls \( O_2 \) at a higher strike price (with the same underlying and expiration). Suppose we buy one option \( O_1 \) with strike \( K_1 \) for the price \( c_1 \) and sell two options \( O_2 \) with strike \( K_2 > K_1 \) for the price \( c_2 \) each. Analyze the profit of this strategy depending on the price \( S_T \) of the stock at expiration \( T \) and represent this in a diagram.

(b) We consider a bull spread involving two call options \( O_1 \) and \( O_2 \) (with the same underlying and expiration) with strike \( K_1 \) and \( K_2 \), respectively, where \( K_1 < K_2 \). Devise a strategy based on put options that gives the same payoff as the bull spread.

(c) Devise a strategy based on options that leads to the same payoff as short selling the stock.

(3) (a) Suppose \( X_1 \) and \( X_2 \) are continuous random variables with probability density functions \( p_1 \) and \( p_2 \), respectively. Suppose that \( X_1 \) and \( X_2 \) are independent. Express the probability density function \( q \) of \( Y = X_1 + X_2 \) in terms of an integral involving the functions \( p_1 \) and \( p_2 \).

(b) Suppose that in (a) the random \( X_i \) is a normal random variable with mean \( \mu_i \) and standard deviation \( \sigma_i \) for \( i = 1, 2 \). Use the derived formula in (a) to find the probability density function of \( Y = X_1 + X_2 \) and to show that \( Y \) is also a normal random variable. What are the mean and the standard deviation of \( Y \)?
(c) Suppose $X_1, \ldots, X_n$ are independent normal random variables with the same mean $\mu$ and standard deviation $\sigma$. Show that $S = X_1 + \cdots + X_n$ is also a normal random variable and find its mean and standard deviation.

(4) The purpose of Problems 4 is to develop some of the theoretical foundations for describing random stock price movements (cf. HW3, Prob. 4).

(a) Let $Z_1$ be a random variable representing a random quantity at time $t = 1$ (such as the random change of the price of an asset from time 0 to time 1). Suppose that $Z_1$ has mean $\mu$ and standard deviation $\sigma$. We divide the time $t = 1$ into a large number $n$ of time intervals and suppose that each of the time intervals contributes independently in the same way to $Z_1$. More precisely, we assume that $Z_1$ is approximately given by a sum

$$X_1 + \cdots + X_n,$$

where $X_1, \ldots, X_n$ are i.i.d. random variables.

What do we have to assume about the mean $a_n$ and the standard deviation $b_n$ of $X_i$, $i = 1, \ldots, n$, so that this can be valid?

(b) Let $n \to \infty$ in (a). What can we then say about the distribution of the random variable $Z_1$?

(c) Suppose we use the same model as in (a) and (b) for an arbitrary time $t \geq 0$. Then we divide $t$ into $k = nt$ time intervals of length $1/n$ and assume that $Z_t$ is approximately given by the sum

$$X_1 + \cdots + X_k.$$

Letting $n \to \infty$, what can we say about the distribution of $Z_t$? In particular, what are the mean $\mu_t$ and the standard deviation $\sigma_t$ of $Z_t$ in terms of $\mu$, $\sigma$, and $t$? What is $Z_0$?