Homework 2 (due: Fr, Oct. 6)

(1) (a) Find the option chain for Facebook (ticker symbol: FB) on one of the days Oct. 2–Oct. 5 after markets close (1pm LA time). Make a table with the following information: last price for call/put options with strikes 160, 170, 180 and expirations Oct. 20, 2017 and Jan. 19, 2018 (a total of 12 values).

(b) Explain the terms volume and open interest that appear in the option chain.

(2) (a) Review the central limit theorem from some source (probability textbook, internet, etc.).

(b) Suppose $X$ is a random variable that only takes the values $X = -1$ and $X = 1$ with equal (= 0.5) probability. What are the expected value $\mu_0$ and the standard deviation $\sigma_0$ of $X$?

(c) Suppose $X_1, X_2, X_3, \ldots$ are random variables that are independent and have the same distribution as the random variable $X$ in (b). Suppose $n$ is a natural number. What is the mean $\mu_n$ and what is the standard deviation $\sigma_n$ of $Z_n = X_1 + \cdots + X_n$?

(d) Suppose $n$ in (c) is very large. Then the central limit theorem suggests that $Z_n$ has a distribution close to the distribution of $a_n + b_n N$, where $N$ is a standard normal random variable. What are $a_n$ and $b_n$ here?

(3) You are collecting money for a charity and want to raise $10,000. For this purpose you send out letters to randomly chosen people asking for a donation of $10. From past experience you know that 10% of the people asked will donate $10 (and 90% will give nothing). The purpose of this problem is to find a number $n$ such that if $n$ letters are sent out, you will collect the desired amount of $10,000 with about 95% certainty. This number $n$ should be as small as possible, because you do not want to send out more letters than necessary.

(a) Define a (discrete) random variable $X$ that models the amount received from one donor.

(b) Suppose that the potential donors receiving your letters do not influence each other in their decision whether to make the $10 donation or not. Find a formula for the total amount $S_n$ raised if $n$ letters are sent out. To find this formula, use the concept of i.i.d. (independent and identically distributed) random variables and relate them to $X$ defined in (a).
(c) Justify why for large $n$ we can assume that $S_n$ has a distribution closely approximated by the distribution of a normal random variable $Z_n$. What are the mean $\mu_n$ and the standard deviation $\sigma_n$ of $Z_n$ here?

(d) Let $N$ be a standard normal random variable. Find a real number $a_n$ such that

$$\Pr(S_n < 10,000) \approx \Pr(Z_n < 10,000) = \Pr(N < a_n).$$

Hint: Find $c_n$ and $d_n$ such that $c_n N + d_n$ has the same distribution as $Z_n$.

(e) Let $N$ be a standard normal random variable. Find the value $a$ such that

$$\Pr(N < a) = 0.05.$$ 

Hint: This is related to the cumulative distribution function of $N$. Its values can be looked up in tables or can be computed from built-in functions in standard programs for statistical computation.

(f) What is the number $n$ of letters that should be sent out?

(4) You play a game that is stacked in your favor as follows. A (fair) coin is flipped. If it comes up tails, then you lose the amount $B$ that you bet, but if it comes up heads then they your money is tripled (i.e., you receive the original amount $B$ back plus $2B$ in winnings). You start with a (large) amount $M_0$ of money. Let $M_n$ be your total amount after you played the game $n$ times. We now consider two scenarios.

(a) Each time you bet $B = 1$. Express $M_n$ in terms of $M_0$ and suitable i.i.d. random variables $X_1, X_2, \ldots$. Describe the distribution of $X_1$!

(b) In this second, more interesting scenario we fix a number $p \in [0, 1]$. Each time we bet the fraction $p$ of the total amount that we have at this point. Express $M_n$ again in terms of $M_0$ and suitable i.i.d. random variables $Y_1, Y_2, \ldots$. Describe the distribution of $Y_1$!

(c) The game in your favor. So should you choose $p = 1$ in (b), i.e., bet all the money you have each time? If not, which $p$ should we choose to optimize the long-term outcome of the game? Hint: Typically, $M_n \approx M_0 e^{an}$ for large $n$, where $a$ depends on $p$. You want to maximize $a$. 