Optimal portfolio strategy

You are an investor and have two investment choices: a risk-free asset E and a riskier, but potentially more rewarding asset F. You choose the following strategy: you fix $p \in [0,1]$ and at any moment you invest the proportion p of your portfolio in the risky investment F and the rest $1-p$ in the risk-free investment E. We assume that the necessary reallocations of assets can be performed instantaneously without transaction costs. We want to find the optimal proportion p that gives the largest growth of the value of your portfolio in the long run.

The first investment choice E gives the risk-free rate r as a return rate. So if E_t is the amount invested in E at time t, then the return dR_E on this investment in a small time interval dt is given by

$$dR_E = rE_t\,dt.$$

The second investment F gives a higher average return $q > r$, but has a non-zero volatility σ. So if F_t is the amount invested in F at time t, then the return dR_F on this investment in a small time interval dt can be modeled by

$$dR_F = qF_t\,dt + \sigma F_t\,dB_t,$$

where $\{B_t\}_{t\in[0,\infty)}$ is a Brownian motion.

If P_t be the value of your portfolio at time t, then

$$E_t = (1-p)P_t \quad \text{and} \quad F_t = pP_t.$$

Let dP_t be the change of the value of your portfolio in a small time interval dt. Then

$$\begin{aligned} dP_t &= dR_E + dR_F = (rE_t + qF_t)\,dt + \sigma F_t\,dB_t \\ &= (r(1-p)+qp)P_t\,dt + \sigma p P_t\,dB_t. \end{aligned}$$

This is an SDE for P_t of the form

$$dP_t = aP_t\,dt + bP_t\,dB_t,$$

where $a = r(1-p)+qp$ and $b = \sigma p$. We know that its solution is given by

$$\begin{aligned} P_t &= P_0 \exp\big((a - \tfrac{1}{2}b^2)t + bB_t\big) \\ &= P_0 \exp\big((r(1-p)+qp - \tfrac{1}{2}\sigma^2p^2)t + \sigma p B_t\big). \end{aligned}$$

We also know that $B_t/t \to 0$ as $t \to +\infty$ almost surely (i.e., with probability 1). This means that in the exponential expression for P_t the term with B_t is negligible compared with the term with t in the long run. So we want to maximize the coefficient of t for optimal growth of our portfolio.

In other words, we are reduced to the following problem: find $p \in [0,1]$ that maximizes

$$r(1-p) + qp - \tfrac{1}{2}\sigma^2 p^2.$$

This can be solved by the usual methods from calculus (set the derivative in p equal to 0 and compare with values at $p = 0$ and $p = 1$). One finds that the optimal value for p is

$$p = \frac{q-r}{\sigma^2}$$

if this is in $[0,1]$ and $p = 1$ otherwise (exercise!).

For example, if $r = 3\%$, $q = 5\%$, and $\sigma = 20\%$, then the optimal p is

$$p = \frac{0.05 - 0.03}{0.20^2} = 0.5.$$

For $r = 3\%$, $q = 5\%$, $\sigma = 10\%$, we have

$$\frac{q-r}{\sigma^2} = \frac{0.05 - 0.03}{0.10^2} = 2 > 1.$$

So in this case the optimal value is $p = 1$.