A PROOF OF EXERCISE 2.5.6

Exercise. Show that if v is a harmonic conjugate of u in a domain D, then uv is harmonic in D.

Proof. We give two proofs of this fact. One is slightly more involved, but gives more insight into what is causing the product to be harmonic. Note that it is not true in general that if u and v are harmonic, then the product is harmonic.

First, the short proof: if v is a harmonica conjugate of u, then $f = u + iv$ is analytic in D. But we know products of analytic functions are analytic, so this shows that f^2 is analytic. On the other hand,

$$f^2 = u^2 - v^2 + 2iuv.$$

We know that the real and imaginary parts of analytic functions are harmonic, and since $\mathrm{Im} f^2 = 2uv$, this shows that $2uv$ is harmonic. Therefore, uv is harmonic as well.

Here's a somewhat longer proof, which proceeds by calculating Δuv. For a function g of two variables, if we let $g_x = \frac{\partial g}{\partial x}$, $g_{xx} = \frac{\partial^2 g}{\partial x^2}$and similarly for g_y and g_{yy}, then

$$\begin{aligned} (uv)_{xx} &= (u_x v + u v_x)_x \\ &= u_{xx} v + u_x v_x + u_x v_x + u v_{xx} \\ &= u_{xx} v + v_{xx} u + 2 u_x v_x. \end{aligned}$$

Hence, by symmetry, we see that $(uv)_{yy} = u_{yy} v + v_{yy} u + 2 u_y v_y$. This shows that

$$\begin{aligned} \Delta uv &= (uv)_{xx} + (uv)_{yy} \\ &= u_{xx} v + u_{yy} v + u v_{xx} + u v_{yy} + 2 u_x v_x + 2 u_y v_y \\ &= v \Delta u + u \Delta v + 2(\nabla u \cdot \nabla v). \end{aligned}$$

Note that since u and v are harmonic, the first two terms are zero, and so $\Delta uv = 2(\nabla u \cdot \nabla v)$. Therefore, we see that the product of two harmonic functions is harmonic if and only if their gradient vectors are orthogonal. Fortunately, this happens whenever the Cauchy Riemann equations are satisfied. Since v is a harmonic conjugate to u, we know that $u_x = v_y$,$u_y = -v_x$, which tells us that

$$\begin{aligned} \nabla u \cdot \nabla v &= u_x v_x + u_y v_y \\ &= u_x(-u_y) + u_y(u_x) = 0. \end{aligned}$$

Therefore, uv is harmonic. $\square$