A PROOF OF EXERCISE 2.2.10 (2.2.12)

Exercise. Show that the function Arg$z$ is discontinuous at each point on the non-positive real axis.

Proof. To prove a function f is discontinuous at a point z_0, we must show that $\lim_{z\to z_0} f(z)$ does not equal $f(z_0)$. In the case of the argument function, this can be shown as follows: take a point $x > 0$, and consider the points $xe^{i\theta} \in \mathbb{C}$. Notice that $\mathrm{Arg} xe^{i\theta} = \theta$ whenever $\theta \in (-\pi, \pi]$. In particular,

$$\lim_{\theta\to\pi} \mathrm{Arg}(xe^{i\theta}) = \lim_{\theta\to\pi} \theta = \pi.$$

On the other hand,

$$\lim_{\theta\to-\pi} \mathrm{Arg}(xe^{i\theta}) = \lim_{\theta\to-\pi} \theta = -\pi.$$

But $\lim_{\theta\to\pi} xe^{i\theta} = \lim_{\theta\to-\pi} xe^{i\theta} = -x$. So, for any $-x < 0$, we can find two circular paths along which the Argument function takes different values in the limit as we approach the negative real axis. Therefore, the limit cannot exist, and so by definition Arg cannot be continuous on the negative real axis. □