

A PROOF OF EXERCISE 1.4.20

Exercise. If $z \neq 1$, then

$$(1) \quad 1 + z + z^2 + \dots + z^n = \frac{z^{n+1} - 1}{z - 1}.$$

Use this result and De Moivre's formula to establish the following identity:

$$1 + \cos \theta + \cos 2\theta + \dots + \cos n\theta = \frac{1}{2} + \frac{\sin \left[\left(n + \frac{1}{2} \right) \theta \right]}{2 \sin \left(\frac{\theta}{2} \right)}.$$

Proof. To show the first equality, let $S_n = 1 + z + z^2 + \dots + z^n$. We see that $(z - 1)S_n = z + z^2 + z^3 + \dots + z^{n+1} - 1 - z - \dots - z^n = z^{n+1} - 1$. Therefore, since $z \neq 1$, we can divide both sides by $1 - z$, so that

$$S_n = \frac{z^{n+1} - 1}{z - 1},$$

as desired.

We'll give two proofs of the second equality.

Proof 1: By applying (1) to the complex number $z = e^{i\theta}$, we see that

$$\begin{aligned} 1 + e^{i\theta} + \dots + e^{in\theta} &= \frac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1} \\ &= \frac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1} \cdot \frac{e^{-i\theta} - 1}{e^{-i\theta} - 1} \\ &= \frac{e^{in\theta} - e^{i(n+1)\theta} - e^{-i\theta} + 1}{2(1 - \cos \theta)}. \end{aligned}$$

Taking real parts, and making liberal use of the trig identities

$$\begin{aligned} \cos(\alpha \pm \beta) &= \cos \alpha \cos \beta \mp \sin \alpha \sin \beta, \\ \sin(\alpha \pm \beta) &= \sin \alpha \cos \beta \pm \sin \beta \cos \alpha, \end{aligned}$$

we see that

$$\begin{aligned} 1 + \cos \theta + \dots + \cos n\theta &= \operatorname{Re} \left(\frac{e^{in\theta} - e^{i(n+1)\theta} - e^{-i\theta} + 1}{2(1 - \cos \theta)} \right) \\ &= \frac{\cos n\theta - \cos(n\theta + \theta) - \cos \theta + 1}{2(1 - \cos \theta)} \\ &= \frac{1}{2} + \frac{\cos n\theta - \cos n\theta \cos \theta + \sin n\theta \sin \theta}{2(1 - \cos \theta)}. \end{aligned}$$

The angle sum formula for cosine tells us, in particular, that $1 - \cos \theta = 1 - \cos^2(\theta/2) + \sin^2(\theta/2) = 2 \sin^2(\theta/2)$, since $\sin^2 x + \cos^2 x = 1$. Therefore, we see that

the above expression is equal to

$$\begin{aligned}
 \frac{1}{2} + \frac{\cos n\theta(2\sin^2(\theta/2)) + \sin n\theta \sin \theta}{[2\sin(\theta/2)]^2} &= \frac{1}{2} + \frac{2\cos n\theta \sin^2(\theta/2) + 2\sin n\theta \sin(\theta/2) \cos(\theta/2)}{[2\sin(\theta/2)]^2} \\
 &= \frac{1}{2} + \frac{\cos n\theta \sin(\theta/2) + \sin n\theta \cos(\theta/2)}{2\sin(\theta/2)} \\
 &= \frac{1}{2} + \frac{\sin\left[\left(n + \frac{1}{2}\right)\theta\right]}{2\sin\left(\frac{\theta}{2}\right)}.
 \end{aligned}$$

Proof 2: This proof relies less on trig identities, and instead uses the expressions for sine and cosine that involve the complex exponential.

Since $\cos n\theta = \frac{e^{in\theta} + e^{-in\theta}}{2}$, we see that

$$\begin{aligned}
 1 + \cos \theta + \cos 2\theta + \dots + \cos n\theta &= \frac{1}{2} (1 + e^{i\theta} + \dots + e^{in\theta}) + \frac{1}{2} (1 + e^{i\theta} + \dots + e^{in\theta}) \\
 &= \frac{1}{2} \left(\frac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1} + \frac{e^{-i(n+1)\theta} - 1}{e^{-i\theta} - 1} \right) \\
 &= \frac{1}{2} \left(\frac{e^{i(n+\frac{1}{2})\theta} - e^{-i\frac{\theta}{2}}}{e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}} + \frac{e^{-i(n+\frac{1}{2})\theta} - e^{i\frac{\theta}{2}}}{e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}}} \right),
 \end{aligned}$$

where in the last step we have divided top and bottom of the first fraction by $e^{i\frac{\theta}{2}}$, and in the second fraction we have multiplied top and bottom by the same number. Now, since the denominators only differ by a factor of -1, we can combine these fractions, which gives

$$\begin{aligned}
 \frac{1}{2} \left(\frac{e^{i(n+\frac{1}{2})\theta} - e^{-i(n+\frac{1}{2})\theta} + e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}}{e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}} \right) &= \frac{1}{2} \left(1 + \frac{e^{i(n+\frac{1}{2})\theta} - e^{-i(n+\frac{1}{2})\theta}}{e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}} \right) \\
 &= \frac{1}{2} \left(1 + \frac{\sin\left[\left(n + \frac{1}{2}\right)\theta\right]}{\sin\left(\frac{\theta}{2}\right)} \right),
 \end{aligned}$$

since $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$. □

Remark. It's worth mentioning that one can also prove equation (1) by using induction on n .