A PROOF OF EXERCISE 1.2.19.

Exercise. Let $a_1, a_2, \ldots, a_n$ be real constants. Then if z_0 is a root of the polynomial equation $z^n + a_1 z^{n-1} + \ldots + a_n = 0$, so is $\overline{z_0}$.

Proof. Let us write the above polynomial as $p(z) = z^n + a_1 z^{n-1} + \ldots + a_n$. We are given that $p(z_0) = 0$, and need to show that $p(\overline{z_0}) = 0$.

Since $p(z_0) = 0$, if we conjugate both sides we find that $\overline{p(z_0)} = \overline{0}$. Since $\overline{c} = c$ whenever $c \in \mathbb{R}$, this tells us that $\overline{p(z_0)} = 0$. On the other hand, we see that

$$\begin{aligned}
\overline{p(z_0)} &= \overline{z_0^n + a_1 z_0^{n-1} + \ldots + a_n} \\
&= \overline{z_0^n} + \overline{a_1 z_0^{n-1}} + \ldots + \overline{a_n} \\
&= \overline{z_0}^n + \overline{a_1} \cdot \overline{z_0}^{n-1} + \ldots + \overline{a_n} \\
&= \overline{z_0}^n + a_1 \overline{z_0}^{n-1} + \ldots + a_n \\
&= p(\overline{z_0}).
\end{aligned}$$

The second equality follows from the fact that $\overline{z + w} = \overline{z} + \overline{w}$. The third equality comes from the fact that $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$, and for the last line we have used the fact that all the coefficients a_i are real, meaning that $\overline{a_i} = a_i$. Therefore, we have shown that $p(\overline{z_0}) = \overline{p(z_0)} = 0$, so $\overline{z_0}$ is a root of the polynomial, as claimed. □