Faithfully flat Lefschetz extensions

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Motivating question

We call a ring of characteristic zero a

Lefschetz ring

if it is isomorphic to an ultraproduct of rings of positive characteristic. (Always: "ring" = "commutative ring with unit 1.")

Examples:

- The field $\mathbb{C}$ of complex numbers;
- in general: any algebraically closed field K of char. zero of cardinality 2^{λ} with $\lambda > \aleph_0$ can be written as an ultraproduct of algebraically closed fields K_p of char. p.

Question: Given a Noetherian ring R of characteristic zero, can we find a *faithfully flat* ring extension D of R which is Lefschetz?

Fact: Every finitely generated algebra A over a field of characteristic zero admits a faithfully flat Lefschetz extension.

Enough to see this for

$$A = K[X], \quad X = (X_1, \dots, X_n).$$

Here and below K is an ultraproduct of alg. closed fields K_p of char. p. Take

$$D := K[X]_\infty := \text{ultraproduct of the } K_p[X].$$

The (images of the) indeterminates $X_1, \dots, X_n$ remain algebraically independent over K in $K[X]_\infty$ (by Łos' Theorem). Hence

$$K[X] \hookrightarrow K[X]_\infty$$

as K-algebras. This embedding is faithfully flat (van den Dries and Schmidt).

Theorem. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring of equicharacteristic zero (i.e.,* $R \supseteq \mathbb{Q}$*). There exists a local Lefschetz ring* $\mathfrak{D}(R)$ *and a faithfully flat embedding* $\eta_R\colon R \to \mathfrak{D}(R)$*.*

Naive idea: For $R = K[[X]]$ put

$$\begin{aligned}\mathfrak{D}(R) &= K[[X]]_\infty \\ &:= \text{the ultraproduct of the } K_p[[X]].\end{aligned}$$

Now $K[X]$ is a subring of $K[[X]]_\infty$, and the local ring $K[[X]]_\infty$ is complete in the X-adic toplogy.

Might try

$$\eta_R(f) := \left\{ \begin{array}{l} \text{limit in } K[[X]]_\infty \text{ of a Cauchy} \\ \text{sequence in } K[X] \text{ approximat-} \\ \text{ing } f. \end{array} \right.$$

But $K[[X]]_\infty$ not Hausdorff!

More subtle problem:

Let L be a field and $i \in \{1, \ldots, n\}$. Let us say that a power series $f \in L[[X]]$ *does not involve the indeterminate* X_i if

$$f \in L[[X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n]].$$

An element of $K[[X]]_\infty$ *does not involve* X_i if it is the ultraproduct of power series in $K_p[[X]]$ not involving X_i.

Fact: There is *no* homomorphism

$$h\colon K[[X_1, \ldots, X_6]] \to K[[X_1, \ldots, X_6]]_\infty$$

such that for $i = 4, 5, 6$, if $f \in K[[X]]$ does not involve the variable X_i, then neither does $h(f)$. (Uses an example due to P. Roberts.)

Nevertheless, the construction in the theorem can be made *functorial* in a way. For this we need some definitions . . .

Define a category $\mathbf{Coh}_K$:

(1) objects are quadruples $\Lambda = (R, \mathbf{x}, k, u)$:

(a) $(R, \mathfrak{m})$ is a Noetherian local ring;

(b) $\mathbf{x}$ is a finite tuple of generators of $\mathfrak{m}$;

(c) k is a quasi-coefficient field of R (i.e., a subfield of R such that $R/\mathfrak{m}$ is algebraic over the image of k under $R \to R/\mathfrak{m}$);

(d) $u\colon R \to K$ is a homom. with $\ker \varphi = \mathfrak{m}$.

(2) morphisms $\Lambda \to \Gamma = (S, \mathbf{y}, l, v)$ are local ring homomorphisms $\alpha\colon R \to S$ such that

(a) $\alpha(\mathbf{x})$ is an initial segment of $\mathbf{y}$,

(b) $\alpha(k) \subseteq l$, and

(c) $v \circ \alpha = u$.

Example. $\Big(K[[X]], X, K, u\Big)$ is an object in $\mathbf{Coh}_K$, where $u(f) = f(0)$.

On the other side, given an *ultraset* $\mathcal{W}$ (= infinite set equipped with a non-principal ultrafilter) let $\mathbf{Lef}_{\mathcal{W}}$ be the category with:

(1) objects: *analytic* Lefschetz rings with respect to $\mathcal{W}$, i.e., ultraproducts w.r.t. $\mathcal{W}$ of complete Noetherian local rings $(R, \mathfrak{m})$ with algebraically closed residue field $R/\mathfrak{m}$ of $\mathrm{char}(R/\mathfrak{m}) = \mathrm{char}(R) > 0$;

(2) morphisms: ultraproducts (with respect to $\mathcal{W}$) of local ring homomorphisms.

Example. $K[[X]]_\infty$ is an analytic Lefschetz ring w.r.t. $\mathcal{W}$ = a non-principal ultrafilter on the set of prime numbers.

Theorem′. *There exists an ultraset $\mathcal{W}$ and a functor*

$$\mathfrak{D}\colon\ \mathbf{Coh}_K \to \mathbf{Lef}_{\mathcal{W}}$$

with the following property: for every $\mathbf{Coh}_K$-object Λ as above there exists a faithfully flat homomorphism $\eta_\Lambda\colon R \to \mathfrak{D}(\Lambda)$ such that for any $\mathbf{Coh}_K$-morphism $\Lambda \to \Gamma$ with underlying homomorphism $\alpha\colon R \to S$ the diagram

$$\begin{tikzcd} R \arrow[r, "\alpha"] \arrow[d, "\eta_\Lambda"] & S \arrow[d, "\eta_\Gamma"] \\ \mathfrak{D}(\Lambda) \arrow[r, "\mathfrak{D}(\alpha)"] & \mathfrak{D}(\Gamma) \end{tikzcd}$$

of local homomorphisms commutes.

Ingredients in the proof of Theorem′.

1. A model-theoretic embedding criterion.
2. An approximation result.
3. A test for flatness.

1. An embedding criterion.

A *nested ring* is a ring R together with a *nest* of subrings:

$$R_0 \subseteq R_1 \subseteq \cdots \subseteq R_n \subseteq \cdots, \qquad R = \bigcup_n R_n.$$

A homomorphism $\varphi\colon S \to R$ between nested rings $R = (R_n)$ and $S = (S_n)$ is a *homomorphism of nested rings* if $\varphi(S_n) \subseteq R_n$ for all n.

A *nested system $\mathcal{S}$ of equations* over S:

$$\begin{array}{lcccccl}
P_{00}(Z_0) & = & \cdots & = & P_{0k}(Z_0) & = & 0, \\
P_{10}(Z_0, Z_1) & = & \cdots & = & P_{1k}(Z_0, Z_1) & = & 0, \\
\vdots & & & & \vdots & & \\
P_{n0}(Z_0, \ldots, Z_n) & = & \cdots & = & P_{nk}(Z_0, \ldots, Z_n) & = & 0,
\end{array}$$

where $k, n \in \mathbb{N}$, $Z_i = (Z_{i1}, \ldots, Z_{ik_i})$, $k_i \in \mathbb{N}$, and $P_{ij} \in S_i[Z_0, \ldots, Z_i]$.

A tuple $(\mathbf{a}_0, \ldots, \mathbf{a}_n)$ with $\mathbf{a}_i \in (R_i)^{k_i}$ is called a *nested solution of $\mathcal{S}$* in R if $P_{ij}(\mathbf{a}_0, \ldots, \mathbf{a}_i) = 0$ for all i, j.

Let A and B be nested S-algebras. Given an ultraset $\mathcal{U}$ consider the S-subalgebra $B^{\langle\mathcal{U}\rangle} := \bigcup_n B_n^{\mathcal{U}}$ of the ultrapower $B^{\mathcal{U}}$ as a nested S-algebra with nest $(B_n^{\mathcal{U}})$.

Theorem. *If each S_n is Noetherian, then the following are equivalent:*

(1) Every nested system of polynomial equations over S which has a nested solution in A has one in B.

(2) There exists a homomorphism of nested S-algebras $\eta\colon A \to B^{\langle\mathcal{U}\rangle}$, for some ultraset $\mathcal{U}$.

2. An approximation result.

Define nested rings S, A, B and $S^\sim$ by

$$\begin{aligned}
S_n &:= K[X_1, \dots, X_n], \\
A_n &:= K[[X_1, \dots, X_n]], \\
B_n &:= K[[X_1, \dots, X_n]]_\infty, \\
S_n^\sim &:= \text{algebraic closure of } S_n \text{ in } A_n.
\end{aligned}$$

Theorem. *Every nested system of polynomial equations over S which has a nested solution in A has one in $S^\sim$, and hence in B.*

Follows from the fact that rings of the form

$$K[[X_1, \dots, X_n]][X_{n+1}, \dots, X_{n+m}]_{(X_{n+1}, \dots, X_{n+m})}$$

are existentially closed in their completion (C. Rotthaus, 1987).

3. A test for flatness.

Let $(R, \mathfrak{m})$ be a Noetherian local ring, $\mathbf{x} = (x_1, \ldots, x_n) \in R^n$. Recall: $\mathbf{x}$ is a *system of parameters* (s.o.p.) for R if $n = \dim R$ and $\dim R/(\mathbf{x}) = 0$, and R is *regular* if it has a s.o.p. generating $\mathfrak{m}$. (E.g., $R = K[[X]]$.)

Let M be an R-module. Then $\mathbf{x}$ is called an *M-regular sequence* if $M/(x_1, \ldots, x_n)M \neq 0$ and x_i is a non-zerodivisor on $M/(x_1, \ldots, x_{i-1})M$ for all i. If M is f.g., then every permutation of an M-regular sequence is M-regular.

Fact. (Hochster & Huneke) If R is regular and

(1) there exists a s.o.p. for R which is M-regular, and

(2) every permutation of an M-regular sequence is again M-regular,

then M is flat.

Approximations.

Let R_w be the complete Noetherian local rings of equicharacteristic $p(w) > 0$ whose ultraproduct is $\mathfrak{D}(\Lambda)$. We think of the R_w as *approximations* of R. They share many properties of R, e.g.:

(1) Almost all R_w have the same Hilbert-Samuel function as R. (Hence almost all R_w have the same dimension, embedding dimension, multiplicity as R.)

(2) Almost all R_w have the same depth as R.

(3) Almost all R_w are regular (Cohen-Macaulay) if and only if R is regular (Cohen-Macaulay, respectively).

(4) ... and much more. (Sometimes need to choose Λ carefully.)

Applications.

The functor $\mathfrak{D}(\,\cdot\,)$ can be used to
1. define a notion of tight closure, and
2. construct big Cohen-Macaulay algebras
in equicharacteristic zero.

Observation: $\mathfrak{D}(R) = \mathfrak{D}(\Lambda)$ comes equipped with the *non-standard Frobenius* $\mathbf{F}_\infty$ = ultraproduct of the

$$R_w \to R_w \colon a \mapsto \mathbf{F}_{p(w)}(a) = a^{p(w)}.$$

Notation: $R^\circ := R \setminus$ (minimal primes of R).

Definition. An element $a \in R$ is in the *non-standard tight closure* $\mathrm{cl}(I) = \mathrm{cl}_\Lambda(I)$ of I if

$$\exists c \in R^\circ : \forall m \gg 0 : c\,\mathbf{F}_\infty^m(a) \in \mathbf{F}_\infty^m(I)\mathfrak{D}(R).$$

(Similar to a definition given by Hochster & Huneke in positive char.)

Theorem 1. *If R is regular, then* $\mathrm{cl}(I) = I$ *for every ideal I of R.*

Proof. Easy to check: the image of an R-regular sequence in R under $\mathbf{F}^m_\infty$ is $\mathfrak{D}(R)$-regular. Hence by the Hochster-Huneke flatness criterion,

$$R \to \mathfrak{D}(R)\colon a \mapsto \mathbf{F}^m_\infty(a) \qquad (*)$$

is flat. Suppose that $a \in \mathrm{cl}(I) \setminus I$. For some non-zero $c \in R$, we have

$$c\,\mathbf{F}^m(a) \in \mathbf{F}^m(I)\mathfrak{D}(R)$$

for m sufficiently large. Thus

$$c \in (\mathbf{F}^m_\infty(I)\mathfrak{D}(R) :_{\mathfrak{D}(R)} \mathbf{F}^m_\infty(a)) = \mathbf{F}^m_\infty(I :_R a)\mathfrak{D}(R)$$

where we used flatness of $(*)$ for the last equality. Since $a \notin I$, we have $(I :_R a) \subseteq \mathfrak{m}$, hence

$$c \in \mathbf{F}^m_\infty(\mathfrak{m})\mathfrak{D}(R) \cap R = (0),$$

contradiction. $\square$

The Briançon-Skoda Theorem.

The *integral closure* $\overline{J}$ of an ideal $J \subseteq S$ in a ring S is the ideal of all $b \in S$ for which

$$b^n + a_1 b^{n-1} + \cdots + a_{n-1} b + a_n = 0$$

with $a_i \in J^i$ for each i.

Lemma. (Huneke) *Let S be a Noetherian local ring, J an ideal of S, $b \in S$. Then*

$$b \in \overline{J} \iff \begin{cases} b \in JV \text{ for every local homo-} \\ \text{morphism } S \to V \text{ to a discrete} \\ \text{valuation ring } V \text{ whose kernel} \\ \text{is a minimal prime of } S. \end{cases}$$

Together with Theorem 1 and functoriality of $\mathfrak{D}$ this yields:

$$\mathrm{cl}(I) \subseteq \overline{I} \qquad \text{for every ideal } I \text{ of } R.$$

(Hence "tight" closure.)

Theorem 2. *If I has positive height and is generated by n elements then $\overline{I^n} \subseteq \mathrm{cl}(I)$.*

(Follows from the definitions and Łos' Theorem. The assumption $\mathrm{ht}(I) > 0$ is needed to get $I^k \cap R^\circ \neq \emptyset$ for all k.)

Corollary. (Briançon-Skoda)
If $f \in \mathbb{C}[[X_1, \dots, X_n]]$ with $f(0) = 0$, then

$$f^n = \frac{\partial f}{\partial X_1} g_1 + \cdots + \frac{\partial f}{\partial X_n} g_n$$

for some $g_1, \dots, g_n \in \mathbb{C}[[X_1, \dots, X_n]]$.

Proof. Let $I :=$ ideal generated by the $\partial f / \partial X_i$. Then $f \in \overline{I}$. (By the lemma: may take $V = K[[t]]$ with $K \supseteq \mathbb{C}$, use Chain Rule.) Hence

$$f^n \in \overline{I^n} \underset{\text{Thm. 2}}{\subseteq} \mathrm{cl}(I) \underset{\text{Thm. 1}}{=} I.$$

$\square$