A CRITERION FOR HNN EXTENSIONS OF FINITE p-GROUPS TO BE RESIDUALLY p

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Abstract. We give a criterion for an HNN extension of a finite p-group to be residually p.

1. Statement of the Main Results

By an *HNN pair* we mean a pair (G,φ) where G is a group and $\varphi\colon A\to B$ is an isomorphism between subgroups A and B of G. Given such an HNN pair (G,φ) we consider the corresponding HNN extension

$$G^* = \langle G, t \,|\, t^{-1}at=\varphi(a),\ a\in A\rangle$$

of G, which we denote, by slight abuse of notation, as $G^* = \langle G, t \,|\, t^{-1}At=\varphi(A)\rangle$. Throughout this paper we fix a prime number p, and by a p-group we mean a finite group of p-power order. We are interested in the question under which conditions an HNN extension of a p-group is residually a p-group. (HNN extensions of finite groups are always residually finite [BT78, Co77].) Recall that given a property $\mathcal{P}$ of groups, a group G is said to be *residually* $\mathcal{P}$ if for any non-trivial $g\in G$ there exists a morphism $\alpha\colon G\to P$ to a group P with property $\mathcal{P}$ such that $\alpha(g)$ is non-trivial.

Given HNN pairs (G,φ) and (G',φ'), a group morphism $\alpha\colon G\to G'$ is a morphism of HNN pairs if $\alpha(A)\subseteq A'$, $\alpha(B)\subseteq B'$, and the diagram

$$\begin{tikzcd}
A' \arrow[r, "\varphi'"] & B' \\
A \arrow[u, "\alpha|_A"] \arrow[r, "\varphi"] & B \arrow[u, "\alpha|_B"']
\end{tikzcd}$$

commutes. (When talking about an HNN pair (G,φ), we always denote the domain and codomain of φ by A respectively B, possibly with decorations.) A morphism $\alpha\colon(G,\varphi)\to(G',\varphi')$ of HNN pairs is called an *embedding of HNN pairs* if α is injective. Given a group G and $g\in G$ we denote the conjugation automorphism $x\mapsto g^{-1}xg$ of G by c_g.

There is a well-known criterion for HNN extensions to be residually p:

Lemma 1.1. *Let (G,φ) be an HNN pair, where G is a p-group. Then the following are equivalent:*

1. *the group $G^* = \langle G, t \,|\, t^{-1}At=\varphi(A)\rangle$ is residually p;*
2. *there exists a p-group X and an automorphism γ of X of p-power order such that (G,φ) embeds into (X,γ);*
3. *there exists a p-group Y and $y\in Y$ such that (G,φ) embeds into (Y,c_y).*

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Proof. For a proof of the equivalence of (1) and (3) we refer to [RV91, Proposition 1] or alternatively to Lemma 5.11 below. Clearly (3) implies (2). On the other hand, let X be a p-group and $\gamma \in \operatorname{Aut}(X)$ such that (G,φ) embeds into (X,γ), and suppose γ has order p^k. Let $Y = \mathbb{Z}/p^k\mathbb{Z} \ltimes X$ where $1 \in \mathbb{Z}/p^k\mathbb{Z}$ acts on X on the right via γ, and let $y = (1,1) \in \mathbb{Z}/p^k\mathbb{Z} \ltimes X$. Then (G,φ) embeds into (Y, c_y). ☐

Example. Suppose $A = B = G$, i.e., $G^* = \langle t \rangle \ltimes G$ where t acts on G on the right via φ. Then G^* is residually p if and only if the automorphism φ of G has order a power of p.

Let G be a group. We say that a finite sequence $\boldsymbol{G} = (G_1, \dots, G_n)$ of normal subgroups of G with

$$G = G_1 \supsetneq G_2 \supsetneq \cdots \supsetneq G_n = \{1\}$$

is a *filtration* of G. Given such a filtration $\boldsymbol{G}$ of G we set $G_i := \{1\}$ for $i > n$. We say that $\boldsymbol{G}$ is *central* if G_i/G_{i+1} is central in G/G_{i+1} for each i. Recall that the lower central series of G is the sequence $(\gamma_i(G))_{i\geq 1}$ defined inductively by $\gamma_1(G) = G$ and $\gamma_{i+1}(G) = [G, \gamma_i(G)]$ for $i \geq 1$. By definition G is nilpotent if and only if $\gamma_n(G) = \{1\}$ for some $n \geq 1$. In that case, taking n minimal such that $\gamma_n(G) = \{1\}$, we obtain a central filtration $\gamma(G) = (\gamma_1(G), \dots, \gamma_n(G))$ of G. In fact, G admits a central filtration if and only if G is nilpotent.

A filtration of a group G is called a *chief filtration* of G if the filtration cannot be refined non-trivially to another filtration of G. It is well-known that a filtration (G_i) of a p-group is a chief filtration if and only if all its non-trivial factors G_i/G_{i+1} are of order p. Note that a chief filtration of a p-group is necessarily a central filtration, since $\mathbb{Z}/p\mathbb{Z}$ has no non-trivial automorphism of p-power order.

We say that an HNN pair (G,φ) and a filtration (G_i) of G as above are *compatible* if φ restricts to an isomorphism $A \cap G_i \to B \cap G_i$, for each i. We recall the following theorem, which gives an intrinsic criterion for an HNN extension of a p-group to be residually p. This theorem was shown in [Ch94, Lemma 1.2] (and later rediscovered in [Mo07]); it can be viewed as a version for HNN extensions of Higman's theorem [Hig64], which gives a criterion for an amalgamated product of two p-groups to be residually p.

Theorem 1.2. *Let (G,φ) be an HNN pair, where G is a p-group. Then the following are equivalent:*

1. *the group $G^* = \langle G, t \,|\, t^{-1}At = \varphi(A)\rangle$ is residually p;*
2. *there exists a chief filtration $(G_1, \dots, G_n)$ of G, compatible with (G,φ), such that $\varphi(a) \equiv a \bmod G_{i+1}$ for all i and $a \in A \cap G_i$.*

The objective of this paper is to give an alternative criterion for G^* to be residually p, employing a certain group $H(G,\varphi)$ associated to every HNN pair (G,φ), and defined as follows: set $H_0 = G$ and inductively $H_{i+1} = \varphi^{-1}(H_i) \cap H_i \cap \varphi(H_i)$ and put $H(G,\varphi) := \bigcap_i H_i$. Note that φ restricts to an automorphism of $H(G,\varphi)$; in fact $H(G,\varphi)$ is the largest subgroup of G which gets mapped onto itself by φ. The group $H(G,\varphi)$ was introduced by Raptis and Varsos in [RV89, RV91]. It had been previously employed in [Hic81], and a slight variant (the largest normal subgroup of G mapped onto itself by φ) also occurs in [Ba93, Section 1.26]. We propose to call $H(G,\varphi)$ the *core* of (G,φ). Indeed, $H(G,\varphi)$ is the core with respect to $\langle t \rangle$ of G construed as a subgroup of G^*, i.e., $H(G,\varphi) = \bigcap_{i\in\mathbb{Z}} t^{-i}Gt^i$; so if G is abelian, then $H(G,\varphi)$ is indeed the core of G in G^* (the largest normal subgroup of G^*

contained in G). See Lemma 2.1, where we give other alternative descriptions of the core of (G,φ) which are oftentimes useful. Note that if $\alpha\colon (G,\varphi)\to(G',\varphi')$ is an embedding of HNN pairs, then $\alpha(H(G,\varphi))\leq H(G',\varphi')$.

If (G_i) is a filtration compatible with the HNN pair (G,φ), then for any $i<j$ the morphism φ induces an isomorphism

$$(A\cap G_i)G_j/G_j \to (B\cap G_i)G_j/G_j,$$

which we denote by φ_{ij}. For $a\in A\cap G_i$, the conjugation automorphism c_a of G induces an automorphism of $(A\cap G_i)G_j/G_j$ which we continue to denote by c_a, and similarly for c_b with $b\in B\cap G_i$. We can now formulate our first result, which gives an obstruction to an HNN extension of a p-group being residually p. The statement of the proposition is inspired by the ideas of [RV91]. *For the rest of this section we fix an HNN pair (G,φ) where G is a p-group, and we let $G^*=\langle G,t\,|\,t^{-1}At=\varphi(A)\rangle$.*

Proposition 1.3. *If G^* is residually p, then there exists a central filtration $\boldsymbol{G}=(G_i)$ of G, compatible with (G,φ), such that for any $i<j$, any $a\in A\cap G_i$ and any $b\in B\cap G_i$ the order of the automorphism of $H(G_i/G_j, c_b\circ\varphi_{ij}\circ c_a)$ induced by $c_b\circ\varphi_{ij}\circ c_a$ is a power of p.*

In [RV91, Theorem 13] it is claimed that the following strong converse to Proposition 1.3 holds: *If there exists a central filtration of G, compatible with (G,φ), and if the order of the automorphism of $H(G,\varphi)$ induced by φ is a power of p, then the HNN extension G^* of G is residually p.* In Section 4 we show that this statement is incorrect; in fact, we give two counterexamples to [RV91, Theorem 13], highlighting the role of a and b and the importance of the filtration $\boldsymbol{G}$ in Proposition 1.3.

Our main theorem is the following converse to Proposition 1.3.

Theorem 1.4. *Suppose there exists a central filtration $\boldsymbol{G}=(G_i)$ of G, compatible with (G,φ), such that for any i the order of the automorphism of $H(G_i/G_{i+1},\varphi_{i,i+1})$ induced by $\varphi_{i,i+1}$ is a power of p. Then G^* is residually p.*

Note that for a chief filtration $\boldsymbol{G}$ of G, the statement of Theorem 1.4 is equivalent to the implication (2) $\Rightarrow$ (1) in Theorem 1.2.

For every filtration (G_i) of G compatible with (G,φ) and any i, the group $H(G_i/G_{i+1},\varphi_{i,i+1})$ is a subgroup of $H(G/G_{i+1},\varphi_{i+1})$; here φ_i is the isomorphism $AG_i/G_i\to BG_i/G_i$ induced by φ. We therefore get the following corollary, which is perhaps of interest in light of the claims in [RV91].

Corollary 1.5. *Assume that there exists a central filtration (G_i) of G, compatible with (G,φ), such that for any i the order of the automorphism of $H(G/G_i,\varphi_i)$ induced by φ_i is a power of p. Then G^* is residually p.*

Remark. Note that the statement of Corollary 1.5 for an abelian p-group with the trivial filtration is the same as the statement of [RV91, Proposition 7]. However, the proof given in [RV91] is seriously flawed. We refer to Section 2 for details.

Conventions. All groups are finitely generated. By a p-group we mean a finite group of p-power order. The identity element of a multiplicatively written group is denoted by 1.

2. Preliminaries on the Core of an HNN Pair

In this section, we let (G,φ) be an HNN pair with corresponding HNN extension G^* of G. For $g\in G$ and $n\in\mathbb{N}$ we say that $\varphi^n(g)$ is defined if g is in the domain of the n-fold compositional iterate of φ thought of as a partially defined map $G\to G$, and similarly we say that $\varphi^{-n}(g)$ is defined if g is in the domain of the n-fold iterate of φ^{-1}. We first prove:

Lemma 2.1. *The group $H=H(G,\varphi)$ is the largest subgroup of G such that $\varphi(H)=H$, and as subgroups of G^*,*

$$H=\bigcap_{i\in\mathbb{Z}} t^{-i}Gt^{i}. \tag{2.1}$$

Moreover,

$$H=\{g\in G:\varphi^{j}(g) \text{ is defined for all } j\in\mathbb{Z}\}. \tag{2.2}$$

If A is finite, then there exists an integer $r\geq 0$ such that

$$H=\{g\in G:\varphi^{j}(g) \text{ is defined for } j=0,\dots,s\} \qquad \text{for all } s\geq r. \tag{2.3}$$

Proof. Recall that we introduced $H=\bigcap_i H_i$ as the intersection of the inductively defined descending sequence of subgroups

$$G=H_0\supseteq H_1\supseteq\cdots\supseteq H_i\supseteq H_{i+1}=\varphi^{-1}(H_i)\cap H_i\cap\varphi(H_i)\supseteq\cdots$$

of G. Clearly $\varphi(H_i)\subseteq H_{i+1}$ for each i, hence $\varphi(H)\subseteq\varphi(\bigcap_i H_{i+1})\subseteq H$; similarly, $\varphi^{-1}(H)\subseteq H$, hence $\varphi(H)=H$. Moreover, given any $H'\leq G$ with $\varphi(H')=H'$, an easy induction on i shows that $H'\subseteq H_i$ for all i, so $H'\subseteq H$. To prove (2.1) and (2.2) we show, by induction on i:

$$H_i=\bigcap_{|j|\leq i} t^{-j}Gt^{j}=\{g\in G:\varphi^{j}(g) \text{ is defined for all } j \text{ with } |j|\leq i\}. \tag{2.4}$$

This is clear for $i=0$. For $i=1$ note that the Normal Form Theorem for HNN extensions [Ro95, Theorem 11.83] yields $G\cap tGt^{-1}=A=tBt^{-1}$, hence

$$t^{-1}Gt\cap G\cap tGt^{-1}=A\cap B=\varphi(G)\cap G\cap\varphi^{-1}(G)=H_1.$$

Moreover, given $g\in G$, clearly both $\varphi(g)$ and $\varphi^{-1}(g)$ are defined precisely if $g\in A\cap B$. Now suppose (2.4) has been shown for some $i\geq 1$. Since $H_i\subseteq A\cap B$ we have

$$H_{i+1}=\varphi(H_i)\cap H_i\cap\varphi^{-1}(H_i)=t^{-1}H_it\cap H_i\cap tH_it^{-1}=\bigcap_{|j|\leq i+1} t^{-i}Gt^{i},$$

where we used the inductive hypothesis for the last equality. Now let $g\in G$. Then $g\in H_{i+1}$ if and only if $\varphi(g),g,\varphi^{-1}(g)\in H_i$. By the inductive hypothesis, this in turn is equivalent to $\varphi^{j}(g)$ being defined for all j with $|j|\leq i+1$.

For $i\geq 0$ we now define

$$H_i'=\{g\in G:\varphi^{j}(g) \text{ is defined for } j=0,\dots,i\}$$

and set $H':=\bigcap_i H_i'$. Then clearly $H\subseteq H'$ and $\varphi(H')\subseteq H'$. Suppose A is finite; then there is an integer $r\geq 0$ with $H'=H_r'=H_{r+1}'=\cdots$. To show (2.3) we prove that $\varphi(H')=H'$ (which yields $H=H'$ by the first part of the lemma). Let $g\in H'$. Since A is finite, there exist $k\geq 0$ and $l>0$ such that $\varphi^{k+l}(g)=\varphi^{k}(g)$. Since φ is injective, $\varphi^{l}(g)=g$, hence $\varphi^{-1}(g)=\varphi^{l-1}(g)$ exists, and clearly $\varphi^{-1}(g)\in H'$. Hence $\varphi^{-1}(H')\subseteq H'$ and thus $\varphi(H')=H'$ as required. □

The main difficulty in dealing with the core of (G, φ) is that it does not behave well under taking quotients. For example, if $H(G, \varphi)$ is trivial, and if $K \leq G$ such that $\varphi(A \cap K) = B \cap K$ and $\overline{\varphi}\colon AK/K \to BK/K$ is the isomorphism induced by φ, then it is not true in general that $H(G/K, \overline{\varphi})$ is trivial. A non-abelian example of this phenomenon is given in Section 4.2. But this can even happen in the abelian case:

Example. Suppose

$$G = \mathbb{F}_p^3, \quad A = \big\{(a_1, a_2, 0) : a_i \in \mathbb{F}_p\big\}, \quad B = \big\{(b_1, 0, b_2) : b_i \in \mathbb{F}_p\big\}.$$

Then $A \cap B = \big\{(c, 0, 0) : c \in \mathbb{F}_p\big\}$. Let $x, y, z \in \mathbb{F}_p$ with $x, z \neq 0$, and suppose φ is the isomorphism $A \to B$ given by

$$\varphi(a_1, a_2, 0) = (xa_1, 0, ya_1 + za_2).$$

If $y \neq 0$, then $\varphi(A \cap B) \cap (A \cap B) = 0$, in particular $H(G, \varphi) = 0$. Now let

$$K = \big\{(0, k_1, k_2) : k_1, k_2 \in \mathbb{F}_p\big\}.$$

Then $\varphi(A \cap K) = B \cap K$, but $H(G/K, \overline{\varphi})$ is non-trivial, in fact it equals $G/K \cong \mathbb{F}_p$ and the automorphism induced by φ is multiplication by x.

A very similar example shows that the proof of [RV91, Lemma 6] does not work in general:

Example. Suppose

$$G = \mathbb{F}_p^4, \quad A = \{(a_1, a_2, a_3, 0) : a_i \in \mathbb{F}_p\}, \quad B = \{(b_1, 0, b_2, b_3) : b_i \in \mathbb{F}_p\},$$

so $A \cap B = \{(c_1, 0, c_2, 0) : c_i \in \mathbb{F}_p\}$. Let $a, b, c \in \mathbb{F}_p$, and suppose φ is the isomorphism $A \to B$ given by

$$\varphi(a_1, a_2, a_3, 0) = (aa_1, 0, ba_1 + a_2, ca_1 + a_3).$$

If $c \neq 0$, then $H = 0$, and for $x = (0, 0, 1, 0) \in A \cap B$ we have $L_x = \{(0, d_1, d_2, d_3) : d_i \in \mathbb{F}_p\}$ (employing the notation of [RV91]), but $H(G/L_x, \overline{\varphi}) \neq 0$, contrary to what is assumed in the inductive step in the proof of [RV91, Lemma 6].

3. Obstructions to HNN Extensions Being Residually p

Before we give the proof of Proposition 1.3 we prove the following lemma. Again, we let (G, φ) be an HNN pair.

Lemma 3.1. *Let Y be a group, $y \in Y$, and $\alpha\colon (G, \varphi) \to (Y, c_y)$ an embedding of HNN pairs.*

1. *If Y is a p-group, then the order of the restriction of φ to $H(G, \varphi)$ is a power of p.*
2. *Let $a \in A$, $b \in B$; then α is an embedding $(G, c_b \circ \varphi \circ c_a) \to (Y, c_{ayb})$.*

Proof. For the first statement write $H = H(G, \varphi)$ and identify G with $\alpha(G) \leq Y$. Then $\varphi|_H = c_y|_H$, hence the order of $\varphi|_H$ divides the order of y. The second statement follows immediately from $c_{ayb} = c_b \circ c_y \circ c_a$. ☐

Proof of Proposition 1.3. Suppose G is a p-group and $\langle G, t \,|\, t^{-1}At = \varphi(A)\rangle$ is residually p. By Lemma 1.1 we can find a p-group Y, $y \in Y$ and an embedding $\alpha\colon (G, \varphi) \to (Y, c_y)$ of HNN pairs. We identify G with its image under α.

Let (Y_i) be any central filtration of Y, and let $G_i = Y_i \cap G$ for each i. Evidently G_i/G_{i+1} is central in G/G_{i+1} for any i. Possibly after renaming we can also achieve that for each i we have $G_{i+1} \subsetneq G_i$, i.e., $\boldsymbol{G} = (G_i)$ is a central filtration of G. Furthermore note that for any i the following holds:

$$\varphi(A \cap G_i) = \varphi(A \cap G \cap Y_i) = c_y(A \cap Y_i) = B \cap Y_i = B \cap G_i$$

since Y_i is normal in Y. This shows that $\boldsymbol{G}$ is compatible with (G, φ).

Finally let $i < j$. Then α gives rise to an embedding

$$(G_i/G_j, \varphi_{ij}) \to (G/G_j, \varphi_j) \to (Y/Y_j, c_{yY_j})$$

of HNN pairs. It follows now from Lemma 3.1 that for any $a \in A \cap G_i$, $b \in B \cap G_i$ the order of the restriction of $c_b \circ \varphi_{ij} \circ c_a$ to $H(G_i/G_j, \varphi_{ij})$ is a power of p. □

4. Examples

In this section we apply Proposition 1.3 to two HNN extensions. The first example highlights the role of a and b in Proposition 1.3, the second one shows the importance of the central series. Both are counterexamples to [RV91, Theorem 13].

4.1. **The first example.**

The multiplicative group

$$P := \langle x, y \,|\, x^3 = y^3 = [x, y] = e \rangle$$

is naturally isomorphic to the additive group $\mathbb{F}_3 \oplus \mathbb{F}_3$. We write $\langle x \rangle$ and $\langle y \rangle$ for the subgroups of P generated by x and y, respectively. We think of elements in the group ring $\mathbb{F}_3[P]$ as polynomials $f(x, y) = \sum_{i,j=0}^{2} v_{ij} x^i y^j$ with coefficients $v_{ij} \in \mathbb{F}_3$. Furthermore $f(x)$ always denotes an element in the subring $\mathbb{F}_3[\langle x \rangle]$ of $\mathbb{F}_3[P]$ and similarly $f(y)$ will denote an element in $\mathbb{F}_3[\langle y \rangle] \subseteq \mathbb{F}_3[P]$.

Let $G = P \ltimes \mathbb{F}_3[P]$ where P acts on its group ring $\mathbb{F}_3[P]$ by multiplication. (Here P is a multiplicative group and $\mathbb{F}_3[P]$ is an additive group. Note that G is in fact just the wreath product $\mathbb{F}_3 \wr P$.) Evidently G is a 3-group. For $f \in \mathbb{F}_3[P]$ we have

$$c_{(x^n y^m, 0)}(1, f(x, y)) = (x^{-n} y^{-m}, 0)(1, f(x, y))(x^n y^m, 0) = (1, x^n y^m f(x, y)).$$

Now consider the subgroups $A = \langle x \rangle \ltimes \mathbb{F}_3[\langle x \rangle]$ and $B = \langle y \rangle \ltimes \mathbb{F}_3[\langle y \rangle]$ of G, and let $\varphi\colon A \to B$ be the map given by

$$\varphi(x^n, f(x)) = (y^n, 2y^{-1} f(y)).$$

It is straightforward to verify that φ is indeed an isomorphism. In fact, φ is the restriction to A of the automorphism ϕ of $G = P \ltimes \mathbb{F}_3[P]$ given by

$$(x^n y^m, f(x, y)) \mapsto (x^m y^n, 2y^{-1} f(y, x)).$$

Claim. The HNN pair (G, φ) is compatible with the lower central series $\gamma(G)$ of G.

The claim follows immediately from the fact that φ is the restriction of an automorphism of G, and the fact that the groups in the lower central series are characteristic. Indeed, we compute

$$\begin{aligned} \varphi(A \cap \gamma_i(G)) &= \phi(A \cap \gamma_i(G)) = \\ &= \phi(A) \cap \phi(\gamma_i(G)) = \phi(A) \cap \gamma_i(G) = B \cap \gamma_i(G). \end{aligned}$$

Claim. The subgroup $H(G, \varphi)$ of G is trivial.

Indeed, first note that $A \cap B = \{(1, v) : v \in \mathbb{F}_3\}$. But $\varphi(1, v) = (1, 2y^{-1} v)$. This shows that $(A \cap B) \cap \varphi(A \cap B) = \{1\}$, hence $H(G, \varphi) = \{1\}$.

If [RV91, Theorem 13] was correct, then $\langle G, t \,|\, t^{-1} A t = \varphi(A) \rangle$ would have to be a residually 3-group. But the combination of the next claim with Proposition 1.3 shows that this is not the case.

Claim. Put $\psi := \varphi \circ c_{(x,0)} \colon A \to B$. Then $H(G,\psi) \neq \{1\}$, and the restriction of ψ to $H(G,\psi)$ has order 2.

We have

$$\psi(1,v) = (\varphi \circ c_{(x,0)})(1,v) = \varphi(1,vx) = (1, 2y^{-1}vy) = (1,2v).$$

This shows that ψ induces an automorphism of $A \cap B$, and the automorphism has order 2. It follows immediately that $H(G,\psi) = \{(1,v) : v \in \mathbb{F}_3\}$ and that ψ restricted to $H(G,\psi)$ has order 2.

4.2. **The second example.** In the following we write elements of $\mathbb{F}_3 \oplus \mathbb{F}_3 \oplus \mathbb{F}_3$ as column vectors. The automorphism of $\mathbb{F}_3 \oplus \mathbb{F}_3 \oplus \mathbb{F}_3$ given by the matrix

$$X := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

clearly descends to an automorphism of

$$V := (\mathbb{F}_3 \oplus \mathbb{F}_3 \oplus \mathbb{F}_3)/\big\{(a,a,a)^{\mathrm{t}} : a \in \mathbb{F}_3\big\}.$$

In the rest of this section a column vector in $\mathbb{F}_3 \oplus \mathbb{F}_3 \oplus \mathbb{F}_3$ will always stand for the element in V it represents. Consider the 3-group $G := \langle x \,|\, x^3 \rangle \ltimes V$ where x acts on V on the right via X. Note that given integers m, n and $u, v \in V$ we have

$$\begin{array}{rcl} [(x^m,u),(x^n,v)] & = & (x^m,u)\cdot(x^n,v)\cdot(x^{-m},-X^{-m}u)\cdot(x^{-n},-X^{-n}v) \\ & = & \big(1, X^{-n}(X^{-m}v - v) - X^{-m}(X^{-n}u - u)\big). \end{array}$$

Since for $r = 1,2$ we have

$$(X^r - \mathrm{id})(V) = \big\{(w_1,w_2,w_3)^{\mathrm{t}} : w_1 + w_2 + w_3 = 0\big\},$$

it follows that

$$\begin{array}{rcl} \gamma_2(G) = [G,G] & = & \big\{(1,w) : w \in (X - \mathrm{id})(V)\big\} \\ & = & \big\{\big(1,(w_1,w_2,w_3)^{\mathrm{t}}\big) : w_1 + w_2 + w_3 = 0\big\}. \end{array}$$

A similar calculation shows that

$$\gamma_3(G) = [G,[G,G]] = \big\{\big(1,(a,a,a)^{\mathrm{t}}\big) : a \in \mathbb{F}_3\big\} = 0.$$

Now let

$$a = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \in V \subseteq G.$$

Note that a and b represent the same element in $G/[G,G]$. Let A and B be the subgroups of $V \leq G$ generated by a and b, respectively. Let $\varphi \colon A \to B$ be the isomorphism given by $\varphi(a) = 2b$. Note that $A \cap \gamma_2(G) = B \cap \gamma_2(G) = \{1\}$. It follows that the HNN pair (G,φ) is compatible with the filtration of G given by the lower central series. Finally note that A and B intersect trivially. In particular $H(G,\varphi)$ is trivial.

If [RV91, Theorem 13] was correct, then $\langle G, t \,|\, t^{-1}At = \varphi(A)\rangle$ would have to be a residually 3-group. The following lemma in conjunction with Proposition 1.3 shows that this is not the case.

Lemma 4.1. *There exists no central filtration (G_i) of G, compatible with the HNN pair (G,φ), such that for any i the order of the automorphism of $H(G/G_i,\varphi_i)$ induced by $\varphi_i \colon AG_i/G_i \to BG_i/G_i$ is a power of* 3.

Proof. Let $G = G_1 \supsetneq G_2 \supsetneq G_3 \supsetneq \cdots \supsetneq G_n = \{1\}$ be a central filtration of G compatible with (G, φ). Denote the natural surjection $G \to G/G_i$ by π_i. Note that π_2 factors through $G/[G,G]$ and therefore $\pi_2(a) = \pi_2(b)$.

First assume that $\pi_2(a) \neq 0$. In that case the subgroup $\pi_2(A)$ of G/G_2 is isomorphic to $\mathbb{F}_3$, and $\pi_2(A) = \pi_2(B)$. It follows that $H(G/G_2, \varphi_2) = \pi_2(A) = \pi_2(B) \cong \mathbb{F}_3$. Furthermore, since $\pi_2(a) = \pi_2(b)$ and $\varphi(a) = 2b$ it follows that the automorphism of $H(G/G_2, \varphi_2) \cong \mathbb{F}_3$ induced by φ_2 is multiplication by 2, which has order 2, hence is not a power of 3.

Now assume that $\pi_2(a) = 0$. In that case we have $G \supsetneq G_2 \supseteq [G,G]$ and $a \in G_2$. Recall that $G/[G,G] \cong \mathbb{F}_3^2$ and $a \notin [G,G]$. It follows easily that $G_2 = \{1\} \times V \subseteq G = \langle x \,|\, x^3 \rangle \ltimes V$. Recall that by the definition of a central series, G_2/G_3 is central in G/G_3. In particular we have $(1, Xv - v) = [(x^{-1}, 0), (1, v)] \in G_3$ for any $v \in V$. Also note that $G_2 \supsetneq G_3$. It now follows immediately that

$$G_3 = \big\{ \big(1, (w_1, w_2, w_3)^{\mathrm{t}}\big) : w_1 + w_2 + w_3 = 0 \big\} = [G,G].$$

We have $\pi_3(a) = \pi_3(b) \neq 0$. We now apply the same argument as above to see that $H(G/G_3, \varphi_3) \cong \mathbb{F}_3$ and that the order of the automorphism of $H(G/G_3, \varphi_3)$ induced by φ_3 is not a power of 3. □

5. Proof of Theorem 1.4

In the first two subsections we collect some basic facts which are needed later on. In subsection 5.3 we then prove a special case of the theorem, and after recalling the notion of a graph of groups in subsection 5.4, we give the proof of the general case of Theorem 1.4 in subsection 5.5.

5.1. **An extension lemma.**

The following lemma will play a prominent role in our proof of Theorem 1.4.

Lemma 5.1. *Let (G, φ) an HNN pair, where G is a p-group. Suppose there exists a central filtration (G_i) compatible with (G, φ) such that for each i there is a p-group Q_i containing $L_i := G_i/G_{i+1}$ as a subgroup such that the isomorphism*

$$A_i := (A \cap G_i)G_{i+1}/G_{i+1} \xrightarrow{\varphi_{i,i+1}} B_i := (B \cap G_i)G_{i+1}/G_{i+1}$$

between subgroups of L_i induced by φ is the restriction of an inner automorphism of Q_i. Then the HNN extension $G^* = \langle G, t \,|\, t^{-1}At = \varphi(A) \rangle$ of G is residually p.

Proof. Throughout the proof we write $\varphi_i = \varphi_{i,i+1}$. (This differs from the use of this notation in Section 1.) Note that by Lemma 1.1 and Theorem 1.2 we can take for each i a chief filtration $(H_{i1}, \dots, H_{im_i})$ of L_i such that $\varphi_i(A_i \cap H_{ij}) = B_i \cap H_{ij}$ for any j and such that

$$\varphi_i(a) \equiv a \bmod H_{i,j+1} \qquad \text{for all } j \text{ and } a \in A_i \cap H_{ij}. \tag{5.1}$$

For each i let $\pi_i \colon G_i \to G_i/G_{i+1} = L_i$ be the natural epimorphism. We set $G_{ij} := \pi_i^{-1}(H_{ij}) \leq G_i$. For each i we have $G_{i1} = G_i$ and $G_{in_i} = G_{i+1}$.

Claim. For any i, j the subgroup G_{ij} is normal in G.

Denote by π the natural surjection $G \to G/G_{i+1}$. Note that if we consider $L_i = G_i/G_{i+1}$ as a subgroup of G/G_{i+1} as usual, then π_i is the restriction of π to G_i, so $G_{ij} = \pi^{-1}(H_{ij})$. It therefore suffices to show that H_{ij} is normal in G/G_{i+1}. But this follows immediately from the fact that G_i/G_{i+1} lies in the center of G/G_{i+1}. This concludes the proof of the claim.

We now get the following filtration of G:

$$G = G_{11} \supsetneq G_{12} \supsetneq \cdots \supsetneq G_{1m_1} = G_2 = G_{21} \supsetneq G_{22} \supsetneq \cdots \supsetneq G_{nm_n} = \{1\}.$$

Evidently each successive non-trivial quotient is isomorphic to $\mathbb{Z}/p\mathbb{Z}$, hence the above is a chief filtration for G. Finally note that (5.1) implies that

$$\varphi(a) \equiv a \bmod G_{i,j+1} \qquad \text{for all } i,\, j \text{ and } a \in A \cap G_{ij}.$$

Hence the chief filtration satisfies condition (2) in Theorem 1.2, and we conclude that the HNN extension G^* is residually p. □

5.2. **Subgroups of homocyclic p-groups.** In this subsection G denotes a homocyclic p-group, i.e., $G = (\mathbb{Z}/p^k\mathbb{Z})^d$ for some d and $k > 0$. The group G has a natural structure of an R-module where R is the ring $\mathbb{Z}/p^k\mathbb{Z}$, and considered as an R-module in this way, G is free, and every subgroup of G is a submodule of G. We denote the canonical basis elements of the R-module G by $e_1, \dots, e_d$. In the following we prove some basic facts about subgroups of G. The main tool is the next lemma, which is an immediate consequence of the Elementary Divisors Theorem for subgroups of $\mathbb{Z}^d$ (cf. [La02, Theorem III.7.8]):

Lemma 5.2. *Let $H \leq G$. Then there exists an automorphism ψ of G and integers $0 \leq k_1 \leq k_2 \leq \cdots \leq k_s < k$, for some $s \in \{0, \dots, d\}$, such that*

$$\psi(H) = p^{k_1} R\, e_1 \oplus \cdots \oplus p^{k_s} R\, e_s.$$

Let H be an abelian p-group. For each $i \geq 0$ we introduce the characteristic subgroup

$$H[p^i] := \{h \in H : p^i h = 0\}$$

of H. For each i we have $H[p^i] \subseteq H[p^{i+1}]$. The subgroup $H[p]$ is called the *socle* of H; it is the additive group of an $\mathbb{F}_p$-linear space. For subgroups of G, we have:

Lemma 5.3. *Let $H \leq G$. Then*

1. *$H[p^i] = H \cap p^{k-i} G$ for $i = 0, \dots, k$;*
2. *$\dim_{\mathbb{F}_p} H[p] = d(H)$, where $d(H)$ denotes the minimal number of generators of H; in particular, $H = 0$ if and only if $H[p] = 0$.*

Proof. By Lemma 5.2 we may assume that H has the form

$$H = p^{k_1} R\, e_1 \oplus \cdots \oplus p^{k_s} R\, e_s \quad \text{where } 0 \leq k_1 \leq \cdots \leq k_s < k. \tag{5.2}$$

Then $H[p] = p^{k-1} R\, e_1 \oplus \cdots \oplus p^{k-1} R\, e_s$, and the lemma follows. □

Recall also that a subgroup K of an abelian p-group H is said to be *pure* in H if $K \cap p^i H = p^i K$ for each i. Suppose $H' \leq H \leq K$; if H' is pure in K, then H' is pure in H, and if H' is pure in H and H is pure in K, then H' is pure in K. If K is a direct summand of H, then K is pure in H. One verifies easily that $R = \mathbb{Z}/p^k\mathbb{Z}$ has no non-trivial pure subgroups. The notion of a pure subgroup of G coincides with that of a direct summand of G; in fact:

Lemma 5.4. *Let $H \leq G$. Then*

1. *there exists a smallest subgroup $\overline{H}$ of G which contains H and which is pure in G, and this subgroup $\overline{H}$ of H is a free R-module and a direct summand of G with $\overline{H}[p] = H[p]$;*
2. *the following are equivalent:*
   - (a) *H is pure in G,*

   (b) *H is a free R-module,*
   (c) *H is a direct summand of G;*
(3) *if $\varphi\colon \overline{H} \to G$ is an injective morphism, then $\varphi(\overline{H}) = \overline{\varphi(H)}$.*

Proof. Again we may assume that H has the form (5.2). Then H is pure in G if and only if $k_i = 0$ for each i. Clearly $\overline{H} := R\,e_1 \oplus \cdots \oplus R\,e_s$ has the properties claimed in (1). In particular, if H is pure, then $H = \overline{H}$ is a free R-module. On the other hand, if the R-module H is free, then

$$|H| = p^{k\,d(H)} = p^{k\,d(\overline{H})} = |\overline{H}|$$

and hence $H = \overline{H}$ is a direct summand of G. This shows (2). For (3) let $\varphi\colon \overline{H} \to G$ be an injective morphism. Note that $\varphi(H) \leq \varphi(\overline{H})$ and $\varphi(\overline{H})$ is pure in G, hence $\overline{\varphi(H)} \leq \varphi(\overline{H})$; similarly, using the injective morphism $\varphi^{-1}|_{\overline{\varphi(H)}}$ in place of φ, we have $\overline{H} \leq \varphi^{-1}(\overline{\varphi(H)})$, and thus $\varphi(\overline{H}) = \overline{\varphi(H)}$. □

Lemma 5.5. *Let $H, K \leq G$. Then*
(1) *$(H \cap K)[p] = H[p] \cap K[p]$;*
(2) *if $H[p] \cap K[p] = 0$, then $H \cap K = 0$;*
(3) *if H and K are pure in G and $H \cap K = 0$, then $H \oplus K$ (internal direct sum of subgroups of G) is pure in G.*

Proof. Part (1) immediate follows from the definition of the socle, and when combined with Lemma 5.3, (2), in turn implies part (2). For (3) note that if H and K are pure in G, then H and K are free R-modules by Lemma 5.4, (2), so if in addition $H \cap K = \{0\}$, then the R-module $H \oplus K$ is also free, so by Lemma 5.4, (2) again, $H \oplus K$ is pure in G. □

Lemma 5.6. *The group G is homogeneous, i.e., every isomorphism $A \to B$ between subgroups A, B of G extends to an automorphism of G.*

Proof. Let $A, B \leq G$ and $\varphi\colon A \to B$ be an isomorphism. Note that for each i, φ restricts to an isomorphism $A[p^i] \to B[p^i]$. By Lemmas 5.2 and 5.4, (1) there are a basis $v_1, \dots, v_s$ of the free R-module $\overline{A}$ and $0 \leq k_1 < \cdots < k_s < k$ such that $A = p^{k_1} R\,v_1 \oplus \cdots \oplus p^{k_s} R\,v_s$. We have $p^{k_i} v_i \in A \cap p^{k_i} G = A[p^{k-k_i}]$, hence there exists $w_i \in G$ such that $\varphi(p^{k_i} v_i) = p^{k_i} w_i \in B$. Let $\psi\colon \overline{A} \to G$ be the R-linear map with $\psi(v_i) := w_i$. Then ψ clearly extends φ. Moreover, ψ is injective: given $a \in \overline{A} \setminus \{0\}$ there exists some i such that $p^i a \in A \setminus \{0\}$, hence $p^i \psi(a) = \psi(p^i a) = \varphi(p^i a) \neq 0$, since φ is injective; thus $\psi(a) \neq 0$. By Lemma 5.4, (3) we have $\psi(\overline{A}) = \overline{B}$. Now take $P, Q \leq G$ with $G = \overline{A} \oplus P = \overline{B} \oplus Q$. Then P and Q are free R-modules of the same rank, hence ψ extends to an automorphism of G with $\psi(P) = Q$. □

(In fact, if p is odd, then the only homogeneous p-groups are the homocyclic ones, cf. [CF00].)

We recall a well-known criterion for an endomorphism of G to be an automorphism of p-power order:

Lemma 5.7. *Let $\psi\colon G \to G$ be a group morphism. The following are equivalent:*
(1) *ψ is an automorphism of G of p-power order;*
(2) *the endomorphism $\overline{\psi}$ of G/pG induced by ψ is an automorphism of G/pG of p-power order;*

(3) *the restriction of* ψ *to an endomorphism of* $G[p]$ *is an automorphism of* $G[p]$ *of* p*-power order.*

Proof. Clearly (1) $\Rightarrow$ (3). For (3) $\Rightarrow$ (2) note that the surjective morphism $G \to G[p] = p^{k-1}G$ given by $g \mapsto p^{k-1}g$ has kernel pG, and hence gives rise to an isomorphism $\mu\colon G/pG \to G[p]$. The following diagram commutes:

$$\begin{tikzcd}
G/pG \arrow[r, "\overline{\psi}"] \arrow[d, "\mu", "\cong"'] & G/pG \arrow[d, "\mu"', "\cong"] \\
G[p] \arrow[r, "\psi"] & G[p].
\end{tikzcd}$$

Hence if ψ restricts to an automorphism of $G[p]$ of p-power order, then $\overline{\psi}$ is an automorphism of G/pG of p-power order. This shows (3) $\Rightarrow$ (2). Let $C \in \mathbb{Z}^{d\times d}$ be a $d \times d$-matrix with integer entries. If $C \equiv \mathrm{id} \mod p^i$, where $i > 0$, so $C = \mathrm{id} + p^i D$ some $D \in \mathbb{Z}^{d\times d}$, then

$$C^p = \mathrm{id} + p \cdot p^i D + \binom{p}{2} \cdot p^{2i} D^2 + \cdots \equiv \mathrm{id} \mod p^{i+1}.$$

It follows inductively that if $C \equiv \mathrm{id} \mod p$, then $C^{p^i} \equiv \mathrm{id} \mod p^{i+1}$ for each $i \geq 0$. In particular, if C represents ψ, and (2) holds, then $C^{p^j} \equiv \mathrm{id} \mod p$ for some $j \geq 0$ and hence $C^{p^{j+k-1}} = (C^{p^j})^{p^{k-1}} \equiv \mathrm{id} \mod p^k$. This shows (2) $\Rightarrow$ (1). ☐

Corollary 5.8. *Let* $\varphi\colon \overline{H} \to G$ *be an injective group morphism such that* $\varphi(H) = H$*, and* $\varphi|_H$ *is an automorphism of* H *of* p*-power order. Then* $\varphi(\overline{H}) = \overline{H}$*, and* φ *is an automorphism of* $\overline{H}$ *of* p*-power order.*

Proof. By Lemma 5.4, (3) we have $\varphi(\overline{H}) = \overline{\varphi(H)} = \overline{H}$. The restriction of φ to $H[p] = \overline{H}[p]$ has p-power order, hence by Lemma 5.7, φ is an automorphism of $\overline{H}$ of p-power order. ☐

5.3. **The proof of a special case of Theorem 1.4.** The following proposition, together with Lemma 1.1, provides a proof for Theorem 1.4 in the special case where G is abelian equipped with the trivial filtration and where furthermore $H(G, \varphi) = A \cap B$. We later reduce the general case to this special case.

Proposition 5.9. *Let* (G, φ) *be an HNN pair with* G *an abelian* p*-group such that* $\varphi(A \cap B) = A \cap B$*, i.e.,* φ *induces an automorphism of* $A \cap B$*, and assume that the order of this automorphism is a power of* p*. Then there exists an abelian* p*-group* X *and an automorphism* α *of* X *of* p*-power order such that* (G, φ) *embeds into* (X, α)*.*

Let (G, φ) be an HNN pair satisfying the hypotheses of the proposition. Then we can take a group embedding $\iota\colon G \to H := (\mathbb{Z}/p^k\mathbb{Z})^d$, for some d and $k > 0$ (cf. [RV91, p. 172]). Then ι is an embedding $(G, \varphi) \to (H, \psi)$ of HNN pairs, where $\psi := \iota \circ \varphi \circ \iota^{-1}\colon \iota(A) \to \iota(B)$. After passing from (G, φ) to (H, ψ), we can therefore assume that G is a homocyclic p-group. We now fix an isomorphism $\overline{A} \to \overline{B}$ extending φ as in Lemma 5.6, and also denote it by φ. By Lemma 5.4, (3) we have $\varphi(\overline{A}) = \overline{B}$. Since $\overline{A \cap B} \leq \overline{A}$ is pure in the homocyclic p-group $\overline{A}$, we may write

$$\overline{A} = \overline{A \cap B} \oplus P \qquad \text{where } P \leq \overline{A}.$$

Note that our assumption that $\varphi(A\cap B)=A\cap B$ and $\varphi|_{A\cap B}$ is an automorphism of $A\cap B$ of p-power order implies that $\varphi(\overline{A\cap B})=\overline{A\cap B}$ and $\varphi|_{\overline{A\cap B}}$ is an automorphhism of p-power order, by Corollary 5.8. We now write $Q:=\varphi(P)$. Note that P is pure in $\overline{A}$ and hence in G, thus Q is also pure in G.

Lemma 5.10. *$Q\cap\overline{A}=0$, and $Q\oplus\overline{A}$ is pure in G.*

Proof. Since φ maps $A\cap B$ onto itself, φ also maps $(A\cap B)[p]$ onto itself. Hence by Lemma 5.5, (1) we have

$$A[p]\cap B[p]=(A\cap B)[p]=\varphi\big((A\cap B)[p]\big)=\varphi\big((\overline{A\cap B})[p]\big).$$

Since $Q\leq\overline{B}$ we also have $Q[p]\leq\overline{B}[p]=B[p]$ and hence

$$\begin{aligned}Q[p]\cap\overline{A}[p] &= Q[p]\cap B[p]\cap A[p]\\ &= \varphi(P)[p]\cap\varphi\big((\overline{A\cap B})[p]\big)\\ &= \varphi\big((P\cap\overline{A\cap B})[p]\big)=0.\end{aligned}$$

By Lemma 5.5, (2) we thus have $Q\cap\overline{A}=0$, and $Q\oplus\overline{A}$ is pure in G by part (3) of that lemma. □

By the previous lemma we may write

$$G=\overline{A}\oplus Q\oplus S=\overline{A\cap B}\oplus P\oplus Q\oplus S\qquad\text{where } S\leq G.$$

We now let

$$X=\overline{A}\oplus Q\oplus Q_1\oplus\cdots\oplus Q_{p-2}\oplus S,$$

where each $Q_i=Q\times i$ is an isomorphic copy of Q. We view G as a subgroup of X in the obvious way. Note that X is a free R-module. Let ψ be the endomorphism of X such that for all $x\in X$,

$$\psi(x)=\begin{cases}\varphi(x) & \text{if } x\in\overline{A},\\ x\times 1 & \text{if } x\in Q,\\ y\times(i+1) & \text{if } x=y\times i \text{ where } y\in Q,\ i=1,\dots,p-3,\\ \varphi^{-1}(y) & \text{if } x=y\times(p-2)\text{ where } y\in Q,\\ x & \text{if } x\in S.\end{cases}$$

(This definition is inspired by the proof of Lemma 5 in [RV91].) Clearly the restriction of ψ to $\overline{A}=\overline{A\cap B}\oplus P$ equals φ. It remains to show that ψ is an automorphism of p-power order. We already observed that $\varphi|_{\overline{A\cap B}}$ is an automorphism of $\overline{A\cap B}$ of p-power order. Moreover, ψ restricts to an automorphism of the subgroup $P\oplus Q\oplus Q_1\oplus\cdots\oplus Q_{p-2}$ of X having order p, and $\psi|_S=\mathrm{id}$. Hence ψ is an automorphism of X of p-power order. □

5.4. **Fundamental groups of graphs of groups.**

In our proof of the general case of Theorem 1.4 we find it convenient to employ the notion of the fundamental group of a graph of groups. We quickly recall the definitions and some basic properties. We refer to [Se80] or [Ba93] for details.

In the following, a *graph* is a pair $\mathcal{Y}=(V,E)$ consisting of a collection $V=V(\mathcal{Y})$ of vertices and a collection $E=E(\mathcal{Y})\subseteq V\times V$ of oriented edges with the property that for every $e\in E(\mathcal{Y})$ the edge with the opposite orientation, denoted by $\overline{e}$, is also an edge of $\mathcal{Y}$. (See [Se80, p. 13].) We call the set $\{e,\overline{e}\}$ a *topological edge* of $\mathcal{Y}$.

Each edge $e \in E(\mathcal{Y})$ has an *initial vertex* $i(e)$ and a *terminal vertex* $t(e)$; we have $i(\overline{e}) = t(e)$ and $t(\overline{e}) = i(e)$.

A *graph of groups* $\mathcal{G}$ *based on a graph* $\mathcal{Y}$ consists of a family $\{G_v\}_{v\in V}$ of groups and a family $\{\varphi_e\}_{e\in E}$ of isomorphisms $\varphi_e\colon A_e \to B_e$ where $A_e \leq G_{i(e)}$ and $B_e \leq G_{t(e)}$, and we demand that $A_e = B_{\overline{e}}$, $B_e = A_{\overline{e}}$ and $\varphi_e = \varphi_{\overline{e}}^{-1}$ for any $e \in E$.

Let $\mathcal{G}$ be a graph of groups based on a finite graph $\mathcal{Y}$ (i.e., $V = V(\mathcal{Y})$ is finite). We assume that $\mathcal{Y}$ is connected: for all $v, w \in V$ there is a path in $\mathcal{Y}$ from v to w, i.e., a finite sequence $(e_1, \dots, e_n)$ of edges of $\mathcal{Y}$ with $i(e_1) = v$, $t(e_n) = w$ and $t(e_i) = i(e_{i+1})$ for $i = 1, \dots, n-1$. We recall the construction of the *fundamental group* $G = \pi_1(\mathcal{G})$ *of* $\mathcal{G}$ from [Se80, I.5.1]. First, consider the group $\pi(\mathcal{G})$ (the *path group of* $\mathcal{G}$) generated by the groups G_v ($v \in V(\mathcal{Y})$) and the elements $e \in E(\mathcal{Y})$ subject to the relations

$$e\varphi_e(g)\overline{e} = g \qquad (e \in E(\mathcal{Y}),\ g \in A_e).$$

Let v_0 be a vertex of $\mathcal{Y}$. Then $\pi_1(\mathcal{G}, v_0)$ is the set of elements of $\pi(\mathcal{G})$ of the form

$$g = g_0 e_1 g_1 e_2 \cdots e_n g_n$$

where

(1) $e_1, \dots, e_n$ are edges of $\mathcal{Y}$ such that $t(e_i) = i(e_{i+1})$ for $i = 1, \dots, n-1$ and $t(e_n) = i(e_1) = v_0$,
(2) $g_i \in G_{v_i}$ for $i = 0, \dots, n$, where $v_i := t(e_i)$ for $i > 0$.

If $v_1 \in V(\mathcal{Y})$ is another base point, then we have an isomorphism $\pi_1(\mathcal{G}, v_0) \to \pi_1(\mathcal{G}, v_1)$, and by abuse of notation we write $\pi_1(\mathcal{G})$ to denote $\pi_1(\mathcal{G}, v_0)$ if the particular choice of base point is irrelevant.

An alternative description of $\pi_1(\mathcal{G}, v_0)$, which is often useful, is as follows: Let $\mathcal{T}$ be a maximal subtree of $\mathcal{Y}$. The fundamental group $\pi_1(\mathcal{G}, \mathcal{T})$ of $\mathcal{G}$ relative to $\mathcal{T}$ is defined by contracting $\mathcal{T}$ to a point:

$$\pi_1(\mathcal{G}, \mathcal{T}) := \pi(\mathcal{G})/(\text{relations } e = 1 \text{ for all } e \in E(\mathcal{T})).$$

The natural projection $\pi(\mathcal{G}) \to \pi_1(\mathcal{G}, \mathcal{T})$ restricts to an isomorphism $\pi_1(\mathcal{G}, v_0) \to \pi_1(\mathcal{G}, \mathcal{T})$, cf. [Se80, I.5.1, Proposition 20].

Example. Suppose that $\mathcal{Y}$ has only one topological edge $\{e, \overline{e}\}$, and let $v_0 := t(e) = i(\overline{e})$ and $v_1 := t(\overline{e}) = i(e)$.

(1) Suppose $v_0 \neq v_1$. Then $\mathcal{Y} = \mathcal{T}$, so $\pi_1(\mathcal{G}, \mathcal{T})$ is the amalgamated product of G_{v_0} and G_{v_1} with the subgroups $A_e \leq G_{v_0}$ and $B_e \leq G_{v_1}$ identified via φ_e, which we simply write as $G_{v_0} *_{A_e = B_e} G_{v_1}$ by abuse of notation.
(2) Suppose $v_0 = v_1$. Then $E(\mathcal{T})$ is empty, so $\pi_1(\mathcal{G}, \mathcal{T}) = \pi(\mathcal{G})$ is the HNN extension
$$\langle G_{v_0}, e \,|\, e^{-1}ae = \varphi_e(a) \text{ for all } a \in A_e\rangle$$
of G_{v_0}.

The following example takes center stage in the next subsection:

Example. Let s be a positive integer, and suppose $\mathcal{Y}$ is the graph with vertex set $V = \mathbb{Z}/s\mathbb{Z}$ and edge set $E = \{e_i, \overline{e_i} : i \in \mathbb{Z}/s\mathbb{Z}\}$ with $i(e_i) = t(\overline{e_i}) = i - 1$ and $t(e_i) = i(\overline{e_i}) = i$ for $i \in \mathbb{Z}/s\mathbb{Z}$. Let $\mathcal{G}$ as above be a graph of groups with underlying graph $\mathcal{Y}$. (See Figure 1.) Then we can describe $\pi_1(\mathcal{G})$ as follows: Let $\mathcal{T} = (V, E \setminus \{e_0, \overline{e_0}\})$, and write

$$K := G_0 *_{A_{e_1} = B_{e_1}} G_1 *_{A_{e_2} = B_{e_2}} G_2 *_{A_{e_3} = B_{e_3}} \cdots *_{A_{e_{s-1}} = B_{e_{s-1}}} G_{s-1}.$$

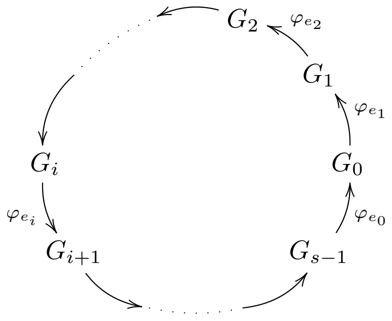

FIGURE 1. Graph of groups whose underlying graph is a loop

Then writing $t = e_0$, we may identify $\pi_1(\mathcal{G}, \mathcal{T})$ with the HNN extension

$$\langle K, t \,|\, t^{-1}at = \varphi_{e_0}(a) \text{ for all } a \in A_{e_0} \rangle$$

of K.

Lemma 5.11. *Let $\mathcal{G}$ be a graph of groups. Suppose each group G_v is a p-group, and let $G = \pi_1(\mathcal{G})$. Then the following are equivalent:*

1. *G has a free normal subgroup N such that G/N is a p-group,*
2. *G is residually p,*
3. *there is a morphism $\psi\colon G \to P$ to a p-group P such that $\psi|G_v$ is injective for every $v \in V(\mathcal{Y})$.*

In the proof we use the following fact (cf. [Gr57, Lemma 1.5]):

Lemma 5.12. *Let G be a group and N be a normal subgroup of G. If G/N is a p-group and N is residually p, then G is residually p.*

Proof of Lemma 5.11. The implication (1) ⇒ (2) follows from the lemma above, since free groups are residually p. Now suppose that G is residually p. Then for every $v \in V(\mathcal{Y})$ and $g \in G_v \setminus \{1\}$ there is a normal subgroup N_g of G with G/N_g a p-group and $g \notin N_g$. Let N be the intersection of all N_g as g ranges over $\bigcup_{v\in V(\mathcal{Y})} G_v \setminus \{1\}$. Then N is a normal subgroup of G of p-power index, and the natural morphism $\psi\colon G \to G/N$ is injective on each G_v. This shows (2) ⇒ (3). Finally, given $\psi\colon G \to P$ as in (3), the kernel N of ψ is a normal subgroup of G with G/N a p-group, and N is free by [Se80, II.2.6, Lemma 8]. This shows (3) ⇒ (1). ☐

5.5. The conclusion of the proof of Theorem 1.4.

In this section we prove another interesting special case of Theorem 1.4:

Theorem 5.13. *Let (G, φ) be an HNN pair where G is an abelian p-group. Then the HNN extension $G^* = \langle G, t \,|\, t^{-1}At = \varphi(A)\rangle$ of G is residually p if and only if the order of the restriction of φ to $H(G, \varphi)$ is a power of p.*

Remark. The HNN extension G^* of G is $\mathbb{Z}$-linear [MRV08, Corollary 3.5] and hence has, for each prime q, a finite-index subgroup which is residually q. In particular, G^* always has a finite-index subgroup which is residually p.

Assuming this theorem for a moment, we are now ready to prove Theorem 1.4 in general:

Proof of Theorem 1.4. Let (G,φ) be an HNN pair where G is a p-group. Assume that there exists a central filtration (G_i) of G compatible with (G,φ), such that for any i the order of the automorphism of $H(G_i/G_{i+1},\varphi_{i,i+1})$ induced by φ is a power of p. By Theorem 5.13 and Lemma 1.1, for each i there is an extension of $\varphi_{i,i+1}$ to an inner automorphism of a p-group containing G_i/G_{i+1} as a subgroup. Now Lemma 5.1 yields that G^* is residually p. □

For the forward direction in Theorem 5.13 note that if G^* is residually p, then so is its subgroup $\langle t\rangle \ltimes H(G,\varphi)$, hence the order of $\varphi|_{H(G,\varphi)}$ is a power of p by the example following Lemma 1.1. The remainder of this section will be occupied by the proof of the backward direction in Theorem 5.13. Throughout this section let (G,φ) be an HNN pair such that G is an abelian p-group. In light of Lemma 2.1 there exists an integer $r\geq 1$ such that for any $s>r$ and $g\in G$ the following holds:

$$g\in H(G,\varphi) \iff \varphi^i(g) \text{ is defined for } i=0,\dots,s-1. \tag{5.3}$$

From now on let s be any integer such that $s>r$. Consider the morphism

$$\phi\colon G^*=\langle G,t\,|\,t^{-1}At=\varphi(A)\rangle\to\mathbb{Z}/s\mathbb{Z} \quad\text{with } \phi(t)=1 \text{ and } \phi(g)=0 \text{ for } g\in G.$$

We let $\mathcal{Y}$ be the graph with vertex set $V=\mathbb{Z}/s\mathbb{Z}$, edge set $E=\{e_i,\overline{e_i}:i\in\mathbb{Z}/s\mathbb{Z}\}$, and $i(e_i)=t(\overline{e_i})=i-1$, $t(e_i)=i(\overline{e_i})=i$ for each $i\in\mathbb{Z}/s\mathbb{Z}$. For $i\in\mathbb{Z}/s\mathbb{Z}$ let

$$G_i:=G\times i,\quad A_i:=A\times i,\quad B_i:=B\times i,$$

and let $\varphi_{e_i}\colon A_{e_i}\to B_{e_i}$ be the isomorphism between $A_{e_i}:=A_{i-1}\leq G_{i-1}$ and $B_{e_i}:=B_i\leq G_i$ given by φ, that is, $\varphi_{e_i}(a\times(i-1))=\varphi(a)\times i$ for $a\in A$. We also endow the opposite edges $\overline{e_1},\dots,\overline{e_s}$ with the corresponding subgroups and isomorphisms, and denote the resulting graph of groups by $\mathcal{G}$. There is a morphism $\Phi\colon\pi(\mathcal{G})\to G^*$ with $g\times i\mapsto g$ and $e_i\mapsto t$ for all $g\in G$ and $i\in\mathbb{Z}/s\mathbb{Z}$. Note that $\phi(\Phi(e))=t(e)-i(e)$ for every $e\in E$.

Lemma 5.14. *The restriction of Φ to $\pi_1(\mathcal{G},0)$ is an isomorphism onto* $\operatorname{Ker}\phi$.

Proof. Let $f_1,\dots,f_n\in E(\mathcal{Y})$, $g_0,\dots,g_n\in G$ and $i_1,\dots,i_n\in\mathbb{Z}/s\mathbb{Z}$ such that

$$g=(g_0\times 0)\,f_1\,(g_1\times i_1)\,f_2\,\cdots f_n\,(g_n\times i_n)\in\pi_1(\mathcal{G},0). \tag{5.4}$$

Then $t(f_j)=i(f_{j+1})=i_j$ for $j=1,\dots,n-1$ and $t(f_n)=i(f_1)=0$. Hence

$$\phi(\Phi(g))=\sum_{j=1}^n\phi(\Phi(f_j))=\sum_{j=1}^n t(f_j)-i(f_j)=0$$

and thus $\operatorname{Im}\Phi\leq\operatorname{Ker}\phi$. Conversely, suppose we are given some $g^*\in\operatorname{Ker}\phi$. Write

$$g^*=g_0t^{\varepsilon_1}g_1t^{\varepsilon_2}\cdots t^{\varepsilon_n}g_n \quad\text{where } g_j\in G \text{ and } \varepsilon_j\in\{\pm1\} \text{ for } j=1,\dots,n.$$

Then

$$0=\phi(g^*)\equiv\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_n \mod s.$$

For $j=1,\dots,n$ let $i_j=\varepsilon_1+\cdots+\varepsilon_n \bmod s$ and $f_j=e_{i_j}$ if $\varepsilon_j=1$, $f_j=\overline{e_{i_j}}$ if $\varepsilon_j=-1$. Then g as in (5.4) is an element of $\pi_1(\mathcal{G},0)$ which maps onto g^* under Φ, showing $\operatorname{Im}\Phi=\operatorname{Ker}\phi$. Using the Normal Form Theorem for fundamental groups of graphs of groups [Se80, Section I.5.2] one also sees easily that Φ is injective. □

Let

$$K := G_0 *_{A_0=B_1} G_1 *_{A_1=B_2} G_2 \cdots *_{A_{s-2}=B_{s-1}} G_{s-1}.$$

By Section 5.4 we have

$$\pi_1(\mathcal{G}) = \langle K, t \,|\, t^{-1}A_{s-1}t = B_0\rangle \qquad \text{where } t = e_0.$$

By a Mayer-Vietoris argument we obtain an exact sequence

$$\begin{array}{cccccccc} & \bigoplus_{i=0}^{s-2} A_i & \xrightarrow{\beta} & \bigoplus_{i=0}^{s-1} G_i & \to & H_1(K;\mathbb{Z}) & \to & 0 \\ & a{\times}i & \mapsto & a{\times}i - \varphi(a){\times}(i+1) & & & & \end{array} \tag{5.5}$$

where the morphism in the middle extends the morphisms $G_i = H_1(G_i;\mathbb{Z}) \to H_1(K;\mathbb{Z})$ induced by the natural inclusions $G_i \to K$.

Lemma 5.15. *Let $j \in \mathbb{Z}/s\mathbb{Z}$. The morphism $G_j \to H_1(K;\mathbb{Z})$ is injective.*

Proof. Let $a \in \bigoplus_{i=0}^{s-2} A_i$. Write $a = \sum_{i=i_1}^{i_2} a_i{\times}i$ where $0 \le i_1 \le i_2 \le s-2$ and $a_i \in A$ for $i = i_1, \dots, i_2$, with $a_{i_1}, a_{i_2} \ne 0$. It is evident that the projections of $\beta(a)$ to G_{i_1} respectively G_{i_2+1} are both non-zero (since $s \ge 2$). It follows that $\operatorname{Im}\beta \cap G_j = 0$. Exactness of (5.5) now yields that $G_j \to H_1(K;\mathbb{Z})$ is injective. □

We now write $G' := H_1(K;\mathbb{Z})$. Note that G' is the quotient of an abelian p-group, in particular G' itself is an abelian p-group. Since $G_j \to G'$ is injective for any j we can view $A' := A_{s-1}$ and $B' := B_0$ as subgroups of G'. We denote by φ' the isomorphism $A' \to B'$ defined by φ, i.e., $\varphi'(a{\times}(s-1)) = \varphi(a){\times}0$ for all $a \in A$.

The following lemma gives in particular a reinterpretation of $H(G,\varphi)$.

Lemma 5.16.

1. *$H(G',\varphi') = A' \cap B'$.*
2. *Let $\Psi\colon G \xrightarrow{\cong} G_0 \le G'$ be the isomorphism given by $\Psi(g) = g{\times}0$ for $g \in G$. Then Ψ restricts to an isomorphism*

$$H(G,\varphi) \to H(G',\varphi'),$$

and the following diagram commutes:

$$\begin{tikzcd} H(G,\varphi) \arrow[r, "\varphi^s"] \arrow[d, "\Psi"] & H(G,\varphi) \arrow[d, "\Psi"] \\ H(G',\varphi') \arrow[r, "\varphi'"] & H(G',\varphi'). \end{tikzcd}$$

Proof. By the exact sequence (5.5) we have $A' \cap B' = \Psi(I)$ where

$$I = \left\{ b \in B : \begin{array}{c} \text{there exist } a \in A \text{ and } a_i \in A,\ i = 0,\dots,s-2, \text{ with} \\ b{\times}0 - a{\times}(s-1) = \sum\limits_{i=0}^{s-2} a_i{\times}i - \varphi(a_i){\times}(i+1) \in \bigoplus\limits_{i=0}^{s-1} G_i \end{array} \right\}.$$

Claim. $I = H(G,\varphi)$.

Proof of the claim. Let $b \in H(G,\varphi)$. By Lemma 2.1 we know that $\varphi^i(b)$ is defined for all i. It is now straightforward to check that $a_i = \varphi^i(b)$, $i = 0,\dots,s-2$ and $a = \varphi^{s-1}(b)$ (all of which lie in A, since $\varphi^i(b)$ is defined for any i) satisfy

$$b{\times}0 - a{\times}(s-1) = \sum_{i=0}^{s-2} a_i{\times}i - \varphi(a_i){\times}(i+1) \in \bigoplus_{i=0}^{s-1} G_i,$$

that is, $b \in I$. On the other hand assume we have $b \in I$. Let $a \in A$ and $a_i \in A$, $i = 0, \dots, s-2$ as in the definition of I. We deduce immediately that $b = a_0$, $a_{i+1} = \varphi(a_i)$ for $i = 0, \dots, s-2$ and $a = \varphi(a_{s-2})$. In particular $\varphi^i(b)$ exists for $i = 1, \dots, s-1$. But by (5.3) this implies that $b \in H(G, \varphi)$. □

Put differently, the above claim shows that Ψ gives rise to an isomorphism $H := H(G, \varphi) \to A' \cap B'$.

Claim. The following diagram commutes:

$$\begin{tikzcd}
H \arrow[r, "\varphi^s"] \arrow[d, "\Psi"] & B \arrow[d, "\Psi"] \\
A' \cap B' \arrow[r, "\varphi'"] & B'.
\end{tikzcd}$$

Proof of the claim. Let $b \in H$. The above discussion shows that $\Psi(b) = b \times 0 \in B'$ and $\varphi^{s-1}(b) \in A \times (s-1) = A'$ represent the same element in G'. We now have

$$\Psi(\varphi^s(b)) = \varphi^s(b) \times 0 = \varphi'(\varphi^{s-1}(b) \times (s-1)) = \varphi'(\Psi(b)) \in G'.$$

This shows that the diagram commutes as claimed. □

Now note that φ^s defines an automorphism of H, hence φ' defines an automorphism of $A' \cap B'$. This shows that $A' \cap B' = H(G', \varphi')$. This concludes the proof of (1). Statement (2) is now an immediate consequence from the above claims. □

We are now in a position to complete the proof of Theorem 5.13 (and hence, of Theorem 1.4). Suppose the order of φ restricted to $H(G, \varphi)$ is a power of p. We continue to use the notations introduced above. Pick k such that $p^k > r$ and set $s := p^k$. Recall that ϕ denotes the morphism

$$G^* = \langle G, t \,|\, t^{-1}At = \varphi(A) \rangle \to \mathbb{Z}/s\mathbb{Z}$$

with $t \mapsto 1$ and $g \mapsto 0$ for $g \in G$. By Lemma 5.12 it suffices to show that $\operatorname{Ker} \phi$ is residually p.

By Lemma 5.16 we have $H(G', \varphi') = A' \cap B'$. Furthermore recall that we assumed that the order of $\varphi|_{H(G,\varphi)}$ is a power of p. We can therefore appeal to Lemma 5.16 to see that the order of the automorphism of $A' \cap B'$ induced by φ' is a power of p. Hence by Proposition 5.9 there exists an abelian p-group X which contains G' as a subgroup, and an automorphism $\alpha \colon X \to X$ which extends the isomorphism $\varphi' \colon A' \to B'$ between subgroups of G', and such that the order of α equals p^l for some l. We can therefore form the semidirect product $\mathbb{Z}/p^l\mathbb{Z} \ltimes X$ where $1 \in \mathbb{Z}/p^l\mathbb{Z}$ acts on X on the right via α. Now consider the composition ψ of the morphisms

$$\begin{array}{rcl}
\pi_1(\mathcal{G}) = \langle K, t \,|\, t^{-1}A_{s-1}t = B_0 \rangle \to \langle G', t \,|\, t^{-1}A't = B' \rangle & \to & \mathbb{Z}/p^l\mathbb{Z} \ltimes X \\
t & \mapsto & (1, 0) \\
g \in G' & \mapsto & (0, g).
\end{array}$$

By Lemma 5.15 the natural morphism $G_i \to G'$ is injective for any $i \in \mathbb{Z}/p^k\mathbb{Z}$; hence the restriction of ψ to G_i is injective for any $i \in \mathbb{Z}/p^k\mathbb{Z}$. It now follows from Lemma 5.11 that $\pi_1(\mathcal{G})$ is residually p. Hence $\operatorname{Ker} \phi$ is residually p by Lemma 5.14. □

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