A SIMPLE PROOF OF THE MARKER-STEINHORN THEOREM FOR EXPANSIONS OF ORDERED ABELIAN GROUPS

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Abstract. We give a short and self-contained proof of the Marker-Steinhorn Theorem for o-minimal expansions of ordered groups, based on an analysis of linear orders definable in such structures.

1. Introduction

Let $\mathcal{M} = (M, \leqslant, \dots)$ be a dense linear order without endpoints, possibly with additional structure, in the language $\mathcal{L}$. A type $p(x)$ over M is said to be *definable* if for every $\mathcal{L}$-formula $\delta = \delta(x, y)$ in the (object) variables $x = (x_1, \dots, x_m)$ and (parameter) variables $y = (y_1, \dots, y_n)$, there is a defining formula for the restriction $p \restriction \delta$ of p to δ, i.e., a formula $\phi(y)$, possibly with parameters from M, such that $\delta(x, b) \in p \Longleftrightarrow \mathcal{M} \models \phi(b)$, for all $b \in M^n$. The Marker-Steinhorn Theorem alluded to in the title of this note gives a condition for certain types over M to be definable, provided that $\mathcal{M}$ is o-minimal.

To explain it, we first recall that a set $C \subseteq M$ is said to be a *cut* in $\mathcal{M}$ if whenever $c \in C$, then $(-\infty, c) := \{a \in M : a < c\}$ is contained in C. Let $\delta(x, y)$ be the formula $x > y$ (in the language of $\mathcal{M}$). It is well known that cuts in $\mathcal{M}$ correspond in a one-to-one way to complete δ-types over M, where to the cut C in $\mathcal{M}$ we associate the complete δ-type

$$p_C(x) := \{\delta(x, b) : b \in C\} \cup \{\neg\delta(x, b) : b \in M \setminus C\}$$

over M. The δ-type p_C is definable if and only if the cut C in $\mathcal{M}$ is definable (as a subset of M). If C is of the form $(-\infty, c] := \{a \in M : a \leqslant c\}$ $(c \in M)$ or $(-\infty, c)$ $(c \in M \cup \{\pm\infty\})$, then C clearly is definable. Cuts of this form are said to be *rational.* The structure $\mathcal{M}$ is definably connected if and only if all definable cuts are rational. If $(M, \leqslant) = (\mathbb{R}, \leqslant)$ is the real line with its usual ordering, then all cuts in $\mathcal{M}$ are rational. This can be used to define the standard part map for elementary extensions. That is, if $(M, \leqslant) = (\mathbb{R}, \leqslant)$ and $\mathcal{M} \preceq \mathcal{M}^* = (M^*, \leqslant, \dots)$, then we can define a map

$$b \mapsto \sup\{a \in M : a \leqslant b\} : M^* \cup \{\pm\infty\} \longrightarrow M \cup \{\pm\infty\},$$

where we declare $\sup \emptyset := -\infty$ and $\sup M := +\infty$. To generalize this, we say that an elementary extension $\mathcal{M} \preceq \mathcal{M}^*$ is *tame* if for every $a \in M^*$ the

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cut $\{b \in M : b \leqslant a\}$ is rational. (Thus if $(M, \leqslant)$ is the usual ordered set of reals, then every elementary extension of $\mathcal{M}$ is tame.) We can then define a standard part map in the same way.

Now $\mathcal{M}$ is o-minimal if and only if every 1-type over M is determined by its restriction to δ, in which case a 1-type over M is definable exactly when the associated cut in $\mathcal{M}$ is rational. It trivially follows that $\mathcal{M} \preceq \mathcal{M}^*$ is tame if and only if for every $a \in M^*$, the type $\mathrm{tp}(a|M)$ is definable. Marker and Steinhorn [4] generalized this to show that if $\mathcal{M}$ is o-minimal and $\mathcal{M} \preceq \mathcal{M}^*$ is tame then for every $a \in (M^*)^m$, the type $\mathrm{tp}(a|M)$ is definable. In particular, if $\mathcal{M}$ is a structure on the real line, then every type over M is definable. See [1] for a survey of geometric applications of this very useful result. The original proof of Marker and Steinhorn uses a complicated inductive proof. Tressl [8] proved the Marker-Steinhorn theorem for o-minimal expansions of real closed fields with a short and clever argument. His proof gives little idea as to the form of the defining formulas of a type. Chernikov and Simon have given a proof using NIP-theoretic machinery [7]. We give a short proof of the Marker-Steinhorn Theorem for o-minimal expansions of ordered groups. The crucial idea behind our proof is to reduce the analysis of n-types to an analysis of cuts in definable linear orderings. Our main tool is Proposition 1, a result about linear orders definable in o-minimal structures admitting elimination of imaginaries. This result is essentially due to Ramakrishnan [6], which is closely related to earlier work of Onshuus-Steinhorn [5]. For the sake of completeness we provide a proof.

By carefully tracking the parameters used to define the type, we actually obtain a uniform version of the Marker-Steinhorn theorem. The pair $(\mathcal{M}^*, \mathcal{M})$ is the structure that consists of $\mathcal{M}^*$ together with a unary predicate for the underlying set of $\mathcal{M}$ and a unary function symbol for the restriction of the standard part map st to the convex hull of M in M^*. The expanded language is called $\mathcal{L}^*$. We denote by $\mathcal{L}(M)$ the expansion of $\mathcal{L}$ by constant symbols naming each element of M, and similarly with $\mathcal{L}^*$ in place of $\mathcal{L}$. We show that if $\delta(x, y)$ is an $\mathcal{L}$-formula then there is an $\mathcal{L}(M)$-formula $\phi(z, y)$ and an $\mathcal{L}^*(M)$-definable map Ω, taking values in a cartesian power of M, such that for any tuples a in M^* and b in M of appropriate lengths,

$$\mathcal{M}^* \models \delta(a, b) \quad \Longleftrightarrow \quad \mathcal{M} \models \phi\big(\Omega(a), b\big).$$

We will prove this by induction on the length of a. See Proposition 13 below for a precise statement and the proof.

Conventions. We let m, n and k range over the set $\mathbb{N} = \{0, 1, 2, \dots\}$ of natural numbers. Given sets A, B and $C \subseteq A \times B$, as well as $a \in A$ and $b \in B$, we let

$$C_a = \{b \in B : (a, b) \in C\}, \qquad C^b = \{a \in A : (a, b) \in C\}.$$

Throughout the paper, $\mathcal{M}$ is an o-minimal expansion of a dense linear order without endpoints, admitting elimination of imaginaries, and $\mathcal{M} \preceq \mathcal{M}^*$ is a tame extension. If $A \subseteq M^m$ is a definable set, then A^* denotes the subset of $(M^*)^m$ defined in $\mathcal{M}^*$ by the same formula. (Since $\mathcal{M} \preceq \mathcal{M}^*$, this does not depend on the choice of defining formula.) Similarly, if $f\colon A \to M^n$, $A \subseteq M^m$, is a definable map, then $f^*\colon A^* \to (M^*)^n$ denotes the map whose graph is defined in $\mathcal{M}^*$ by the same formula as the graph of f. Unless said otherwise, "definable" means "definable, possibly with parameters," and the adjective "definable" applied to subsets of M^m or maps $A \to M^n$, $A \subseteq M^m$, will mean "definable in $\mathcal{M}$." Let $A, B \subseteq M^m$ be definable. By $\dim(A)$ we denote the usual o-minimal dimension of A. If $A \subseteq B$, then we say that A is *almost all* of B if $\dim(B \setminus A) < \dim(B)$, and we say that a property of elements of B is true of *almost all* $b \in B$ if it holds on a definable subset of B which is almost all of B. Let $\sim$ be a definable equivalence relation on A. Then for $a \in A$ we let $[a]_\sim$ denote the $\sim$-class of a, and we let

$$A/{\sim} := \big\{[a]_\sim : a \in A\big\}$$

be the set of equivalence classes of $\sim$. We tacitly assume that (by elimination of imaginaries) we are given a definable set $S \subseteq A$ of representatives of $\sim$, and identify S with $A/{\sim}$. The basic facts about o-minimal structures that we use can be found in [2]. If $\mathcal{M}$ expands an ordered abelian group, given a bounded definable $A \subseteq M$ we let $\mu(A)$ be the sum of the lengths of the components of A. If $A \subseteq M^m \times M$ is such that every A_x is bounded then there is a definable $f\colon M^m \longrightarrow M$ such that $f(x) = \mu(A_x)$. We call $\mu(A)$ the *measure* of A. (Indeed, μ is a finitely additive measure on the collection of bounded definable subsets of M.)

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2. Definable Linear Orders

In this section we establish a key result about definable linear orders in $\mathcal{M}$. As mentioned earlier, this fact is a very weak version of a result due to Ramakrishnan (related to earlier work of Onshuus-Steinhorn). It can fairly easily be proved directly; for sake of completeness, and since we also need to investigate the uniformities in the construction, we include a proof.

We fix a definable linear order $(P, \leqslant_P)$; i.e., P is a definable subset of M^m, for some m, and $\leqslant_P$ is a definable binary relation on P which is a linear ordering (possibly with endpoints). Sometimes we suppress $\leqslant_P$ from the notation. We let a, b, c range over P. A map $\rho\colon P \longrightarrow Q$, where Q is a definable linear order, is said to be monotone if $a \leqslant_P b \Rightarrow \rho(a) \leqslant_Q \rho(b)$, for all a, b.

Proposition 1. *Suppose $l = \dim(P) \geqslant 2$. Then there is a definable linear order $(Q, \leqslant_Q)$ such that $\dim(Q) = l-1$ and a definable surjective monotone map $\rho\colon P \longrightarrow Q$ all of whose fibers have dimension at most 1.*

We first reduce the proof of this proposition to constructing a certain definable equivalence relation on P. Suppose that $\sim$ is a definable equivalence relation on P whose equivalence classes are convex (with respect to $\leqslant_P$), have dimension at most 1, and for almost all a, $[a]_\sim$ is infinite. The first condition ensures that the linear order on P pushes forward to a definable linear order $\leqslant_Q$ on $Q := P/{\sim}$, so that the quotient map $\rho\colon P \longrightarrow P/{\sim}$ becomes monotone. The third condition ensures that $\dim(P/{\sim}) = l - 1$. Then $\rho\colon P \longrightarrow Q$ satisfies the conditions of Proposition 1.

If all intervals

$$(a, b)_P := \{c : a <_P c <_P b\} \qquad (a <_P b)$$

in P are infinite then a convex subset of P is infinite if it has at least two elements. The next lemma allows us to assume that all intervals in P are infinite.

Lemma 2. *There is a definable surjective monotone map $P \longrightarrow R$ to a definable linear order R in which all intervals are infinite and $\dim(P) = \dim(R)$.*

Proof. Take $N \in \mathbb{N}$ such that if $(a, b)_P$ is finite then $|(a, b)_P| < N$. Every finite interval in P is contained in a maximal finite interval. Define $a \sim_f b$ if a and b are contained in the same maximal finite interval. This is a definable equivalence relation on P with convex equivalence classes. For each a, $[a]_{\sim_f}$ is a finite interval and so has cardinality strictly less than N. Let $R = P/{\sim_f}$, equipped with the definable linear order making the natural projection $P \longrightarrow R$ monotone. As the quotient map is finite-to-one, $\dim(R) = \dim(P)$. Suppose $[a]_{\sim_f}$, $[b]_{\sim_f}$ are distinct elements of R with $a <_P b$. Then $(a, b)_P$ is infinite and so contains infinitely many $\sim_f$-classes. Thus $([a]_{\sim_f}, [b]_{\sim_f})_R$ is infinite. □

If $P \longrightarrow R$ is as in the previous lemma, and if we have a map $\rho\colon R \longrightarrow Q$ which satisfies the conditions on ρ in Proposition 1 with R replaced by P, then the composition of ρ with the map $P \longrightarrow R$ satisfies the conditions on ρ in Proposition 1. *We henceforth assume that all intervals in P are infinite.*

We now define the required equivalence relation. Let $d \leqslant l$ be a natural number. We say that $a \sim_d b$ if $\dim\,(a, b)_P < d$. It is very easy to see that $\sim_d$ is a definable equivalence relation on P, and even easier to see that its equivalence classes are convex. Lemma 3 below will be used to show that $\dim\,[a]_{\sim_d} < d$ for all a. It is more difficult to show that almost all $[a]_{\sim_d}$ are infinite. Our desired equivalence relation is $\sim_1$; we will show that the quotient map $P \longrightarrow P/{\sim_1}$ satisfies the conditions of Proposition 1. This proof uses Lemma 5.

Lemma 3. *The ordered set P contains an l-dimensional interval.*

Proof. Let $D \subseteq P^3$ be the set of triples in P^3 with pairwise distinct components. Then clearly $\dim(D) = 3l$. Let E be the set of $(a, b, c) \in D$ such that $a <_P b <_P c$. Let $f\colon D \longrightarrow D$ be the map given by $f(a_1, a_2, a_3) = (a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})$, where σ is the permutation of $\{1, 2, 3\}$ such that $a_{\sigma(1)} <_P a_{\sigma(2)} <_P a_{\sigma(3)}$. Clearly f is finite-to-one, and $E = f(D)$, so $\dim(E) = 3l$. Since

$$E = \bigcup_{a,b} \{a\} \times (a, b)_P \times \{b\},$$

there are a, b such that $\dim\,(a, b)_P = l$. □

Corollary 4. $\dim\,[a]_{\sim_d} < d$ *for all a.*

Proof. The set $[a]_{\sim_d}$ with the order induced by $\leqslant_P$ is a definable linear order. Applying the lemma above, there exists $b_1, b_2 \in [a]_{\sim_d}$ such that $\dim\,(b_1, b_2)_P = \dim\,[a]_{\sim_d}$. As $b_1 \sim_d b_2$, $\dim\,(b_1, b_2)_P < d$. □

Lemma 5. *The set C consisting of all a such that $(-\infty, a]_P$ and $[a, +\infty)_P$ are both closed (in M^m) is at most one-dimensional.*

Proof. Let $d = \dim(C)$ and let $C' \subseteq C$ be a d-dimensional cell. Let a_1, b, a_2 be distinct elements of C' with $a_1 < b < a_2$. Then $a_1 \in (-\infty, b)_P \cap C'$ and $a_2 \in (b, +\infty)_P \cap C'$. Thus $(-\infty, b)_P \cap C'$ and $(b, +\infty)_P \cap C'$ form a nontrivial partition of $C' \setminus \{b\}$ into disjoint closed sets. Thus $C' \setminus \{b\}$ is not definably connected, and so $d = \dim(C') \leqslant 1$. □

Lemma 6. $[a]_{\sim_l}$ *is infinite, for almost all a.*

Proof. Let O be the set of $(a, b) \in P \times P$ such that $a >_P b$. Note that $O_c = (-\infty, c)_P$ and $O^c = (c, +\infty)_P$, for each c. We let D be the boundary of O in $P \times P$. As $\dim(D) < 2l$, for almost all $c \in P$ we have $\dim(D_c), \dim(D^c) < l$. Let E be the set of c such that $\dim(D_c) \geqslant l$ or $\dim(D^c) \geqslant l$.

Note that $[a]_{\sim_l}$ is finite if and only if it equals $\{a\}$. Let A be the set of a such that $[a]_{\sim_l} = \{a\}$. Suppose that $a <_P c$ and c is in the closure of $(-\infty, a)_P$. Let $b \in (a, c)_P$. So $(c, b) \in O$, and (c, b) is a limit point of $(-\infty, a) \times \{b\} \subseteq [P \times P] \setminus O$. Hence $(c, b) \in D$. This holds for any element of $(a, c)_P$, so $\{c\} \times (a, c)_P \subseteq D$. Hence $(a, c)_P \subseteq D_c$, and as $c \in A$, $\dim\,(a, c)_P = l$, so $\dim\, D_c \geqslant l$. Thus $c \in E$. An analogous argument shows that if there is an a such that $a >_P c$ and c is in the closure of $(a, +\infty)_P$, then $c \in E$. It follows from what we have shown that if $c_1, c_2 \in A \setminus E$ and $c_1 <_P c_2$ then c_1 is not in the closure of $(c_2, +\infty)_P$ and c_2 is not in the closure of $(-\infty, c_1)_P$. Consider $A \setminus E$ as a definable linear order with the order induced from P. For all $c \in A \setminus E$, both $(-\infty, c]_{A\setminus E}$ and $[c, +\infty)_{A\setminus E}$ are closed. From Lemma 5 we obtain $\dim(A \setminus E) = 1$. So either $\dim(A) = 1$ or $\dim(A) = \dim(E) < l$. In either case $\dim(A) < l$. □

With the following lemma we now finish the proof of Proposition 1:

Lemma 7. *$[a]_{\sim_1}$ is infinite, for almost all a.*

Proof. We show this by induction on $l = \dim(P)$. If $l = 1$, then this is trivially true. Suppose this statement holds for all smaller values of l. For almost all a, $[a]_{\sim_l}$ is infinite, by the previous lemma. As $\dim [a]_{\sim_l} < l$ for almost all $b \sim_l a$, there are infinitely many $c \in [a]_{\sim_l}$ such that $c \sim_1 b$. The fiber lemma for o-minimal dimension now implies that $[a]_{\sim_1}$ is infinite for almost all a. $\square$

Note that these constructions are done uniformly in the parameters defining $(P, \leqslant_P)$. Namely, if $P \subseteq M^k \times M^m$ and $\leqslant_P \subseteq M^k \times (M^m \times M^m)$ are definable sets such that for each $a \in M^k$, $\leqslant_{P_a} := (\leqslant_P)_a$ is a linear order on P_a, then there are definable sets $Q \subseteq M^k \times M^n$, $\leqslant_Q \subseteq M^k \times (M^n \times M^n)$ and $R \subseteq M^k \times (M^m \times M^n)$ such that for each $a \in M^k$ with $\dim(P_a) \geqslant 2$, $\leqslant_{Q_a} := (\leqslant_Q)_a$ is a linear order on Q_a and R_a is the graph of a monotone map $(P_a, \leqslant_{P_a}) \to (Q_a, \leqslant_{Q_a})$ satisfying the conditions of Proposition 1.

3. Rational Cuts in Definable Linear Orders

From now on until the end of the paper we assume that $\mathcal{M}$ expands an ordered abelian group. As in the previous section, we let $(P, \leqslant_P)$ be a definable linear order. We now give an application of Proposition 1 used in our proof of the Marker-Steinhorn Theorem in the next section. Recall that we assume $P \subseteq M^m$.

Proposition 8. *If $V \subseteq P^*$ is definable in $\mathcal{M}^*$ and $W = V \cap P$ is a cut in P, then W is definable in $\mathcal{M}$.*

The proof of this proposition is the most difficult part of this paper. The difficulty largely lies in the fact that V is not assumed to be a cut in P^*. If V was a cut, then we could try to prove the result in the following way: argue by induction on $\dim(P)$, let $\rho\colon P \longrightarrow Q$ be the map given by Proposition 1, let $B \subseteq Q^*$ be the set of q such that $\rho^{-1}(q) \subseteq V$, argue inductively that $B \cap Q$ is $\mathcal{M}$-definable, and use this to show that W is $\mathcal{M}$-definable. It is natural to try to apply this arguement to our situation by replacing V with its convex hull V' in Q. However W can be a proper subset of $V' \cap P$. For example let $P = (M, \leqslant)$, let t be an element of M^* larger then every element of M, and let $V = (0, 1) \cup \{t\}$.

In the proof of Proposition 8 we also need the following two lemmas. The first is the base case of the Marker-Steinhorn theorem.

Lemma 9. *Let $A \subseteq M^m$ be a definable one-dimensional subset of M^m, and let $B \subseteq (M^*)^m$ be definable in $\mathcal{M}^*$. Then $B \cap A$ is definable in $\mathcal{M}$.*

Proof. By Cell Decomposition, A is the union of finitely many sets of the form $f(M)$, where $f\colon M \longrightarrow A$ is a definable map. We may thus reduce to the case that A itself is of this form. It suffices to show that $f^{-1}(B \cap$

$A) = (f^*)^{-1}(B) \cap M$ is definable. So we may assume that $m = 1$ and $A = M$. Then B is a boolean combination of rays of the type $(-\infty, b)_{M^*}$ or $(-\infty, b]_{M^*}$, where $b \in M^*$. Let $b \in M^*$; if $b > M$, then $M \cap (-\infty, b)_{M^*} = M \cap (-\infty, b]_{M^*} = M$; otherwise, $M \cap (-\infty, b)_{M^*}$ and $M \cap (-\infty, b]_{M^*}$ each equal one of $(-\infty, \mathrm{st}(b))_M$ or $(-\infty, \mathrm{st}(b)]_M$. □

Lemma 10. *Let $A \subseteq M$ be bounded, infinite and definable, and let $B \subseteq A^*$ be definable in $\mathcal{M}^*$. If $A \subseteq B$, then $\mathrm{st}(\mu(B)) = \mu(A) > 0$. If $A \cap B = \emptyset$ then $\mathrm{st}(\mu(B)) = 0$.*

Proof. Let $c < d$ be elements of M^* contained in the convex hull of M. If $\mathrm{st}(d - c) > 0$ then $(c, d)_{M^*}$ must contain infinitely many elements of M. Therefore if $\mathrm{st}(\mu(B)) > 0$, then B contains infinitely many elements of A (as then B contains an interval whose length is not infinitesimal); so $A \cap B \neq \emptyset$. If $\mathrm{st}(\mu(B)) < \mu(A) = \mu(A^*)$ then $\mathrm{st}(\mu(A^* \setminus B)) > 0$, so as before $A^* \setminus B$ contains infinitely many points in A, therefore A is not a subset of B. □

Proof of Proposition 8. We use induction on $l = \dim(P)$. If $l = 1$, then this is a special case of the preceding Lemma 9. Suppose that $l \geqslant 2$. Take $(Q, \leqslant_Q)$ and $\rho\colon P \longrightarrow Q$ as in Proposition 1. We fix a positive element 1 of M and identify $\mathbb{Q}$ with its image under the embedding $\mathbb{Q} \to M$ of (additive) ordered abelian groups which sends $1 \in \mathbb{Q}$ to $1 \in M$. We shall specify an integer $N \geqslant 1$ and a definable injective map

$$\iota\colon P \longrightarrow Q \times M \times \{1, \dots, m\} \times \{1, \dots, N\} \subseteq Q \times M \times M \times M$$

with the property that $\iota(p) = (\rho(p), \dots)$ for each $p \in P$. We let i range over $\{1, \dots, m\}$, and for each i we let $\pi_i\colon P \longrightarrow M$ be the restriction to P of the projection $M^m \to M$ onto the ith coordinate. For each $q \in Q$ define P_q^i inductively as the set of $a \in \rho^{-1}(q) \setminus (P_q^1 \cup \dots \cup P_q^{i-1})$ such that there are only finitely many $b \in \rho^{-1}(q)$ with $\pi_i(a) = \pi_i(b)$. For each $q \in Q$, $\rho^{-1}(q)$ is then the disjoint union of the P_q^i. Let $N \in \mathbb{N}$ be such that for all q, i, the fibers of $\pi_i|_{P_q^i}$ have cardinality bounded by N. If $p \in P_{\rho(p)}^i$ is the jth element of $\pi_i^{-1}(\rho(p)) \cap P_{\rho(p)}^i$ in the lexiographic order induced from M^m, then we set

$$\iota(p) = (\rho(p), \pi_i(p), i, j).$$

Below, we let j range over $\{1, \dots, N\}$.

Let now $V \subseteq P^*$ be definable in $\mathcal{M}^*$ such that $W = V \cap P$ is a cut in P. As ρ is monotone, $\rho(W)$ is a cut in Q. We construct a set $B \subseteq Q^*$, definable in $\mathcal{M}^*$, such that $B \cap Q = \rho(W)$. (It will then follow from the inductive hypothesis that $\rho(W)$ is definable.) It is easily seen that if q is a non-maximal element of $\rho(W)$ then $\rho^{-1}(q)$ is contained in W. It is also easily seen that if $q \in Q$ is not in $\rho(W)$ then $\rho^{-1}(q)$ is disjoint from W. For $q \in Q^*$ we define

$$P(q) := \iota^*\big((\rho^*)^{-1}(q)\big), \qquad W(q) := \iota\big(W \cap (\rho^*)^{-1}(q)\big).$$

and

$$V(q) = \iota^*\big(V \cap (\rho^*)^{-1}(q)\big),$$

so that $V(q) \cap (Q \times M^3) = W(q)$ if $q \in Q$. Again, for all $q \in Q$, if q is a non-maximal element of $\rho(W)$ then $P(q) \subseteq W(q)$, and if $q \notin \rho(W)$ then $W(q) = \emptyset$.

For $q \in Q^*$ let $P(q, i, j)$ be the set of $s \in M^*$ such that $(q, s, i, j) \in P(q)$, and define $V(q, i, j) \subseteq M^*$ likewise. Now we list some consequences of Lemma 10. For this, let $q \in Q$ and $c, d \in M^*$ with $c < d$. If $P(q) \subseteq W(q)$, then:

i. If $(c, +\infty)_{M^*}$ is the interior of a component of $P(q, i, j)$ then
$$\operatorname{st}\big(\mu(V(q, i, j) \cap [c, c+1]_{M^*})\big) = 1.$$
ii. If $(-\infty, c)_{M^*}$ is the interior of a component of $P(q, i, j)$ then
$$\operatorname{st}\big(\mu(V(q, i, j) \cap [c-1, c]_{M^*})\big) = 1.$$
iii. If $(c, d)_{M^*}$ is the interior of a component of $P(q, i, j)$ then
$$\operatorname{st}\big(\mu(V(q, i, j) \cap [c, d]_{M^*})\big) = d - c.$$

On the other hand, if $W(q) = \emptyset$, then in each of the preceding cases the standard part of the measure of the intersection of $V(q, i, j)$ with the appropriate segment in M^* is zero. Let now $\Lambda \in Q$ be the maximal element of $\rho(W)$ if this exists, and some fixed element of Q otherwise. We let B be the set of $q \in Q^*$ such that for all i, j and all $c < d$ in M^*,

i. if $(c, +\infty)_{M^*}$ is the interior of a component of $P(q, i, j)$ then
$$\mu\big(V(q, i, j) \cap [c, c+1]_{M^*}\big) < \frac{1}{2};$$
ii. if $(-\infty, c)_{M^*}$ is the interior of a component of $P(q, i, j)$ then
$$\mu\big(V(q, i, j) \cap [c, c-1]_{M^*}\big) < \frac{1}{2};$$
iii. if $(c, d)_{M^*}$ is the interior of a component of $P(q, i, j)$ then
$$\mu\big(V(q, i, j) \cap [c, d]_{M^*}\big) < \frac{1}{2}(d - c).$$

The set B is definable in $\mathcal{M}^*$, and $B \cap Q$ is the set of all nonmaximal elements of $\rho(W)$, possibly together with Λ. This is a cut in Q. By induction, $B \cap Q$ is definable in $\mathcal{M}$. Let $p \in P$. If $\rho(W)$ has a maximal element then p is in W if $\rho(p) < \Lambda$ or if $p \in W \cap \rho^{-1}(\Lambda)$. By Lemma 9, $W \cap \rho^{-1}(\Lambda)$ is definable in $\mathcal{M}$. If $\rho(W)$ does not have a maximal element then $p \in W$ if and only if $\rho(p) \in B$. $\square$

By carefully keeping track of the parameters used in the proof of Lemma 9, we see that we have in fact proven the following uniform version of the lemma, which also provides the base case of the uniform Marker-Steinhorn Theorem.

Lemma 11. *Let $A \subseteq M^k \times M^m$ be definable with $\dim(A_x) = 1$ for every $x \in M^k$, and let $B \subseteq (M^*)^j \times (M^*)^m$ be definable in $\mathcal{M}^*$. Then there is* *a definable $E \subseteq M^l \times M^m$, for some l, and a map $\Omega\colon M^k \times (M^*)^j \to M^l$, definable in the $\mathcal{L}^*$-structure $(\mathcal{M}^*, \mathcal{M})$, such that $A_x \cap B_a = E_{\Omega(x,a)}$.*

Similarly, by carefully keeping track of the parameters used to define B and W and strengthening the inductive assumption in the natural way, we can see that we have in fact proven a uniform version of Proposition 8: ι can be defined uniformly in the same way as ρ; B can be defined uniformly from W; and if $W \subseteq (M^*)^k \times P^*$ is definable in $\mathcal{M}^*$ then the map $\Lambda\colon (M^*)^k \longrightarrow Q$ that takes a to the maximum of $\rho(W_a \cap P)$ if such exists and to some fixed element of Q otherwise, is definable in $(\mathcal{M}^*, \mathcal{M})$.

Proposition 12. *Let $P \subseteq M^l \times M^m$ and $\leqslant$ be a subset of $M^l \times [M^m \times M^m]$ such that for every $a \in M^l$, $\leqslant_a$ is a linear order on P_a. Let $V \subseteq (M^*)^k \times P^*$ be definable in $\mathcal{M}^*$. Then there is some j and a map $\Omega\colon (M^*)^k \times M^l \longrightarrow M^j$, definable in the $\mathcal{L}^*$-structure $(\mathcal{M}^*, \mathcal{M})$, and a definable $W \subseteq M^j \times M^m$ such that for each $x \in (M^*)^k$ and $a \in M^l$, if $(V_x \cap P)_a$ is a cut in P_a then $(V_x \cap P)_a = W_{\Omega(x,a)}$.*

We remark that the use of the function ι in the proof of Proposition 8 may be avoided by using Ramakrishnan's theorem [6] on embedding definable linear orders into lexicographic orders. Moreover, the only point in our proof of the Marker-Steinhorn Theorem where we need to assume that $\mathcal{M}$ expands an ordered abelian group is in Proposition 8.

4. Proof of the Marker-Steinhorn Theorem

We now prove the uniform Marker-Steinhorn Theorem. Recall our standing assumption that $\mathcal{M} \preceq \mathcal{M}^*$ is a tame extension.

Proposition 13. *Let $\delta(x, y)$ be an $\mathcal{L}$-formula, where $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$. Then there is an $\mathcal{L}(M)$-formula $\phi(z, w)$, where $z = (z_1, \ldots, z_k)$, and a map $\Omega\colon (M^*)^m \to M^k$, definable in the $\mathcal{L}^*$-structure $(\mathcal{M}^*, \mathcal{M})$, such that for all $a \in (M^*)^m$, $b \in M^n$:*

$$\mathcal{M}^* \models \delta(a, b) \quad \Longleftrightarrow \quad \mathcal{M} \models \phi(\Omega(a), b).$$

We use induction on m. Lemma 11 treats the base case $m = 1$. Suppose that $m \geqslant 2$. Let $\hat{a} = (a_1, \ldots, a_{m-1})$; inductively, $\mathrm{tp}(\hat{a}|M)$ is definable. We construct a defining formula for the restriction $\mathrm{tp}(a|M) \restriction \delta$ of $\mathrm{tp}(a|M)$ to δ. It is a direct consequence of the Cell Decomposition Theorem that $\delta(x; y)$ is a boolean combination of formulas $\delta_i(x; y)$ such that the set of tuples $(a, b) = (\hat{a}, a_m, b) \in (M^*)^{m-1} \times M^* \times (M^*)^n$ defined by δ_i in $\mathcal{M}^*$ has one of the following forms:

i. $(\hat{a}, b) \in X^*$ and $a_m \geqslant f^*(\hat{a}, b)$,
ii. $(\hat{a}, b) \in X^*$ and $a_m \leqslant f^*(\hat{a}, b)$,
iii. $(\hat{a}, b) \in X^*$,

where $X \subseteq M^{m+n-1}$ and $f\colon M^{m+n-1} \longrightarrow M$ is definable. The defining formula of $\mathrm{tp}(a|M) \restriction \delta$ is the corresponding boolean combination of the defining formulas of $\operatorname{tp}(a|M)\restriction\delta_i$. We therefore assume that δ is of one these forms. The last case is rendered trivial by the inductive assumption. We now suppose that δ is of the first form. Thus

$$\mathcal{M}^* \models \delta(a,b) \Longleftrightarrow (\hat{a},b) \in X^* \text{ and } f^*(\hat{a},b) \leqslant a_m.$$

By the induction hypothesis we take a definable $B \subseteq M^k \times M^n$, for some k, and map $\Omega_1\colon (M^*)^{m-1} \longrightarrow M^k$, definable in the pair $(\mathcal{M}^*,\mathcal{M})$, such that for $\hat{a} \in (M^*)^{m-1}$ and $b \in M^n$ we have

$$(\hat{a},b) \in X^* \quad \Longleftrightarrow \quad b \in B_{\Omega_1(\hat{a})}.$$

For $\hat{a} \in (M^*)^{m-1}$, $b_1, b_2 \in M^n$ with $(\hat{a},b_i) \in X^*$ $(i=1,2)$, we define

$$b_1 \lesssim_{\hat{a}} b_2 \quad :\Longleftrightarrow \quad f^*(\hat{a},b_1) \leqslant f^*(\hat{a},b_2).$$

Again, the inductive hypothesis gives a definable $C \subseteq M^l \times (M^n \times M^n)$ and a map $\Omega_2\colon (M^*)^{m-1} \longrightarrow M^l$ which is definable in $(\mathcal{M}^*,\mathcal{M})$ and such that $b_1 \lesssim_{\hat{a}} b_2$ if and only if $(b_1,b_2) \in C_{\Omega_2(\hat{a})}$, for all $\hat{a} \in (M^*)^{m-1}$ and $b_1,b_2 \in M^n$. It is easy to check that each $\lesssim_{\hat{a}}$ is a quasi-order on M^n in which any two elements are comparable. For $b_1, b_2 \in M^n$ set

$$\begin{aligned} b_1 \sim_{\hat{a}} b_2 \quad &:\Longleftrightarrow \quad b_1 \lesssim_{\hat{a}} b_2 \text{ and } b_2 \lesssim_{\hat{a}} b_1 \\ &\Longleftrightarrow \quad f^*(\hat{a},b_1) = f^*(\hat{a},b_2) \\ &\Longleftrightarrow \quad (b_1,b_2),(b_2,b_1) \in C_{\Omega_2(\hat{a})}. \end{aligned}$$

This is a definable equivalence relation on M^n. Let

$$C' := \big\{(b,b_1,b_2) \in M^l \times [M^n \times M^n] : (b_1,b_2),(b_2,b_1) \in C_b\big\}.$$

If $b \in M^l$ is of the form $\Omega_2(\hat{a})$ then C'_b is a definable equivalence relation on M^n. By uniform elimination of imaginaries let $A \subseteq M^l \times M^n$ be definable such that for all $b \in M^l$ we have $A_b = M^n/C'_b$ whenever C'_b a definable equivalence relation, and $A_b = \emptyset$ otherwise. So we have that $A_{\Omega_2(\hat{a})} = M^n/{\sim_{\hat{a}}}$ for all $\hat{a} \in (M^*)^{m-1}$. The relation $\lesssim_{\hat{a}}$ pushes forward to a linear order on $A_{\Omega_2(\hat{a})}$, which we denote by $\leqslant_{\hat{a}}$. For $(\hat{a},x) \in (M^*)^{m-1} \times M^*$ let $V_{(\hat{a},x)}$ be the set of $b \in (M^*)^n$ such that $f^*(\hat{a},b) \leqslant x$. Then $V_{(\hat{a},x)}/{\sim_{\hat{a}}}$ is easily seen to be a cut in the definable linear order $(A_{\Omega_2(\hat{a})}, \leqslant_{\hat{a}})$, and hence definable (in $\mathcal{M}$), by Proposition 8. In fact, by Proposition 12, there is a definable $D \subseteq M^p \times A$, for some p, and a map $\Omega_3\colon (M^*)^{n-1} \times (M^*) \longrightarrow M^p$, definable in $(\mathcal{M}^*,\mathcal{M})$, such that

$$[b]_{\sim_{\hat{a}}} \in V_{(\hat{a},x)}/{\sim_{\hat{a}}} \quad \Longleftrightarrow \quad b \in D_{\Omega_3(\hat{a},x)}.$$

Hence

$$\begin{aligned} \mathcal{M}^* \models \delta(a;b) \quad &\Longleftrightarrow \quad (\hat{a},b) \in X^* \wedge \big[f^*(\hat{a},b) \leqslant a_m\big] \\ &\Longleftrightarrow \quad \big[b \in B_{\Omega_1(\hat{a})}\big] \wedge \big[b \in D_{\Omega_3(\hat{a},x)}\big] \end{aligned}$$

Therefore $\operatorname{tp}(a|M)\restriction\delta$ is definable in the way indicated in the proposition. $\square$

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