On Numbers, Germs, and Transseries

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Abstract

Germs of real-valued functions, surreal numbers, and transseries are three ways to enrich the real continuum by infinitesimal and infinite quantities. Each of these comes with naturally interacting notions of *ordering* and *derivative*. The category of H-fields provides a common framework for the relevant algebraic structures. We give an exposition of our results on the model theory of H-fields, and we report on recent progress in unifying germs, surreal numbers, and transseries from the point of view of asymptotic differential algebra.

Contemporaneous with Cantor's work in the 1870s but less well-known, P. du Bois-Reymond [10]–[15] had original ideas concerning non-Cantorian infinitely large and small quantities [34]. He developed a "calculus of infinities" to deal with the growth rates of functions of one real variable, representing their "potential infinity" by an "actual infinite" quantity. The reciprocal of a function tending to infinity is one which tends to zero, hence represents an "actual infinitesimal".

These ideas were unwelcome to Cantor [39] and misunderstood by him, but were made rigorous by F. Hausdorff [46]–[48] and G. H. Hardy [42]–[45]. Hausdorff firmly grounded du Bois-Reymond's "orders of infinity" in Cantor's set-theoretic universe [38], while Hardy focused on their differential aspects and introduced the *logarithmico-exponential functions* (short: *LE-functions*). This led to the concept of a *Hardy field* (Bourbaki [22]), developed further mainly by Rosenlicht [63]–[67] and Boshernitzan [18]–[21]. For the role of Hardy fields in *o-minimality* see [61].

Surreal numbers were discovered (or created?) in the 1970s by J. H. Conway [23] and popularized by M. Gardner, and by D. E. Knuth [55] who coined the term "surreal number". The surreal numbers form a proper class containing all reals as well as Cantor's ordinals, and come equipped with a natural ordering and arithmetic operations turning them into an ordered field. Thus with ω the first infinite ordinal, $\omega - \pi$, $1/\omega$, $\sqrt{\omega}$ make sense as surreal numbers. In contrast to non-standard real numbers, their construction is completely canonical, naturally generalizing both Dedekind cuts and von Neumann's construction of the ordinals. (In the words of their creator [24, p. 102], the surreals are "the only correct extension of the notion of real number to the infinitely large and the infinitesimally small.") The surreal universe is very rich, yet shares many properties with the real world. For example, the ordered field of surreals is real closed and hence, by

2010 *Mathematical Subject Classification.* Primary 03C64, Secondary 12H05, 12J15
Date. November 16, 2017
The first-named author was partially support by NSF Grant DMS-1700439.

Tarski [72], an elementary extension of its ordered subfield of real numbers. (In fact, every set-sized real closed field embeds into the field of surreal numbers.) M. Kruskal anticipated the use of surreal numbers in asymptotics, and based on his ideas Gonshor [40] extended the exponential function on the reals to one on the surreals, with the same first-order logical properties [29]. Rudiments of analysis for functions on the surreal numbers have also been developed [1, 26, 68].

Transseries generalize LE-functions in a similar way that surreals generalize reals and ordinals. Transseries have a precursor in the *generalized power series* of Levi-Civita [57, 58] and Hahn [41], but were only systematically considered in the 1980s, independently by Écalle [32] and Dahn-Göring [27]. Écalle introduced transseries as formal counterparts to his "analyzable functions", which were central to his work on Dulac's Problem (related to Hilbert's 16th Problem on polynomial vector fields). Dahn and Göring were motivated by Tarski's Problem on the model theory of the real field with exponentiation. Transseries have since been used in various parts of mathematics and physics; their formal nature also makes them suitable for calculations in computer algebra systems. Key examples of transseries are the *logarithmic-exponential series* (*LE-series* for short) [30, 31]; more general notions of transseries have been introduced in [49, 69]. A transseries can represent a function of a real variable using exponential and logarithmic terms, going beyond the more prevalent asymptotic expansions in terms of powers of the independent variable. Transseries can be manipulated algebraically—added, subtracted, multiplied, divided—and like power series, can be differentiated termwise: they comprise a differential field. However, they carry much more structure: for example, by virtue of its construction, the field of LE-series comes with an exponential function; there is a natural notion of composition for transseries; and differential-compositional equations in transseries are sometimes amenable to functional-analytic techniques [50].

The logical properties of the *exponential* field of LE-series have been well-understood since the 1990s: by [73] and [30] it is model-complete and o-minimal. In our book [4] we focused instead on the *differential* field of LE-series, denoted below by $\mathbb{T}$, and obtained some decisive results about its model theory. Following A. Robinson's general ideas we placed $\mathbb{T}$ into a suitable category of *H-fields* and, by developing the extension theory of H-fields, showed that $\mathbb{T}$ is existentially closed as an H-field: each system of algebraic differential equations and inequalities over $\mathbb{T}$ which has a solution in an H-field extension of $\mathbb{T}$ already has one in $\mathbb{T}$ itself. In [4] we also prove the related fact that $\mathbb{T}$ is model-complete; indeed, we obtain a quantifier elimination (in a natural language) for $\mathbb{T}$. As a consequence, the elementary theory of $\mathbb{T}$ is decidable, and model-theoretically "tame" in various ways: for example, it has Shelah's *non-independence property* (NIP).

Results from [4] about existential closedness, model completeness, and quantifier elimination substantiate the intuition, expressed already in [32], that $\mathbb{T}$ plays the role of a *universal domain* for the part of asymptotic differential algebra that steers clear of oscillations. How far does this intuition lead us? Hardy's field of LE-functions embeds into $\mathbb{T}$, as an ordered differential field, but this fails for other Hardy fields. The natural question here is: *Are all maximal Hardy fields elementarily equivalent to $\mathbb{T}$?* It would mean that any maximal Hardy field instantiates

Hardy's vision of a maximally inclusive and well-behaved algebra of oscillation-free real functions. Related is the issue of embedding Hardy fields into more general differential fields of transseries. Positive answers to these questions would tighten the link between germs of functions (living in Hardy fields) and their transseries expansions. We may also ask how surreal numbers fit into the picture: *Is there a natural isomorphism between the field of surreal numbers and some field of generalized transseries?* This would make it possible to differentiate and compose surreal numbers as if they were functions, and confirm Kruskal's premonition of a connection between surreals and the asymptotics of functions.

We believe that answers to these questions are within grasp due to advances in our understanding during the last decade as represented in our book [4]. We discuss these questions with more details in Sections 3, 4, 5. In Section 1 we set the stage by describing Hardy fields and transseries as two competing approaches to the asymptotic behavior of non-oscillatory real-valued functions. (Section 5 includes a brief synopsis of the remarkable surreal number system.) In Section 2 we define H-fields and state the main results of [4].

We let m, n range over $\mathbb{N} = \{0, 1, 2, \dots\}$. Given an (additive) abelian group A we let $A^{\neq} := A \setminus \{0\}$. In some places below we assume familiarity with very basic model theory, for example, on the level of [4, Appendix B]. "Definable" will mean "definable with parameters".

1 Orders of Infinity and Transseries

Germs of continuous functions

Consider continuous real-valued functions whose domain is a subset of $\mathbb{R}$ containing an interval $(a, +\infty)$, $a \in \mathbb{R}$. Two such functions *have the same germ* (at $+\infty$) if they agree on an interval $(a, +\infty)$, $a \in \mathbb{R}$, contained in both their domains; this defines an equivalence relation on the set of such functions, whose equivalence classes are called *germs.* Addition and multiplication of germs is defined pointwise, giving rise to a commutative ring $\mathcal{C}$. For a germ g of such a function we also let g denote that function if the resulting ambiguity is harmless. With this convention, given a property P of real numbers and $g \in \mathcal{C}$ we say that $P\big(g(t)\big)$ holds *eventually* if $P\big(g(t)\big)$ holds for all sufficiently large real t in the domain of g. We identify each real number r with the germ of the constant function $\mathbb{R} \to \mathbb{R}$ with value r. This makes the field $\mathbb{R}$ into a subring of $\mathcal{C}$.

Following Hardy we define for $f, g \in \mathcal{C}$,

$$\begin{aligned} f \preccurlyeq g \quad &:\Longleftrightarrow \quad \text{for some } c \in \mathbb{R}^{>0} \text{ we have } |f(t)| \leqslant c|g(t)| \text{ eventually,} \\ f \prec g \quad &:\Longleftrightarrow \quad \text{for every } c \in \mathbb{R}^{>0} \text{ we have } |f(t)| < c|g(t)| \text{ eventually.} \end{aligned}$$

The reflexive and transitive relation $\preccurlyeq$ yields an equivalence relation $\asymp$ on $\mathcal{C}$ by setting $f \asymp g :\Longleftrightarrow f \preccurlyeq g$ and $g \preccurlyeq f$, and $\preccurlyeq$ induces a partial ordering on the set of equivalence classes of $\asymp$; these equivalence classes are essentially du Bois-Reymond's "orders of infinity". Thus with x the germ of the identity function on $\mathbb{R}$:

$$0 \prec 1 \prec \log\log x \prec \log x \prec \sqrt{x} \prec x \asymp -2x + x\sin x \prec x^2 \prec \mathrm{e}^x .$$

One way to create interesting subrings of $\mathcal{C}$ is via expansions of the field of real numbers: any such expansion $\widetilde{\mathbb{R}}$ gives rise to the subring $H(\widetilde{\mathbb{R}})$ of $\mathcal{C}$ consisting of the germs of the continuous functions $\mathbb{R} \to \mathbb{R}$ that are definable in $\widetilde{\mathbb{R}}$.

Hausdorff fields

A *Hausdorff field* is by definition a subfield of $\mathcal{C}$. Simple examples are

$$\mathbb{Q}, \quad \mathbb{R}, \quad \mathbb{R}(x), \quad \mathbb{R}(\sqrt{x}), \quad \mathbb{R}(x, \mathrm{e}^x, \log x). \tag{1.1}$$

That $\mathbb{R}(x, \mathrm{e}^x, \log x)$ is a Hausdorff field, for instance, follows from two easy facts: first, an element f of $\mathcal{C}$ is a unit iff $f(t) \neq 0$ eventually (and then either $f(t) > 0$ eventually or $f(t) < 0$ eventually), and if $f \neq 0$ is an element of the subring $\mathbb{R}[x, \mathrm{e}^x, \log x]$ of $\mathcal{C}$, then $f \asymp x^k \, \mathrm{e}^{lx} (\log x)^m$ for some $k, l, m \in \mathbb{N}$. Alternatively, one can use the fact that an expansion $\widetilde{\mathbb{R}}$ of the field of reals is o-minimal iff $H(\widetilde{\mathbb{R}})$ is a Hausdorff field, and note that the examples above are subfields of $H(\mathbb{R}_{\exp})$ where $\mathbb{R}_{\exp}$ is the exponential field of real numbers, which is well-known to be o-minimal by Wilkie [73].

Let H be a Hausdorff field. Then H becomes an ordered field with (total) ordering given by: $f > 0$ iff $f(t) > 0$ eventually. Moreover, the set of orders of infinity in H is totally ordered by $\preccurlyeq$: for $f, g \in H$ we have $f \preccurlyeq g$ or $g \preccurlyeq f$. In his landmark paper [48], Hausdorff essentially proved that H has a unique algebraic Hausdorff field extension that is real closed. (Writing before Artin and Schreier [2], of course he doesn't use this terminology.) He was particularly interested in "maximal" objects and their order type. By Hausdorff's Maximality Principle (a form of Zorn's Lemma) every Hausdorff field is contained in one that is maximal with respect to inclusion. By the above, maximal Hausdorff fields are real closed. Hausdorff also observed that maximal Hausdorff fields have uncountable cofinality; indeed, he proved the stronger result that the underlying ordered set of a maximal Hausdorff field H is η_1: *if A, B are countable subsets of H and $A < B$, then $A < h < B$ for some $h \in H$.* A real closed ordered field is $\aleph_1$-saturated iff its underlying ordered set is η_1. Standard facts from model theory (or [36]) now yield an observation that could have been made by Hausdorff himself in the wake of Artin and Schreier [2]:

Corollary 1.1. *Assuming* CH (*the Continuum Hypothesis*), *all maximal Hausdorff fields are isomorphic.*

This observation was in fact made by Ehrlich [35] in the more specific form that under CH any maximal Hausdorff field is isomorphic to the field of surreal numbers of countable length; see Section 5 below for basic facts on surreals. We don't know whether here the assumption of CH can be omitted. (By [37], the negation of CH implies the existence of non-isomorphic real closed η_1-fields of size $2^{\aleph_0}$.) It may also be worth mentioning that the intersection of all maximal Hausdorff fields is quite small: it is just the field of real algebraic numbers.

Hardy fields

A Hardy field is a Hausdorff field whose germs can be differentiated. This leads to a much richer theory. To define Hardy fields formally we introduce the subring

$$\mathcal{C}^n := \{f \in \mathcal{C} : f \text{ is eventually } n \text{ times continuously differentiable}\}$$

of $\mathcal{C}$, with $\mathcal{C}^0 = \mathcal{C}$. Then each $f \in \mathcal{C}^{n+1}$ has derivative $f' \in \mathcal{C}^n$. A *Hardy field* is a subfield of $\mathcal{C}^1$ that is closed under $f \mapsto f'$; Hardy fields are thus not only ordered fields but also differential fields. The Hausdorff fields listed in (1.1) are all Hardy fields; moreover, for each o-minimal expansion $\widetilde{\mathbb{R}}$ of the field of reals, $H(\widetilde{\mathbb{R}})$ is a Hardy field. As with Hausdorff fields, each Hardy field is contained in a maximal one. For an element f of a Hardy field we have either $f' > 0$, or $f' = 0$, or $f' < 0$, so f is either eventually strictly increasing, or eventually constant, or eventually strictly decreasing. (This may fail for f in a Hausdorff field.) Each element of a Hardy field is contained in the intersection $\bigcap_n \mathcal{C}^n$, but not necessarily in its subring $\mathcal{C}^\infty$ consisting of those germs which are eventually infinitely differentiable. In a Hardy field H, the ordering and derivation interact in a pleasant way: if $f \in H$ and $f > \mathbb{R}$, then $f' > 0$. Asymptotic relations in H can be differentiated and integrated: for $0 \neq f, g \not\asymp 1$ in H, $f \preccurlyeq g$ iff $f' \preccurlyeq g'$.

Extending Hardy fields

Early work on Hardy fields focussed on solving algebraic equations and simple first order differential equations: Borel [17], Hardy [43, 44], Bourbaki [22], Marić [59], Sjödin [71], Robinson [62], Rosenlicht [63]. As a consequence, every Hardy field H has a smallest real closed Hardy field extension $\operatorname{Li}(H) \supseteq \mathbb{R}$ that is also closed under integration and exponentiation; call $\operatorname{Li}(H)$ the *Hardy-Liouville closure* of H. (Hardy's field of LE-functions mentioned earlier is contained in $\operatorname{Li}(\mathbb{R})$.) Here is a rather general result of this kind, due to Singer [70]:

Theorem 1.2. *If $y \in \mathcal{C}^1$ satisfies a differential equation $y'P(y) = Q(y)$ where $P(Y)$ and $Q(Y)$ are polynomials over a Hardy field H and $P(y)$ is a unit in $\mathcal{C}$, then y generates a Hardy field $H\langle y\rangle = H(y, y')$ over H.*

Singer's theorem clearly does not extend to second order differential equations: the nonzero solutions of $y'' + y = 0$ in $\mathcal{C}^2$ do not belong to any Hardy field. The solutions in $\mathcal{C}^2$ of the differential equation

$$y'' + y \quad = \quad \mathrm{e}^{x^2} \tag{1.2}$$

form a two-dimensional affine space $y_0 + \mathbb{R}\sin x + \mathbb{R}\cos x$ over $\mathbb{R}$, with y_0 any particular solution. Boshernitzan [21] proved that any of these continuum many solutions generates a Hardy field. Since no Hardy field can contain more than one solution, there are at least continuum many different maximal Hardy fields. By the above, each of them contains $\mathbb{R}$, is real closed, and closed under integration and exponentiation. *What more can we say about maximal Hardy fields?* To give an answer to this question, consider the following conjectures about Hardy fields H:

A. *For any differential polynomial $P(Y) \in H\{Y\} = H[Y, Y', Y'', \dots]$ and $f < g$ in H with $P(f) < 0 < P(g)$ there exists y in a Hardy field extension of H such that $f < y < g$ and $P(y) = 0$.*

B. *For any countable subsets $A < B$ in H there exists y in a Hardy field extension of H such that $A < y < B$.*

Conjecture A for $P \in H[Y, Y']$ holds by [28]. Conjecture A implies that *all maximal Hardy fields are elementarily equivalent* as we shall see in in Section 2. Conjecture B was first raised as a question by Ehrlich [35]. The conjectures together imply that, under CH, *all maximal Hardy fields are isomorphic* (the analogue of Corollary 1.1). We sketch a program to prove A and B in Section 3.

Transseries

Hardy made the point that the LE-functions seem to cover all orders of infinity that occur naturally in mathematics [42, p. 35]. But he also suspected that the order of infinity of the compositional inverse of $(\log x)(\log\log x)$ differs from that of any LE-function [43]; this suspicion is correct. For a more revealing view of orders of infinity and a more comprehensive theory we need transseries. For example, transseries lead to an easy argument to confirm Hardy's suspicion [30, 49]. Here we focus on the field $\mathbb{T}$ of LE-series and in accordance with [4], simply call its elements *transseries,* bearing in mind that many variants of formal series, such as those appearing in [69] (see Section 4 below), can also rightfully be called "transseries".

Transseries are formal series $f = \sum_{\mathfrak{m}} f_{\mathfrak{m}} \mathfrak{m}$ where the $f_{\mathfrak{m}}$ are real coefficients and the $\mathfrak{m}$ are "transmonomials" such as

$$x^r \ (r \in \mathbb{R}), \quad x^{-\log x}, \quad \mathrm{e}^{x^2 \mathrm{e}^x}, \quad \mathrm{e}^{\mathrm{e}^x}.$$

One can get a sense by considering an example like

$$7\,\mathrm{e}^{\mathrm{e}^x + \mathrm{e}^{x/2} + \mathrm{e}^{x/4} + \cdots} - 3\,\mathrm{e}^{x^2} + 5x^{\sqrt{2}} - (\log x)^{\pi} + 42 + x^{-1} + x^{-2} + \cdots + \mathrm{e}^{-x}.$$

Here think of x as positive infinite: $x > \mathbb{R}$. The transmonomials in this series are arranged from left to right in decreasing order. The reversed order type of the set of transmonomials that occur in a given transseries can be any countable ordinal. (In the example above it is $\omega + 1$ because of the term e^{-x} at the end.) Formally, $\mathbb{T}$ is an ordered subfield of a Hahn field $\mathbb{R}[[G^{\mathrm{LE}}]]$ where G^{LE} is the ordered group of transmonomials (or LE-monomials). More generally, let $\mathfrak{M}$ be any (totally) ordered commutative group, multiplicatively written, the $\mathfrak{m} \in \mathfrak{M}$ being thought of as monomials, with the ordering denoted by $\preccurlyeq$. The Hahn field $\mathbb{R}[[\mathfrak{M}]]$ consists of the formal series $f = \sum_{\mathfrak{m}} f_{\mathfrak{m}} \mathfrak{m}$ with real coefficients $f_{\mathfrak{m}}$ whose support $\operatorname{supp} f := \{\mathfrak{m} \in \mathfrak{M} : \ f_{\mathfrak{m}} \neq 0\}$ is *well-based*, that is, well-ordered in the reversed ordering $\succcurlyeq$ of $\mathfrak{M}$. Addition and multiplication of these Hahn series works just as for ordinary power series, and the ordering of $\mathbb{R}[[\mathfrak{M}]]$ is determined by declaring a nonzero Hahn series to be positive if its leading coefficient is positive (so the series above, with leading coefficient 7, is positive). Both $\mathbb{R}[[G^{\mathrm{LE}}]]$ and its ordered subfield $\mathbb{T}$ are real closed. Informally, each transseries is obtained, starting with the powers x^r $(r \in \mathbb{R})$, by applying the following operations finitely many times:

(1) multiplication with real numbers;

(2) infinite summation in $\mathbb{R}[[G^{\mathrm{LE}}]]$;

(3) exponentiation and taking logarithms of positive transseries.

To elaborate on (2), a family $(f_i)_{i\in I}$ in $\mathbb{R}[[\mathfrak{M}]]$ is said to be *summable* if for each $\mathfrak{m}$ there are only finitely many $i \in I$ with $\mathfrak{m} \in \operatorname{supp} f_i$, and $\bigcup_{i\in I} \operatorname{supp} f_i$ is well-based; in this case we define the *sum* $f = \sum_{i\in I} f_i \in \mathbb{R}[[\mathfrak{M}]]$ of this family by $f_{\mathfrak{m}} = \sum_{i\in I}(f_i)_{\mathfrak{m}}$ for each $\mathfrak{m}$. One can develop a "strong" linear algebra for this notion of "strong" (infinite) summation [52, 69]. As to (3), it may be instructive to see how to exponentiate a transseries f: decompose f as $f = g + c + \varepsilon$ where $g := \sum_{\mathfrak{m}\succ 1} f_{\mathfrak{m}}\mathfrak{m}$ is the infinite part of f, $c := f_1$ is its constant term, and ε its infinitesimal part (in our example $c = 42$ and $\varepsilon = x^{-1} + x^{-2} + \cdots + \mathrm{e}^{-x}$); then

$$\mathrm{e}^f \;=\; \mathrm{e}^g \cdot \mathrm{e}^c \cdot \sum_n \frac{\varepsilon^n}{n!}$$

where $\mathrm{e}^g \in \mathfrak{M}$ is a transmonomial, and $\mathrm{e}^c \in \mathbb{R}$, $\sum_n \frac{\varepsilon^n}{n!} \in \mathbb{R}[[G^{\mathrm{LE}}]]$ have their usual meaning. The story with logarithms is a bit different: taking logarithms may also create transmonomials, such as $\log x$, $\log\log x$, etc.

The formal definition of $\mathbb{T}$ is inductive and somewhat lengthy; see [31, 33, 52] or [4, Appendix A] for detailed expositions. We only note here that by virtue of the construction of $\mathbb{T}$, series like $\frac{1}{x} + \frac{1}{\mathrm{e}^x} + \frac{1}{\mathrm{e}^{\mathrm{e}^x}} + \cdots$ or $\frac{1}{x} + \frac{1}{x\log x} + \frac{1}{x\log x\log\log x} + \cdots$ (involving "nested" exponentials or logarithms of unbounded depth), though they are legitimate elements of $\mathbb{R}[[G^{\mathrm{LE}}]]$, do not appear in $\mathbb{T}$; moreover, the sequence $x, \mathrm{e}^x, \mathrm{e}^{\mathrm{e}^x}, \dots$ is cofinal in $\mathbb{T}$, and the sequence $x, \log x, \log\log x, \dots$ is coinitial in the set $\{f \in \mathbb{T} : f > \mathbb{R}\}$. The map $f \mapsto \mathrm{e}^f$ is an isomorphism of the ordered additive group of $\mathbb{T}$ onto its multiplicative group of positive elements, with inverse $g \mapsto \log g$. As an ordered exponential field, $\mathbb{T}$ turns out to be an elementary extension of $\mathbb{R}_{\exp}$ [30].

Transseries can be differentiated termwise; for instance, $\left(\sum_n n!\frac{\mathrm{e}^x}{x^{n+1}}\right)' = \frac{\mathrm{e}^x}{x}$. We obtain a derivation $f \mapsto f'$ on the field $\mathbb{T}$ with constant field $\{f \in \mathbb{T} : f' = 0\} = \mathbb{R}$ and satisfying $(\exp f)' = f' \exp f$ and $(\log g)' = g'/g$ for $f, g \in \mathbb{T}$, $g > 0$. Moreover, each $f \in \mathbb{T}$ has an antiderivative in $\mathbb{T}$, that is, $f = g'$ for some $g \in \mathbb{T}$. As in Hardy fields, $f > \mathbb{R} \Rightarrow f' > 0$, for transseries f. We also have a dominance relation on $\mathbb{T}$: for $f, g \in \mathbb{T}$ we set

$$\begin{aligned} f \preccurlyeq g \;:&\Longleftrightarrow\; |f| \leqslant c|g| \text{ for some } c \in \mathbb{R}^{>0} \\ &\Longleftrightarrow\; (\text{leading transmonomial of } f) \;\preccurlyeq\; (\text{leading transmonomial of } g), \end{aligned}$$

and as in Hardy fields we declare $f \asymp g :\Longleftrightarrow f \preccurlyeq g$ and $g \preccurlyeq f$, as well as $f \prec g :\Longleftrightarrow f \preccurlyeq g$ and $g \not\preccurlyeq f$. As in Hardy fields we can also differentiate and integrate asymptotic relations: for $0 \neq f, g \not\asymp 1$ in $\mathbb{T}$ we have $f \preccurlyeq g$ iff $f' \preccurlyeq g'$.

Hardy's ordered exponential field of (germs of) logarithmic-exponential functions embeds uniquely into $\mathbb{T}$ so as to preserve real constants and to send the germ x to the transseries x; this embedding also preserves the derivation. However, the field of LE-series enjoys many closure properties that the field of LE-functions lacks. For instance, $\mathbb{T}$ is not only closed under exponentiation and integration, but also comes with a natural operation of composition: for $f, g \in \mathbb{T}$ with $g > \mathbb{R}$ we can substitute g for x in $f = f(x)$ to obtain $f \circ g = f(g(x))$. The Chain Rule holds: $(f \circ g)' = (f' \circ g) \cdot g'$. Every $g > \mathbb{R}$ has a compositional inverse in $\mathbb{T}$: a transseries $f > \mathbb{R}$ with $f \circ g = g \circ f = x$. As shown in [52], a Newton diagram method can be used to solve any "feasible" algebraic differential equation in $\mathbb{T}$ (where the meaning of feasible can be made explicit).

Thus it is not surprising that soon after the introduction of $\mathbb{T}$ the idea emerged that it should play the role of a *universal domain* (akin to Weil's use of this term in algebraic geometry) for asymptotic differential algebra: that it *is truly the algebra-from-which-one-can-never-exit and that it marks an almost impassable horizon for "ordered analysis"* [32, p. 148]. Model theory provides a language to make such an intuition precise, as we explain in our survey [3] where we sketched a program to establish the basic model-theoretic properties of $\mathbb{T}$, carried out in [4]. Next we briefly discuss our main results from [4].

2 *H*-Fields

We shall consider $\mathbb{T}$ as an $\mathcal{L}$-structure where the language $\mathcal{L}$ has the primitives 0, 1, $+$, $-$, $\cdot$, ∂ (derivation), $\leqslant$ (ordering), $\preccurlyeq$ (dominance). More generally, let K be any ordered differential field with constant field $C = \{f \in K : f' = 0\}$. This yields a dominance relation $\preccurlyeq$ on K by

$$f \preccurlyeq g \quad :\Longleftrightarrow \quad |f| \leqslant c|g| \text{ for some positive } c \in C$$

and we view K accordingly as an $\mathcal{L}$-structure. The convex hull of C in K is the valuation ring $\mathcal{O} = \{f \in K : f \preccurlyeq 1\}$ of K, with its maximal ideal $\mathcal{o} := \{f \in K : f \prec 1\}$ of infinitesimals.

Definition 2.1. An *H-field* is an ordered differential field K such that (with the notations above), $\mathcal{O} = C + \mathcal{o}$, and for all $f \in K$ we have: $f > C \Rightarrow f' > 0$.

Examples include all Hardy fields that contain $\mathbb{R}$, and all ordered differential subfields of $\mathbb{T}$ that contain $\mathbb{R}$. In particular, $\mathbb{T}$ is an H-field, but $\mathbb{T}$ has further basic elementary properties that do not follow from this: its derivation is small, and it is Liouville closed. An H-field K is said to have *small derivation* if it satisfies $f \prec 1 \Rightarrow f' \prec 1$, and to be *Liouville closed* if it is real closed and for every $f \in K$ there are $g, h \in K$, $h \neq 0$, such that $g' = f$ and $h' = hf$. Each Hardy field H has small derivation, and $\mathrm{Li}(H)$ is Liouville closed.

Inspired by the familiar characterization of real closed ordered fields via the intermediate value property for one-variable polynomial functions, we say that an H-field K has the *Intermediate Value Property* (IVP) if for all differential polynomials $P(Y) \in K\{Y\}$ and all $f < g$ in K with $P(f) < 0 < P(g)$ there is some $y \in K$ with $f < y < g$ and $P(y) = 0$. Van der Hoeven showed that a certain variant of $\mathbb{T}$, namely its H-subfield of gridbased transseries, has IVP; see [51].

Theorem 2.2. *The $\mathcal{L}$-theory of $\mathbb{T}$ is completely axiomatized by the requirements: being an H-field with small derivation; being Liouville closed; and having IVP.*

Actually, IVP is a bit of an afterthought: in [4] we use other (but equivalent) axioms that will be detailed below. We mention the above variant for expository reasons and since it explains why Conjecture A from Section 1 yields that all maximal Hardy fields are elementarily equivalent. Let us define an *H-closed field* to be an H-field that is Liouville closed and has the IVP. All H-fields embed into H-closed fields, and the latter are exactly the existentially closed H-fields. Thus:

Theorem 2.3. *The theory of H-closed fields is model complete.*

Here is an unexpected byproduct of our proof of this theorem:

Corollary 2.4. *H-closed fields have no proper differentially algebraic H-field extensions with the same constant field.*

IVP refers to the ordering, but the valuation given by $\preccurlyeq$ is more robust and more useful. IVP comes from two more fundamental properties: ω-*freeness* and *newtonianity* (a differential version of henselianity). These concepts make sense for any differential field with a suitable dominance relation $\preccurlyeq$ in which the equivalence $f \preccurlyeq g \Longleftrightarrow f' \preccurlyeq g'$ holds for $0 \neq f, g \prec 1$.

To give an inkling of these somewhat technical notions, let K be an H-field and assume that for every $\phi \in K^\times$ for which the derivation $\phi\partial$ is small (that is, $\phi\partial\mathcal{o} \subseteq \mathcal{o}$), there exists $\phi_1 \prec \phi$ in $K^\times$ such that $\phi_1\partial$ is small. (This assumption is satisfied for Liouville closed H-fields.) Let $P(Y) \in K\{Y\}^{\neq}$. We wish to understand how the function $y \mapsto P(y)$ behaves for $y \preccurlyeq 1$. It turns out that this function only reveals its true colors after rewriting P in terms of a derivation $\phi\partial$ with suitable $\phi \in K^\times$.

Indeed, this rewritten P has the form $a \cdot (N + R)$ with $a \in K^\times$ and where $N(Y) \in C\{Y\}^{\neq}$ is independent of ϕ for sufficiently small $\phi \in K^\times$ with respect to $\preccurlyeq$, subject to $\phi\partial$ being small, and where the coefficients of $R(Y)$ are infinitesimal. We call N the *Newton polynomial* of P. Now K is said to be ω-*free* if for all P as above its Newton polynomial has the form $A(Y) \cdot (Y')^n$ for some $A \in C[Y]$ and some n. We say that K is *newtonian* if for all P as above with $N(P)$ of degree 1 we have $P(y) = 0$ for some $y \in \mathcal{O}$. For H-fields, IVP $\Longrightarrow$ ω-free and newtonian; for Liouville closed H-fields, the converse also holds.

Our main result in [4] refines Theorem 2.3 by giving quantifier elimination for the theory of H-closed fields in the language $\mathcal{L}$ above augmented by an additional unary function symbol ι and two extra unary predicates Λ and Ω. These have defining axioms in terms of the other primitives. Their interpretations in $\mathbb{T}$ are as follows: $\iota(f) = 1/f$ if $f \neq 0$, $\iota(0) = 0$, and with $\ell_0 := x$, $\ell_{n+1} := \log \ell_n$,

$$\begin{aligned} \Lambda(f) \quad &\Longleftrightarrow \quad f < \lambda_n := \tfrac{1}{\ell_0} + \tfrac{1}{\ell_0\ell_1} + \cdots + \tfrac{1}{\ell_0\cdots\ell_n} \text{ for some } n, \\ \Omega(f) \quad &\Longleftrightarrow \quad f < \omega_n := \tfrac{1}{(\ell_0)^2} + \tfrac{1}{(\ell_0\ell_1)^2} + \cdots + \tfrac{1}{(\ell_0\cdots\ell_n)^2} \text{ for some } n. \end{aligned}$$

Thus Λ and Ω define downward closed subsets of $\mathbb{T}$. The sequence (ω_n) also appears in classical non-oscillation theorems for second-order linear differential equations.

The ω-freeness of $\mathbb{T}$ reflects the fact that (ω_n) has no pseudolimit in the valued field $\mathbb{T}$. Here are some applications of this quantifier elimination:

Corollary 2.5.

(1) *"O-minimality at infinity": if $S \subseteq \mathbb{T}$ is definable, then for some $f \in \mathbb{T}$ we either have $g \in S$ for all $g > f$ in $\mathbb{T}$ or $g \notin S$ for all $g > f$ in $\mathbb{T}$.*

(2) *All subsets of $\mathbb{R}^n$ definable in $\mathbb{T}$ are semialgebraic.*

Corollaries 2.4 and 2.5 are the departure point for developing a notion of (differential-algebraic) dimension for definable sets in $\mathbb{T}$; see [5].

The results reported on above make us confident that the category of H-fields is the right setting for asymptotic differential algebra. To solidify this impression we return to the motivating examples—Hardy fields, ordered differential fields of transseries, and surreal numbers—and consider how they are related. We start with Hardy fields, which historically came first.

3 *H*-Field Elements as Germs

After Theorem 1.2 and Boshernitzan [19, 21], the first substantial "Hardy field" result on more general differential equations was obtained by van der Hoeven [53]. In what follows we use "d-algebraic" to mean "differentially algebraic" and "d-transcendental" to mean "differentially transcendental".

Theorem 3.1. *The differential subfield $\mathbb{T}^{\mathrm{da}}$ of $\mathbb{T}$ whose elements are the* d-*algebraic transseries is isomorphic over $\mathbb{R}$ to a Hardy field.*

The proof of this theorem is in the spirit of model theory, iteratively extending by a single d-algebraic transseries. The most difficult case (immediate extensions) is handled through careful construction of suitable solutions as convergent series of iterated integrals. We are currently trying to generalize Theorem 3.1 to d-algebraic extensions of arbitrary Hardy fields. Here is our plan:

Theorem 3.2. *Every Hardy field has an ω-free Hardy field extension.*

Theorem 3.3 (in progress)**.** *Every ω-free Hardy field has a newtonian* d-*algebraic Hardy field extension.*

These two theorems, when established, imply that all maximal Hardy fields are H-closed. Hence (by Theorem 2.2) they will all be elementarily equivalent to $\mathbb{T}$, and since H-closed fields have the IVP, Conjecture A from Section 1 will follow.

In order to get an even better grasp on the structure of maximal Hardy fields, we also need to understand how to adjoin d-transcendental germs to Hardy fields. An example of this situation is given by d-transcendental series such as $\sum_n n!!x^{-n}$. By an old result by É. Borel [16] every formal power series $\sum_n a_n t^n$ over $\mathbb{R}$ is the Taylor series at 0 of a $\mathcal{C}^\infty$-function f on $\mathbb{R}$; then $\sum_n a_n x^{-n}$ is an asymptotic expansion of the function $f(x^{-1})$ at $+\infty$, and it is easy to show that if this series is d-transcendental, then the germ at $+\infty$ of this function does generate a Hardy field. Here is a far-reaching generalization:

Theorem 3.4 (in progress)**.** *Every pseudocauchy sequence* (y_n) *in a Hardy field* H *has a pseudolimit in some Hardy field extension of* H*.*

The proof of this for H-closed $H \supseteq \mathbb{R}$ relies heavily on results from [4], using also intricate glueing techniques. For extensions that increase the value group, we need very different constructions. If succesful, these constructions in combination with Theorem 3.4 will lead to a proof of Conjecture B from Section 1:

Theorem 3.5 (in progress)**.** *For any countable subsets* $A < B$ *of a Hardy field* H *there exists an element* y *in a Hardy field extension of* H *with* $A < y < B$*.*

The case $H \subseteq \mathcal{C}^\infty$, $B = \emptyset$ was already dealt with by Sjödin [71]. The various "theorems in progress" together with results from [4] imply that any maximal Hardy fields H_1 and H_2 are back-and-forth equivalent, which is considerably stronger than H_1 and H_2 being elementarily equivalent. It implies for example

> *Under* CH *all maximal Hardy fields are isomorphic.*

This would be the Hardy field analogue of Corollary 1.1. (In contrast to maximal Hausdorff fields, however, maximal Hardy fields cannot be $\aleph_1$-saturated, since their constant field is $\mathbb{R}$.) When we submitted this manuscript, we had finished the proof of Theorem 3.2, and also the proof of Theorem 3.4 in the relevant H-closed case.

Related problems

Some authors (such as [71]) prefer to consider only Hardy fields contained in $\mathcal{C}^\infty$. Theorem 3.2 and our partial result for Theorem 3.4 go through in the $\mathcal{C}^\infty$-setting. All the above "theorems in progress" are plausible in that setting.

What about real analytic Hardy fields (Hardy fields contained in the subring $\mathcal{C}^\omega$ of $\mathcal{C}$ consisting of all real analytic germs)? In that setting Theorem 3.2 goes through. Any d-algebraic Hardy field extension of a real analytic Hardy field is itself real analytic, and so Theorem 3.3 (in progress) will hold in that setting as well. However, our glueing technique employed in the proof of Theorem 3.4 doesn't work there.

Kneser [54] obtained a real analytic solution E at infinity to the functional equation $E(x+1) = \exp E(x)$. It grows faster than any finite iteration of the exponential function, and generates a Hardy field. See Boshernitzan [20] for results of this kind, and a proof that Theorem 3.5 holds for $B = \emptyset$ in the real analytic setting. So in this context we also have an abundant supply of Hardy fields.

Similar issues arise for germs of quasi-analytic and "cohesive" functions [32]. These classes of functions are somewhat more flexible than the class of real analytic functions. For instance, the series $x^{-1} + \mathrm{e}^{-x} + \mathrm{e}^{-\mathrm{e}^x} + \cdots$ converges uniformly for $x > 1$ to a cohesive function that is not real analytic.

Accelero-summation

The definition of a Hardy field ensures that the differential field operations never introduce oscillatory behavior. Does this behavior persist for operations such as composition or various integral transforms? In this connection we note that the Hardy field $H(\widetilde{\mathbb{R}})$ associated to an o-minimal expansion $\widetilde{\mathbb{R}}$ of the field of reals is always closed under composition (see [61]).

To illustrate the problem with composition, let α be a real number > 1 and let $y_0 \in \mathcal{C}^2$ be a solution to (1.2). Then $z_0 := y_0(\alpha x)$ satisfies the equation

$$\alpha^{-2} z'' + z \ = \ \mathrm{e}^{\alpha^2 x^2}. \tag{3.1}$$

It can be shown that $\{y_0 + \sin x, z_0\}$ generates a Hardy field, but it is clear that no Hardy field containing both $y_0 + \sin x$ and z_0 can be closed under composition.

Adjoining solutions to (1.2) and (3.1) "one by one" as in the proof of Theorem 3.1 will not prevent the resulting Hardy fields to contain both $y_0 + \sin x$ and z_0. In order to obtain closure under composition we therefore need an alternative device. Écalle's theory of *accelero-summation* [32] is much more than that. Vastly extending Borel's summation method for divergent series [17], it associates to each *accelero-summable* transseries an *analyzable* function. In this way many non-oscillating real-valued functions that arise naturally (e.g., as solutions of algebraic differential equations) can be represented faithfully by transseries. This leads us to conjecture an improvement on Theorem 3.1:

Conjecture 3.6. Consider the real accelero-summation process where we systematically use the organic average whenever we encounter singularities on the positive real axis. This yields a composition-preserving H-field isomorphism from $\mathbb{T}^{\mathrm{da}}$ onto a Hardy field contained in $\mathcal{C}^{\omega}$.

There is little doubt that this holds. The main difficulty here is that a full proof will involve many tools forged by Écalle in connection with accelero-summation, such as resurgent functions, well-behaved averages, cohesive functions, etc., with some of these tools requiring further elaboration; see also [25, 60].

The current theory of accelero-summation only sums transseries with coefficients in $\mathbb{R}$. Thus it is not clear how to generalize Conjecture 3.6 in the direction of Theorem 3.3. Such a generalization might require introducing transseries over a Hardy field H with suitable additional structure, as well as a corresponding theory of accelero-summation over H for such transseries. In particular, elements of H should be accelero-summable over H in this theory, by construction.

4 *H*-Field Elements as Generalized Transseries

Next we discuss when H-fields embed into differential fields of formal series. A classical embedding theorem of this type is due to Krull [56]: any valued field has a spherically complete immediate extension. As a consequence, any real closed field containing $\mathbb{R}$ is isomorphic over $\mathbb{R}$ to a subfield of a Hahn field $\mathbb{R}[[\mathfrak{M}]]$ with divisible monomial group $\mathfrak{M}$, such that the subfield contains $\mathbb{R}(\mathfrak{M})$. We recently proved an analogue of this theorem for valued differential fields [7]. Here a *valued differential field* is a valued field of equicharacteristic zero equipped with a derivation that is continuous with respect to the valuation topology.

Theorem 4.1. *Every valued differential field has a spherically complete immediate extension.*

For a real closed H-field K with constant field C this theorem gives a Hahn field $\widehat{K} = C[[\mathfrak{M}]]$ with a derivation ∂ on $\widehat{K}$ making it an H-field with constant field C such that K is isomorphic over C to an H-subfield of $\widehat{K}$ that contains $C(\mathfrak{M})$. A shortcoming of this result is that there is no guarantee that ∂ preserves infinite summation. In contrast, the derivation of $\mathbb{T}$ is *strong* (does preserve infinite summation). An abstract framework for even more general notions of transseries is due to van der Hoeven and his former student Schmeling [69].

Fields of transseries

To explain this, consider an (ordered) Hahn field $\mathbb{R}[[\mathfrak{M}]]$ with a partially defined function exp obeying the usual rules of exponentiation; see [52, Section 4.1] for details. In particular, exp has a partially defined inverse function log. We say that $\mathbb{R}[[\mathfrak{M}]]$ is a *field of transseries* if the following conditions hold:

(T1) the domain of the function log is $\mathbb{R}[[\mathfrak{M}]]^{>0}$;

(T2) for each $\mathfrak{m} \in \mathfrak{M}$ and $\mathfrak{n} \in \operatorname{supp} \log \mathfrak{m}$ we have $\mathfrak{n} \succ 1$;

(T3) $\log(1+\varepsilon) = \varepsilon - \frac{1}{2}\varepsilon^2 + \frac{1}{3}\varepsilon^3 + \cdots$ for all $\varepsilon \prec 1$ in $\mathbb{R}[[\mathfrak{M}]]$; and

(T4) for every sequence $(\mathfrak{m}_n)$ in $\mathfrak{M}$ with $\mathfrak{m}_{n+1} \in \operatorname{supp} \log \mathfrak{m}_n$ for all n, there exists an index n_0 such that for all $n \geqslant n_0$ and all $\mathfrak{n} \in \operatorname{supp} \log \mathfrak{m}_n$, we have $\mathfrak{n} \succcurlyeq \mathfrak{m}_{n+1}$ and $(\log \mathfrak{m}_n)_{\mathfrak{m}_{n+1}} = \pm 1$.

The first three axioms record basic facts from the standard construction of transseries. The fourth axiom is more intricate and puts limits on the kind of "nested transseries" that are allowed. Nested transseries such as

$$y \quad = \quad \sqrt{x} + \mathrm{e}^{\sqrt{\log x} + \mathrm{e}^{\sqrt{\log \log x} + \mathrm{e}^{\cdot^{\cdot^{\cdot}}}}} \tag{4.1}$$

are naturally encountered as solutions of functional equations, in this case

$$y(x) \quad = \quad \sqrt{x} + \mathrm{e}^{y(\log x)}. \tag{4.2}$$

Axiom (T4) does allow nested transseries as in (4.1), but excludes series like

$$u \quad = \quad \sqrt{x} + \mathrm{e}^{\sqrt{\log x} + \mathrm{e}^{\sqrt{\log \log x} + \mathrm{e}^{\cdot^{\cdot^{\cdot}}} + \log \log \log x} + \log \log x} + \log x,$$

which solves the functional equation $u(x) = \sqrt{x} + \mathrm{e}^{u(\log x)} + \log x$; in some sense, u is a perturbation of the solution y in (4.1) to the equation (4.2).

Schmeling's thesis [69] shows how to extend a given field of transseries $K = \mathbb{R}[[\mathfrak{M}]]$ with new exponentials and nested transseries like (4.1), and if K also comes with a strong derivation, how to extend this derivation as well. Again, (T4) is crucial for this task: naive termwise differentiation leads to a huge infinite sum that turns out to be summable by (T4). A *transserial derivation* is a strong derivation on K such that nested transseries are differentiated in this way. Such a transserial derivation is uniquely determined by its values on the *log-atomic* elements: those $\lambda \in K$ for which $\lambda, \log \lambda, \log \log \lambda, \ldots$ are all transmonomials in $\mathfrak{M}$.

We can now state a transserial analogue of Krull's theorem. This analogue is a consequence of Theorem 5.3 below, proved in [6].

Theorem 4.2. *Every H-field with small derivation and constant field $\mathbb{R}$ can be embedded over $\mathbb{R}$ into a field of transseries with transserial derivation.*

For simplicity, we restricted ourselves to transseries over $\mathbb{R}$. The theory naturally generalizes to transseries over ordered exponential fields [52, 69] and it should be possible to extend Theorem 4.2 likewise.

Hyperseries

Besides derivations, one can also define a notion of composition for generalized transseries [49, 69]. Whereas certain functional equations such as (4.2) can still be solved using nested transseries, solving the equation $E(x+1) = \exp E(x)$ where $E(x)$ is the unknown, requires extending $\mathbb{T}$ to a field of transseries with composition containing an element $E(x) = \exp_\omega x > \mathbb{T}$, called the *iterator* of $\exp x$. Its compositional inverse $\log_\omega x$ should then satisfy $\log_\omega \log x = (\log_\omega x) - 1$, providing us with a primitive for $(x \log x \log_2 x \cdots)^{-1}$:

$$\log_\omega x \quad = \quad \int \frac{\mathrm{d}x}{x \log x \log_2 x \cdots}.$$

It is convenient to start with iterated logarithms rather than iterated exponentials, and to introduce transfinite iterators $\log_\alpha x$ recursively using

$$\log_\alpha x \quad = \quad \int \frac{\mathrm{d}x}{\prod_{\beta<\alpha} \log_\beta x} \qquad (\alpha \text{ any ordinal}).$$

By Écalle [32] the iterators $\log_\alpha x$ with $\alpha < \omega^\omega$ and their compositional inverses $\exp_\alpha x$ suffice to resolve all pure composition equations of the form

$$f^{\circ k_1} \circ \phi_1 \circ \cdots \circ f^{\circ k_n} \circ \phi_n \;=\; x \quad \text{where } \phi_1, \ldots, \phi_n \in \mathbb{T} \text{ and } k_1, \ldots, k_n \in \mathbb{N}.$$

The resolution of more complicated functional equations involving differentiation and composition requires the introduction of fields of *hyperseries*: besides exponentials and logarithms, hyperseries are allowed to contain iterators $\exp_\alpha x$ and $\log_\alpha x$ of any strength α. For $\alpha < \omega^\omega$, the necessary constructions were carried out in [69]. The ultimate objective is to construct a field $\mathbf{Hy}$ of hyperseries as a proper class, similar to the field of surreal numbers, endow it with its canonical derivation and composition, and establish the following:

Conjecture 4.3. Let Φ be any partial function from $\mathbf{Hy}$ into itself, constructed from elements in $\mathbf{Hy}$, using the field operations, differentiation and composition. Let $f < g$ be hyperseries in $\mathbf{Hy}$ such that Φ is defined on the closed interval $[f, g]$ and $\Phi(f)\Phi(g) < 0$. Then for some $y \in \mathbf{Hy}$ we have $\Phi(y) = 0$ and $f < y < g$.

One might then also consider H-fields with an additional composition operator and try to prove that these structures can always be embedded into $\mathbf{Hy}$.

5 Growth Rates as Numbers

Turning to surreal numbers, how do they fit into asymptotic differential algebra?

The *H*-field of surreal numbers

The totality **No** of surreal numbers is not a set but a proper class: a surreal $a \in$ **No** is uniquely represented by a transfinite sign sequence $(a_\lambda)_{\lambda<\ell(a)} \in \{-,+\}^{\ell(a)}$ where $\ell(a)$ is an ordinal, called the *length* of a; a surreal b is said to be *simpler than* a (notation: $b <_s a$) if the sign sequence of b is a proper initial segment of that of a. Besides the (partial) ordering $<_s$, **No** also carries a natural (total) lexicographic ordering $<$. For any *sets* $L < R$ of surreals there is a unique simplest surreal a with $L < a < R$; this a is denoted by $\{L \mid R\}$ and called the *simplest* or *earliest* surreal between L and R. In particular, $a = \{L_a \mid R_a\}$ for any $a \in$ **No**, where $L_a := \{b <_s a : \ b < a\}$ and $R_a = \{b <_s a : \ b > a\}$. We let a^L range over elements of L_a, and a^R over elements of R_a.

A rather magical property of surreal numbers is that various operations have natural inductive definitions. For instance, we have ring operations given by

$$\begin{aligned} a+b &:= \{a^L+b, a+b^L \mid a^R+b, a+b^R\} \\ ab &:= \{a^Lb+ab^L-a^Lb^L, a^Rb+ab^R-a^Rb^R \mid \\ &\qquad a^Lb+ab^R-a^Lb^R, a^Rb+ab^L-a^Rb^L\}. \end{aligned}$$

Remarkably, these operations make **No** into a real closed field with $<$ as its field ordering and with $\mathbb{R}$ uniquely embedded as an initial subfield. (A set $A \subseteq$ **No** is said to be *initial* if for all $a \in A$ all $b <_s a$ are also in A.)

Can we use such magical recursions to introduce other reasonable operations? Exponentiation was dealt with by Gonshor [40]. But it remained long open how to define a "good" derivation ∂ on **No** such that $\partial(\omega) = 1$. (An ordinal α is identified with the surreal of length α whose sign sequence has just plus signs.) A positive answer was given recently by Berarducci and Mantova [8]. Their construction goes in two parts. They first analyze **No** as an exponential field, and show that it is basically a field of transseries in the sense of Section 4. A transserial derivation on **No** is determined by its values at log-atomic elements. There is some flexibility here, but [8] presents a "simplest" way to choose these derivatives. Most important, that choice indeed leads to a derivation ∂_{BM} on **No**. In addition:

Theorem 5.1 (Berarducci-Mantova [8])**.** *The derivation ∂_{BM} is transserial and makes* **No** *a Liouville closed H-field with constant field $\mathbb{R}$.*

This result was further strengthened in [6], using key results from [4]:

Theorem 5.2. **No** *with the derivation ∂_{BM} is an H-closed field.*

Embedding *H*-fields into No

In the remainder of this section we consider **No** as equipped with the derivation ∂_{BM}, although Theorems 5.1 and 5.2 and much of what follows hold for other transserial derivations. Returning to our main topic of embedding H-fields into specific H-fields such as $\mathbf{No}$, we also proved the following in [6]:

Theorem 5.3. *Every H-field with small derivation and constant field $\mathbb{R}$ can be embedded as an ordered differential field into* $\mathbf{No}$.

How "nice" can we take the embeddings in Theorem 5.3? For instance, when can we arrange the image of the embedding to be initial? The image of the natural embedding $\mathbb{T} \to \mathbf{No}$ is indeed initial, as has been shown by Elliot Kaplan.

For further discussion it is convenient to introduce, given an ordinal α, the *set* $\mathbf{No}(\alpha) := \{a \in \mathbf{No} : \ell(a) < \alpha\}$. It turns out that for uncountable cardinals κ, $\mathbf{No}(\kappa)$ is closed under the differential field operations, and in [6] we also show:

Theorem 5.4. *The H-subfield $\mathbf{No}(\kappa)$ of $\mathbf{No}$ is an elementary submodel of* $\mathbf{No}$.

In particular, the H-field $\mathbf{No}(\omega_1)$ of surreal numbers of countable length is an elementary submodel of $\mathbf{No}$. It has the η_1-property: for any countable subsets $A < B$ of $\mathbf{No}(\omega_1)$ there exists $y \in \mathbf{No}(\omega_1)$ with $A < y < B$. This fact and the various "theorems in progress" from Section 3 imply:

> *Under* CH *all maximal Hardy fields are isomorphic to* $\mathbf{No}(\omega_1)$.

This would be an analogue of Ehrlich's observation about maximal Hausdorff fields.

Hyperseries as numbers and *vice versa*

The similarities in the constructions of the field of hyperseries $\mathbf{Hy}$ and the field of surreal numbers $\mathbf{No}$ led van der Hoeven [52, p. 6] to the following:

Conjecture 5.5. There is a natural isomorphism between $\mathbf{Hy}$ and $\mathbf{No}$ that associates to any hyperseries $f(x) \in \mathbf{Hy}$ its value $f(\omega) \in \mathbf{No}$.

The problem is to make sense of the value of a hyperseries at ω. Thanks to Gonshor's exponential function, it is clear how to evaluate ordinary transseries at ω. The difficulties start as soon as we wish to represent surreal numbers that are not of the form $f(\omega)$ with $f(x)$ an ordinary transseries. That is where the iterators $\exp_\omega$ and $\log_\omega$ come into play:

$$\begin{aligned}
\exp_\omega \omega &:= \{\omega, \exp\omega, \exp_2\omega, \dots \mid \} \\
\log_\omega \omega &:= \{\mathbb{R} \mid \dots, \log_2\omega, \log\omega, \omega\} \\
\exp_{1/2}\omega &:= \exp_\omega\left(\log_\omega\left(\omega + \tfrac{1}{2}\right)\right) \\
&:= \left\{\omega^2, \exp\log^2\omega, \exp_2\log_2^2\omega, \dots \,\middle|\, \dots, \exp_2\sqrt{\log\omega}, \exp\sqrt{\omega}\right\}
\end{aligned}$$

The intuition behind Conjecture 5.5 is that all "holes in $\mathbf{No}$ can be filled" using suitable nested hyperseries and suitable iterators of exp and log. It reconciles two *a priori* very different types of infinities: on the one hand, we have growth orders corresponding to smooth functional behavior; on the other side, we have numbers. Being able to switch between functions (more precisely: formal series acting as functions) and numbers, we may also transport any available structure in both directions: we immediately obtain a canonical derivation ∂_{c} (with constant field $\mathbb{R}$) and composition $\circ_{\mathrm{c}}$ on $\mathbf{No}$, as well as a notion of simplicity on $\mathbf{Hy}$.

Does the derivation ∂_{BM} coincide with the canonical derivation ∂_{c} induced by the conjectured isomorphism? A key observation is that any derivation ∂ on $\mathbf{No}$ with a distinguished right inverse ∂^{-1} naturally gives rise to a definition of $\log_\omega$:

$$\log_\omega a \;:=\; \partial^{-1}(\partial a \log'_\omega a) \quad \text{where}$$
$$\log'_\omega a := 1 \Big/ \prod_n \log_n a \qquad (a \in \mathbf{No},\ a > \mathbb{R}).$$

(For a family (a_i) of positive surreals, $\prod_i a_i := \exp \sum_i \log a_i$ if $\sum_i \log a_i$ is defined.) Since ∂_{BM} is transserial, it does admit a distinguished right inverse $\partial_{\mathrm{BM}}^{-1}$. According to [8, Remark 6.8], $\partial_{\mathrm{BM}}\lambda = 1/\log'_\omega \lambda$ for log-atomic λ with $\lambda > \exp_n \omega$ for all n. For $\lambda = \exp_\omega \omega$ and setting $\exp'_\omega(a) := \prod_n \log_n \exp_\omega a$ for $a \in \mathbf{No}^{>0}$, this yields $\partial_{\mathrm{BM}}\lambda = \exp'_\omega \omega$, which is also the value we expect for $\partial_{\mathrm{c}}\lambda$. However, for $\lambda = \exp_\omega(\exp_\omega \omega)$ we get $\partial_{\mathrm{BM}}\lambda = \exp'_\omega(\exp_\omega \omega)$ whereas we expect $\partial_{\mathrm{c}}\lambda = (\exp'_\omega \omega) \cdot \exp'_\omega(\exp_\omega \omega)$. Thus the "simplest" derivation ∂_{BM} making $\mathbf{No}$ an H-field probably does *not* coincide with the ultimately "correct" derivation ∂_{c} on $\mathbf{No}$. Berarducci and Mantova [9] use similar considerations to conclude that ∂_{BM} is incompatible with any reasonable notion of composition for surreal numbers.

The surreal numbers from a model theoretic perspective

We conclude with speculations motivated by the fact that various operations defined by "surreal" recursions have a nice model theory. Examples: $(\mathbf{No}; \leqslant, +, \cdot)$ is a model of the theory of real closed fields; $(\mathbf{No}; \leqslant, +, \cdot, \exp)$ is a model of the theory of $\mathbb{R}_{\exp}$; and $(\mathbf{No}; \leqslant, +, \cdot, \partial_{\mathrm{BM}})$ is a model of the theory of H-closed fields. Each of these theories is model complete in a natural language. Is there a model theoretic reason that explains why this works so well?

Let us look at this in connection with the last example. Our aim is to define a derivation ∂ on $\mathbf{No}$ making it an H-field. Let $a \in \mathbf{No}$ be given for which we wish to define ∂a, and assume that ∂b has been defined for all $b \in L_a \cup R_a$. Let Δ_a be the class of all surreals b for which there exists a derivation ∂ on $\mathbf{No}$ with $\partial a = b$ and taking the prescribed values on $L_a \cup R_a$. Assembling all conditions that should be satisfied by ∂a, it is not hard to see that there exist sets $L, R \subseteq \mathbf{No}$ such that $\Delta_a = \{b \in \mathbf{No} : L < b < R\}$. We are left with two main questions: *When do we have $L < R$, thereby allowing us to define $\partial a = \{L \mid R\}$? Does this lead to a global definition of ∂ on $\mathbf{No}$ making it an H-closed field?* It might be of interest to isolate reasonable model theoretic conditions that imply the success of this type of construction. If the above construction does work, yet another question is whether the resulting derivation coincides with ∂_{BM}.

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