LIOUVILLE CLOSED H-FIELDS

MATTHIAS ASCHENBRENNER AND LOU VAN DEN DRIES

Abstract. H-fields are fields with an ordering and a derivation subject to some compatibilities. (Hardy fields extending $\mathbb{R}$ and fields of transseries over $\mathbb{R}$ are H-fields.) We prove basic facts about the location of zeros of differential polynomials in Liouville closed H-fields, and study various constructions in the category of H-fields: closure under powers, constant field extension, completion, and building H-fields with prescribed constant field and H-couple. We indicate difficulties in obtaining a good model theory of H-fields, including an undecidability result. We finish with open questions that motivate our work.

Contents

Date: June 2004.
2000 *Mathematics Subject Classification.* 12H05, 12J15.
Key words and phrases. H-fields, Hardy fields, fields of transseries.
Corresponding author: Aschenbrenner.
Van den Dries acknowledges support from NSF grant DMS 01-00979.

Introduction

In [2] we introduced H-fields as an abstraction of Hardy fields [7], [26], and as a step towards a model-theoretic understanding of the differential field $\mathbb{R}((t))^{\mathrm{LE}}$ of logarithmic-exponential series [12]. Here we develop the subject of H-fields further. Recall from [2] that an H-field is an ordered differential field K with constant field C such that for every $f \in K$:

(1) if $f > c$ for all $c \in C$, then $f' > 0$;
(2) if $|f| < d$ for some positive $d \in C$, then there exists $c \in C$ such that $|f - c| < d$ for all positive $d \in C$.

Every Hardy field $K \supseteq \mathbb{R}$ is an H-field, as is every ordered differential subfield $K \supseteq \mathbb{R}$ of $\mathbb{R}((t))^{\mathrm{LE}}$. In the rest of the paper we assume familiarity with [2], including its notational conventions.

Our first aim is to prove some basic facts on zeros of differential polynomials over Liouville closed H-fields, such as the following two results. (An H-field K is said to be *Liouville closed* if K is real closed, and for any $a \in K$ there exist $y, z \in K$ with $y' = a$ and $z \neq 0, z'/z = a$.) Let K be a Liouville closed H-field with constant field C and let $P(Y) \in K\{Y\}$ be a non-zero differential polynomial.

Theorem. *Suppose the coefficients of $P(Y)$ lie in some H-subfield of K with a smallest comparability class. Then there exists $a > C$ in K such that $P(y) \neq 0$ for all y in all H-field extensions L of K with $C_L < y < a$, where C_L is the constant field of L.* (See Section 1 for "comparability class.")

More precise versions are in Section 2, with preliminaries on asymptotic relations involving exponentiation in Section 1. The hypothesis in this theorem is always satisfied for $K = \mathbb{R}((t))^{\mathrm{LE}}$, see Section 2. This hypothesis can be omitted if P is of order 1, see Proposition 2.7, or homogeneous of order 2, see Corollary 12.14. An example in [3] shows that the hypothesis cannot be omitted for differential polynomials of order 3. (This example also produces a differentially algebraic "gap" over a Liouville closed H-field, answering a question formulated at the end of [2].)

While the previous theorem concerns nonexistence of "small" infinite zeros, the next result claims nonexistence of "large" infinite zeros.

Theorem. *There exists $b \in K$ such that $P(y) \neq 0$ for all y in all H-field extensions of K with $y > b$.*

This is shown in a stronger form in Section 3. Next we prove in Section 4 an intermediate value property for differential polynomials of order 1 over H-fields. Section 5 concerns the valuation of higher derivatives, and is used in Section 6 to study *simple* zeros of differential polynomials over H-fields.

For deeper results on solving algebraic differential equations in H-fields we shall need to adapt the Newton polygon methods developed for fields of transseries by J. van der Hoeven in Chapters 3–5 of his Thèse [16]. Here we focus on what can be done by cruder methods under weaker assumptions on the H-fields considered.

Sections 7–11 have a different character, and contain topics that should be part of any systematic development of the subject of H-fields: introducing exponential maps and power functions on Liouville closed H-fields (Section 7), adjoining powers to H-fields (Section 8), constant field extension (Section 9), completion (Section 10), and building H-fields with given constant field and asymptotic couple via a generalized series construction (Section 11).

In Section 12 we study "gaps" in H-fields and fill in the details of an example in Section 6 of [2]. In Section 13 we show that the set of integers is existentially definable in the differential field $\mathbb{R}(\!(x^{-1})\!)^{\mathrm{E}}$ of exponential series. (Hence the theory of this differential field is undecidable.) In Section 14 we summarize what we know about *existentially closed* H-fields, and list open problems.

Notations. The notations and conventions introduced in [1] and [2] remain in force; in particular, m and n range over $\mathbb{N} = \{0, 1, 2, \dots\}$.

Let K be a differential field of characteristic 0. For $\boldsymbol{i} = (i_0, \dots, i_n) \in \mathbb{N}^{n+1}$ we put $|\boldsymbol{i}| := i_0 + i_1 + \cdots + i_n$ (the **degree of $\boldsymbol{i}$**), $w\boldsymbol{i} := i_1 + 2i_2 + 3i_3 + \cdots + ni_n$ (the **weight of $\boldsymbol{i}$**), and we set $Y^{\boldsymbol{i}} := Y^{i_0}(Y')^{i_1} \cdots (Y^{(n)})^{i_n}$ for a differential indeterminate Y, and $y^{\boldsymbol{i}} := y^{i_0}(y')^{i_1} \cdots (y^{(n)})^{i_n}$ for an element y of K. Following a suggestion by J. van der Hoeven, we denote the logarithmic derivative y'/y of $y \in K^\times = K \setminus \{0\}$ by $y^\dagger$. Let the differential polynomial $P \in K\{Y\}$ be of order at most n. Thus

$$P(Y) = \sum_{\boldsymbol{i}} a_{\boldsymbol{i}} Y^{\boldsymbol{i}},$$

where the sum is understood to range over all $\boldsymbol{i} \in \mathbb{N}^{n+1}$, and $a_i \in K$ for every $\boldsymbol{i}$, with $a_{\boldsymbol{i}} \neq 0$ for only finitely many $\boldsymbol{i}$. For $P \neq 0$ the (total) **degree** of P is the largest natural number d such that $d = |\boldsymbol{i}|$ for some $\boldsymbol{i} \in \mathbb{N}^{n+1}$ with $a_{\boldsymbol{i}} \neq 0$. We also set, for $\boldsymbol{i} = (i_0, \dots, i_n) \in \mathbb{N}^{n+1}$,

$$P^{(\boldsymbol{i})} = \frac{\partial^{|\boldsymbol{i}|} P}{\partial Y^{\boldsymbol{i}}} := \frac{\partial^{i_0}}{\partial (Y^{(0)})^{i_0}} \cdots \frac{\partial^{i_n}}{\partial (Y^{(n)})^{i_n}} P,$$

a differential polynomial of order $\leq n$.

An element y of a differential field extension L of K is said to be *differentially algebraic over K* if it satisfies an algebraic differential equation $P(y) = 0$ with $P(Y) \in K\{Y\} \setminus \{0\}$. An extension $L|K$ of differential fields is called *differentially algebraic* if every element y of L is differentially algebraic over K.

1. Asymptotic Relations and Exponentiation

Asymptotic relations among elements of H-fields can be expressed in terms of the valuation and the function ψ induced on the value group, as we did in [2]. In the present paper we often use the following shorter notations suggested to us by J. van der Hoeven. Let K be a pre-differential-valued field and $f,g\in K$. Then

(1) $f\preceq g :\Longleftrightarrow v(f)\geq v(g)$,
(2) $f\asymp g :\Longleftrightarrow f\preceq g$ and $g\preceq f \Longleftrightarrow v(f)=v(g)$,
(3) $f\prec g :\Longleftrightarrow v(f)>v(g)$,
(4) $f\sim g :\Longleftrightarrow f-g\prec g$,
(5) $f\mathrel{\preceq\!\!\!\prec} g :\Longleftrightarrow f,g\neq 0$ and $f^\dagger\preceq g^\dagger$,
(6) $f\mathrel{-\!\!\!\!\asymp} g :\Longleftrightarrow f,g\neq 0$ and $f^\dagger\asymp g^\dagger$,
(7) $f\mathrel{-\!\!\!\!\prec\!\!\!\prec} g :\Longleftrightarrow f,g\neq 0$ and $f^\dagger\prec g^\dagger$.

In particular, if K is a pre-H-field, then

$$f\preceq g \quad\Longleftrightarrow\quad |f|\leq a|g| \text{ for some } a\in\mathcal{O}^{>0}.$$

We also write $f\preceq g$ as $g\succeq f$, and $f\prec g$ as $g\succ f$. To negate any of the above relations we use a slash; for example, $f\not\asymp g$ means that $v(f)\neq v(g)$. These relations among elements of K are all preserved (in both directions) when K is replaced by a pre-differential-valued field extension. Note that $\asymp$ and $\sim$ are equivalence relations on K and $K^\times$ respectively. When using $\mathrel{\preceq\!\!\!\prec}$, $\mathrel{-\!\!\!\!\asymp}$, and $\mathrel{-\!\!\!\!\prec\!\!\!\prec}$ it is often convenient to exclude elements $f\asymp 1$. Indeed, if $f,g\neq 0$ and $f,g\not\asymp 1$, then we have the equivalences

$$\begin{aligned} f\mathrel{\preceq\!\!\!\prec} g &\quad\Longleftrightarrow\quad \psi\big(v(f)\big)\geq\psi\big(v(g)\big),\\ f\mathrel{-\!\!\!\!\asymp} g &\quad\Longleftrightarrow\quad \psi\big(v(f)\big)=\psi\big(v(g)\big),\\ f\mathrel{-\!\!\!\!\prec\!\!\!\prec} g &\quad\Longleftrightarrow\quad \psi\big(v(f)\big)>\psi\big(v(g)\big). \end{aligned}$$

Thus we say that f,g are **comparable** if $f,g\neq 0$, $f,g\not\asymp 1$ and $f\mathrel{-\!\!\!\!\asymp} g$. Comparability is an equivalence relation on $\{f\in K: 0\neq f\not\asymp 1\}$. The corresponding equivalence class of f with $0\neq f\not\asymp 1$ is called its **comparability class**, and written as $\operatorname{Cl}(f)$. The set of comparability classes is linearly ordered by setting

$$\operatorname{Cl}(f)\leq\operatorname{Cl}(g) \quad:\Longleftrightarrow\quad f\mathrel{\preceq\!\!\!\prec} g.$$

We have an order reversing bijection $\operatorname{Cl}(f)\mapsto\psi(v(f))=v(f^\dagger)$ from the set of comparability classes onto the subset Ψ of Γ. For $0\neq f\not\asymp 1$ the elements f, $-f$, $1/f$ and $-1/f$ are comparable, so if K is a pre-H-field, then each comparability class contains positive infinite elements (elements $>\mathcal{O}$), and usually we take such elements when dealing with comparability. If K is even a Hardy field and f,g are positive infinite, then $f\mathrel{-\!\!\!\!\prec\!\!\!\prec} g$ if and only if $f^n<g$ for all n, so our use of the term "comparability class" and the ordering on the set of comparability classes agrees with Rosenlicht's use for Hardy fields in [27]. For Liouville closed H-fields we shall similarly characterize comparability in terms of "powers" f^c with $f>C$ and $c\in C$, see §7.

The following lemma lists some simple rules about these relations.

Lemma 1.1. *Let $f,g\in K$. Then*

(1) *If $f\not\asymp 1$ and $g\not\asymp 1$, then $f\preceq g$ if and only if $f'\preceq g'$,*
(2) *If $f\prec g\not\asymp 1$, then $f'\prec g'$,*

(3) *If* $f \preceq g \not\asymp 1$ *and* $f' \sim g'$, *then* $f \sim g$.
(4) *If* $1 \prec f \preceq g$, *then* $f \preceq\!\!\!\prec g$.
(5) *If* K *is an* H*-field and* $f, g > C$, $f \prec\!\!\!\prec g$, *then* $f^n < g$ *for all* n.

Proof. Apart from notation and terminology, (1), (2) and (4) are in [2]. For (3), consider the case $f - g \not\asymp 1$, where we can apply (1), and the case $f - g \asymp 1$, which under the hypothesis of (3) implies $f \asymp g$, hence $g \succ 1$, and thus $f - g \prec g$. With the hypothesis of (5), suppose $f^n \geq g$, $n > 0$. Then $1 \prec g \preceq f^n$, so $g \preceq\!\!\!\prec f^n \preceq\!\!\!\prec f$ by (4), contradiction. □

Exponentials in Liouville closed H-fields. In this subsection K will denote a Liouville closed H-field. We shall need a crude substitute for an exponential function on K, and accordingly we choose for every $f \in K$ an element $\mathrm{E}(f) \in K^{>0}$ such that $\mathrm{E}(f)^\dagger = f'$. (So for $g \in K^{>0}$ we have $g^\dagger = f'$ if and only if $g = c\,\mathrm{E}(f)$ for some positive constant c.) Here are some simple rules about E. Let $f, g \in K$; then

(E1) $\mathrm{E}(f+g) = c\,\mathrm{E}(f)\,\mathrm{E}(g)$ and $\mathrm{E}(-f) = d\,\mathrm{E}(f)^{-1}$, where $c, d \in C^{>0}$;
(E2) $f \preceq 1 \Longleftrightarrow \mathrm{E}(f) \asymp 1$;
(E3) $f > C \Longleftrightarrow \mathrm{E}(f) \succ 1$; $f < C \Longleftrightarrow \mathrm{E}(f) \prec 1$;
(E4) $1 \prec f \Longrightarrow f \prec\!\!\!\prec \mathrm{E}(f)$;
(E5) $f > C \Longrightarrow \mathrm{E}(f) > f^n$; $f < C \Longrightarrow 0 < \mathrm{E}(f) < |f|^{-n} < C^{>0}$;
(E6) If $f, g \not\asymp 1$, then $f \prec g \Longleftrightarrow \mathrm{E}(f) \prec\!\!\!\prec \mathrm{E}(g)$.

In Section 7 we show that if $C = \mathbb{R}$, then the map $f \mapsto \mathrm{E}(f)$ can be chosen such that the constants c and d in (E1) are always equal to 1. Proof of (E2):

$$f \preceq 1 \Longleftrightarrow v(f) \geq 0 \Longleftrightarrow v(f') = v\big(\mathrm{E}(f)^\dagger\big) > \Psi$$
$$\Longleftrightarrow v(\mathrm{E}(f)) = 0 \Longleftrightarrow \mathrm{E}(f) \asymp 1.$$

Proof of (E3): suppose $f > C$; then $f' = \mathrm{E}(f)^\dagger > 0$, so $\mathrm{E}(f) \not\prec 1$, and hence $\mathrm{E}(f) \succ 1$ by (E2). For $f < C$, use the second part of (E1) to conclude $\mathrm{E}(f) \prec 1$. Proof of (E4): suppose $1 \prec f$; then $v(f) < 0$, so

$$\psi\big(v(f)\big) = v(f'/f) > v(f') = \psi\big(v\big(\mathrm{E}(f)^\dagger\big)\big),$$

hence $f \prec\!\!\!\prec \mathrm{E}(f)$. Now (E5) follows from (E3), (E4) and part (5) of the last lemma. Proof of (E6): with $f, g \not\asymp 1$ we have

$$f \prec g \Longleftrightarrow f' \prec g' \Longleftrightarrow \mathrm{E}(f)^\dagger \prec \mathrm{E}(g)^\dagger \Longleftrightarrow \mathrm{E}(f) \prec\!\!\!\prec \mathrm{E}(g).$$

Similarly, for all $f \in K$ with $f > 0$ we choose $\mathrm{L}(f) \in K$ such that $\mathrm{L}(f)' = f^\dagger$. (Thus $g \in K$ satisfies $g' = f^\dagger$ if and only if $g = \mathrm{L}(f) + c$ for some constant c.) Clearly $\mathrm{L}\big(\mathrm{E}(f)\big) = c + f$ and $\mathrm{E}\big(\mathrm{L}(f)\big) = d \cdot f$ for some constants c, d of K. For $f, g \in K^{>0}$, we have

(L1) $\mathrm{L}(f \cdot g) = c + \mathrm{L}(f) + \mathrm{L}(g)$ and $\mathrm{L}(f^{-1}) = d - \mathrm{L}(f)$, where $c, d \in C$;
(L2) $f \asymp 1 \Longleftrightarrow \mathrm{L}(f) \preceq 1$;
(L3) $f \succ 1 \Longleftrightarrow \mathrm{L}(f) > C$; $f \prec 1 \Longleftrightarrow \mathrm{L}(f) < C$;
(L4) $f \not\asymp 1 \Longrightarrow \mathrm{L}(f) \prec\!\!\!\prec f$;
(L5) $f \succ 1 \Longrightarrow \big(\mathrm{L}(f)\big)^n < f$; $f \prec 1 \Longrightarrow C < |\mathrm{L}(f)|^n < f^{-1}$;

(L6) If $f,g \not\asymp 1$, then $\mathrm{L}(f) \prec \mathrm{L}(g) \iff f \prec\!\!\prec g$.

(Here, (L2) and (L3) immediately follow from (E2) and (E3), respectively. For (L4), note that $f \not\asymp 1$ implies $1 \prec \mathrm{L}(f)$ by (L2), (L3); so $\mathrm{L}(f) \prec\!\!\prec \mathrm{E}\big(\mathrm{L}(f)\big) \asymp f$ by (E4). Now (L5) follows from (L3), (L4) and part (5) of Lemma 1.1, and (L6) follows from (E6).)

Let E_n denote the n-th iterate of the map $f \mapsto \mathrm{E}(f)\colon K \to K^{>0}$. So $\mathrm{E}_0 = \mathrm{id}_K$, $\mathrm{E}_1 = \mathrm{E}$, $\mathrm{E}_2 = \mathrm{E}\circ\mathrm{E}$, and so on. The function L maps $K^{>C}$ into itself, by (L3). Let $\mathrm{L}_n\colon K^{>C} \to K^{>C}$ be the n-th iterate of

$$f \mapsto \mathrm{L}(f)\colon K^{>C} \to K^{>C},$$

so L_0 is the identity map on $K^{>C}$. In the next two lemmas, used in Section 3, we assume that x is an element of K with $x > C$ and $x' = 1$.

Lemma 1.2. *Let $y \in K$, $y \succeq \mathrm{E}(x^2)$. Then $C < y^{(n)}/y \prec\!\!\prec y$, for each $n > 0$.*

Proof. We may assume $y > 0$ (replacing y by $-y$, if $y < 0$). From $y \succeq \mathrm{E}(x^2) \succ 1$ we get $y \succeq\!\!\succeq \mathrm{E}(x^2)$, that is, $y^\dagger \succeq \mathrm{E}(x^2)^\dagger = 2x$, hence $y^\dagger > C$. Moreover, $1/y^\dagger \preceq 1/x \prec 1$, so by Lemma 1.1, (1) we get $-y^{\dagger\dagger}/y^\dagger = (1/y^\dagger)' \preceq (1/x)' = -1/x^2 \prec 1$, and thus $y^\dagger \prec\!\!\prec y$. This proves the desired inequalities for $n = 1$. For $n > 1$, write

$$y^{(n)}/y = \big(y^{(n-1)}\big)^\dagger \cdot y^{(n-1)}/y,$$

and use a straightforward induction argument. □

Lemma 1.3. *Let $y, f \in K$, $y \neq 0$ and $f \geq x^2$.*

1. *If $y^\dagger < f$, then $|y| < \mathrm{E}(xf)$.*
2. *If $y^\dagger < \mathrm{E}_n(f)$, then $|y| < \mathrm{E}_{n+1}(f)$, for all $n > 0$.*

Proof. We may as well assume that $y > 0$ (by replacing y by $-y$, if $y < 0$). Suppose $y^\dagger < f$. Since $f \geq x^2 \succ x > C$, we have $f' \succ x' = 1$ and $f' > 0$, that is, $f' > C$. Therefore

$$\big(xf - \mathrm{L}(y)\big)' = xf' + f - y^\dagger > xf' > C,$$

hence in particular $\big(xf - \mathrm{L}(y)\big)' \succ 1 = x'$, so $xf - \mathrm{L}(y) \succ x \succ 1$. It follows that $xf - \mathrm{L}(y) > C$, so

$$\mathrm{E}(xf)/y \asymp \mathrm{E}\big(xf - \mathrm{L}(y)\big) > xf - \mathrm{L}(y) \succ 1,$$

by (E1) and (E5). Hence $\mathrm{E}(xf) > y$, showing (1).

For (2), note that it suffices to consider the case $n = 1$, because $\mathrm{E}_{n-1}(f) > f \geq x^2$ for $n > 1$, by (E5). We prove the stronger statement

$$y \succeq \mathrm{E}_2(f) \quad\Longrightarrow\quad y^\dagger \succ \mathrm{E}(f).$$

Suppose $y \succeq \mathrm{E}_2(f)$. Since $f > C$, we have $\mathrm{E}_2(f) \succ 1$ by (E3), so $y^\dagger \succeq \mathrm{E}_2(f)^\dagger = \mathrm{E}(f)'$ by Lemma 1.1, (4). Now $f \succ x \succ 1$ implies $\mathrm{E}(f)^\dagger = f' \succ x' = 1$, hence $y^\dagger \succeq \mathrm{E}(f)' = \mathrm{E}(f)\,\mathrm{E}(f)^\dagger \succ \mathrm{E}(f)$ as required. □

2. Nonexistence of Small Infinite Zeros

A *small infinite* element of an H-field is one that is just a bit larger than all constants. Many difficulties in the subject arise from properties of small infinite elements. In this section we focus on the property of being a zero of a given differential polynomial.

Since the above description of *small infinite element* is not a precise definition, we shall avoid this term below, but it might be helpful to keep in the back of one's mind.

Lemma 2.1. *Let $E \subseteq F$ be an extension of pre-H-fields with* $\operatorname{trdeg}(F|E) \leq n$*. Then there are at most n comparability classes of F without representative in E.*

This is proved just like Proposition 5 in [27] about Hardy fields.

Lemma 2.2. *Let E be a pre-H-field contained in a Liouville closed H-field F, and let $a \in E$ be positive infinite, such that E has comparability classes smaller than that of a. Then there exists a positive infinite $b \in E$ such that $b \asymp \mathrm{L}(a)$ in F.*

This is proved just like Proposition 6 of Rosenlicht's paper [27].

Lemma 2.3. *Let $E \subseteq F$ be pre-H-subfields of a Liouville closed H-field L, such that E has a smallest comparability class, and* $\operatorname{trdeg}(F|E) \leq n$*. Then there are integers $r, s \geq 0$ with $r + s \leq n$ such that*

(1) *F has a smallest comparability class, and it contains an element $\asymp \mathrm{L}_r(a)$ for any positive infinite $a \in E$ of smallest comparability class in E;*
(2) *for each $b \in F$ there is $a \in E$ such that $b < \mathrm{E}_s(a)$.*

This is again proved like Theorem 3 of [27], using Lemmas 2.1 and 2.2. Note that a pre-H-field E has a smallest comparability class if and only if the set

$$\Psi_E = \left\{ v(a^\dagger) : a \in E^\times, a \not\asymp 1 \right\}$$

has a largest element.

In the rest of this section K is an H-field and $P(Y) = \sum_{\boldsymbol{i}} a_{\boldsymbol{i}} Y^{\boldsymbol{i}} \in K\{Y\}$ a non-zero differential polynomial of order at most n. The last lemma immediately implies:

Theorem 2.4. *Suppose the coefficients of P lie in some pre-H-subfield of K with a smallest comparability class $\mathrm{Cl}(f)$, $f \in K^{>C}$. Then $P(y) \neq 0$ for all y in all Liouville closed H-field extensions L of K with $C_L < y < \mathrm{L}_{n+1}(f)$.*

The hypothesis on the coefficients of P is automatically satisfied if K is a directed union of pre-H-subfields each of which has a smallest comparability class. For example,

$$\mathbb{R}(\!(x^{-1})\!)^{\mathrm{LE}} = \bigcup_n \mathbb{R}\left(\!\!\left(\tfrac{1}{\ell_n}\right)\!\!\right)^{\mathrm{E}}$$

is such a representation of $\mathbb{R}((x^{-1}))^{\mathrm{LE}}$ as directed union, with

$$\ell_n := \log_n x = \log\log\cdots\log x \qquad (n \text{ times})$$

representing the smallest comparability class of $\mathbb{R}\left(\left(\frac{1}{\ell_n}\right)\right)^{\mathrm{E}}$. Thus we may conclude:

Corollary 2.5. *Let $K = \mathbb{R}((x^{-1}))^{\mathrm{LE}}$. There is no element b in any differentially algebraic H-field extension L of K such that $C_L < b < a$ for all $a \in K^{>\mathbb{R}}$.* $\square$

Such a representation as directed union is not always possible. Indeed, we cannot omit the condition on P in the last theorem, see [3]. The following is much weaker than the conclusion of the last theorem, but at least it holds unconditionally:

Corollary 2.6. *Let K be Liouville closed. Then there is $a > C$ in K such that P has no zero $y \in K$ with $C < y < a$.*

Proof. Suppose not. Then there is an elementary extension L of K and a $u \in L$ with $C_L < u < K^{>C}$ and $P(u) = 0$. By Lemma 2.1, the pre-H-subfield $K\langle u\rangle = K(u, u', \dots)$ of L has a smallest comparability class, since $K\langle u\rangle$ has finite transcendence degree over K and $\mathrm{Cl}(u) < \mathrm{Cl}(f)$ for all $f \in K$, $0 \neq f \not\asymp 1$. Hence, by the last theorem, there is a positive infinite $g \in K\langle u\rangle$ such that any zero $> C_L$ of P in L is $\geq \mathrm{L}_{n+1}(g)$. But L is an elementary extension of K, so K must then have an element $a > C$ such that P has no zeros $y \in K$ with $C < y < a$. $\square$

We say that a term $a_{\boldsymbol{j}} Y^{\boldsymbol{j}}$ (with $\boldsymbol{j} \in \mathbb{N}^{n+1}$ such that $a_{\boldsymbol{j}} \neq 0$) is **the dominating term** of P at a point $y \in K$, if

$$a_{\boldsymbol{i}} y^{\boldsymbol{i}} \prec a_{\boldsymbol{j}} y^{\boldsymbol{j}} \qquad \text{for all } \boldsymbol{i} \neq \boldsymbol{j} \text{ in } \mathbb{N}^{n+1}.$$

(In that case $P(y) \sim a_{\boldsymbol{j}} y^{\boldsymbol{j}}$ and hence $\operatorname{sign} P(y) = \operatorname{sign} a_{\boldsymbol{j}} y^{\boldsymbol{j}}$.)

As in [2], we say that K is **closed under asymptotic integration** if for each $a \in K$ there is $b \in K$ with $b' \sim a$; equivalently, $(\mathrm{id} + \psi)(\Gamma^*) = \Gamma$.

Proposition 2.7. *Suppose K is closed under asymptotic integration, Ψ is not bounded from below in Γ, and $P(Y)$ is of order at most 1. Then there exist $a, b \in K$ with $C < a$ and $i, j, k, l \in \mathbb{N}$ such that for all y in all H-field extensions of K:*

1. *if $C < y < a$, then $a_{(i,j)} Y^i (Y')^j$ is the dominating term of P at y,*
2. *if $y > b$, then $a_{(k,l)} Y^k (Y')^l$ is the dominating term of P at y.*

Proof. This follows from the fact that the function

$$\Gamma^{<0} \to \Gamma \colon \gamma \mapsto r\gamma + s\psi(\gamma),$$

for given integers r, s, not both zero, is monotone and does not assume a largest or a smallest value. See [2], Section 2, for details. $\square$

Example. Let $K = \mathbb{R}((x^{-1}))^{\mathrm{LE}}$, and consider the differential polynomial $P(Y, Y') = xY' + YY' - Y \in K\{Y\}$. Let $y \in K^{>0}$. Then $P(y, y') \sim -y$ if $1 \prec y \lll x$, and $P(y, y') \sim yy'$, if $x \prec y$.

Remark. Proposition 2.7 does not generalize to differential polynomials $P(Y)$ of order > 1: consider the differential polynomial $P(Y) = Y''Y + (Y')^2$ of order 2 over $K = \mathbb{R}((x^{-1}))^{\mathrm{LE}}$; then for all sufficiently large $y \in K^{>0}$, we have $y''y \asymp (y')^2$, by Lemma 5.2, (1) below. Similarly, for $P(Y) = Y'' + tY' \in K\{Y\}$ we have $y'' \asymp ty'$ for all sufficiently small $y > \mathbb{R}$ in K, by Corollary 5.2, (2).

3. Nonexistence of Large Infinite Zeros

Let K be a pre-differential-valued field, with corresponding asymptotic couple (Γ, ψ).

Lemma 3.1. *The derivation $a \mapsto a'\colon K \to K$ is continuous with respect to the valuation topology of K.*

Proof. The result being obvious if $\Gamma = \{0\}$, we may assume $\Gamma \neq \{0\}$. Since the derivation is additive we only have to show continuity at 0. Let $\gamma \in \Gamma$. By the proof of Corollary 2 in [24] there exists $x \in K^{\times}$ such that $v(x) > 0$ and $v(x') > \gamma$. It follows that for all $y \in K$ with $v(y) > v(x)$ we have $v(y') > \gamma$. □

In fact, the derivation being additive, it is *uniformly continuous* in the following sense: for each $\gamma \in \Gamma$ there is $\delta \in \Gamma$ such that whenever $x, y \in K$ and $v(x - y) > \delta$, then $v(x' - y') > \gamma$.

An obvious consequence of this continuity property is that each differential polynomial $P(Y) \in K\{Y\}$ gives rise to a continuous function $y \mapsto P(y)\colon K \to K$. Here continuity is with respect to the valuation topology. Note that if K is a pre-H-field and the valuation of K is non-trivial, this topology coincides with the order topology of K.

Lemma 3.2. *Let $P(Y) = a_0 Y + \cdots + a_n Y^{(n)}$, all $a_i \in K$, $a_n \neq 0$. Then each level set $P^{-1}(s)$ ($s \in K$) is a discrete subset of K.*

Proof. Suppose $y \in P^{-1}(s)$ is not isolated in $P^{-1}(s)$. After a translation we can assume that $y = s = 0$. Then there exist zeros $y_0, \dots, y_n \in K^{\times}$ of P such that $1 \succ y_0 \succ y_1 \succ \cdots \succ y_n$, but then $y_0, y_1, \dots, y_n$ are linearly independent over C, a contradiction. □

Lemma 3.3. *No differential polynomial $P(Y) \in K\{Y\} \setminus \{0\}$ vanishes identically on any non-empty open subset of K.*

Proof. We proceed by induction on $d(P) := \sum_i \deg_{Y^{(i)}} P$. If $d(P) = 0$, then $P \in K^{\times}$, and the result holds trivially. Let $d(P) > 0$, and let U be a nonempty open subset of K. For $a \in U$ and $y \in K$ we have by Taylor expansion

$$P(a+y) = P(a) + \sum_i \frac{\partial P}{\partial Y^{(i)}}(a) \cdot y^{(i)} + \text{ terms of higher degree in } (y, y', \dots).$$

Let $a_i := \frac{\partial P}{\partial Y^{(i)}}(a)$. Since $d\left(\frac{\partial P}{\partial Y^{(i)}}\right) < d(P)$ for all i and $\frac{\partial P}{\partial Y^{(i)}} \neq 0$ for some i, we may assume inductively that $a \in U$ has been chosen such that $a_i \neq 0$ for some i. By the previous lemma we can then choose arbitrarily small $y \neq 0$ in K such that $\sum_i a_i y^{(i)} \neq 0$. Then, with $c \in C$, we have

$$P(a+cy) = P(a) + c\sum_i a_i y^{(i)} + \text{ terms of higher degree in } c,$$

which can vanish for only finitely many values of c. $\square$

The case of H-fields. In the rest of this section we assume that K is an H-field with a distinguished element $x > C$ such that $x' = 1$. We want to show:

Theorem 3.4. *Let $P(Y) \in K\{Y\} \setminus \{0\}$ have order at most n. There exists an element f of the subfield of K generated by x and the coefficients of P such that either $P(y) > 0$ for all $y > \mathrm{E}_n(f)$ in all Liouville closed H-field extensions of K, or $P(y) < 0$ for all $y > \mathrm{E}_n(f)$ in all Liouville closed H-field extensions of K.*

Corollary 3.5. *Suppose K is Liouville closed, $P(Y) \in K\{Y\} \setminus \{0\}$, and $a \in K$. Then there exists $\varepsilon \in K^{>0}$ such that either $P(y) > 0$ for all y in each H-field extension of K with $a < y < a + \varepsilon$, or $P(y) < 0$ for all y in each H-field extension of K with $a < y < a+\varepsilon$. (In particular, the zero set of P in K is discrete.)*

Proof. We may assume $a = 0$. Then apply the last theorem to the differential polynomial $Q(Y) := Y^{2d}P(1/Y)$ with d the (total) degree of P. $\square$

Remark. In Corollary 3.5 we cannot omit the condition that K is Liouville closed: the conclusion fails for the Hardy field $K = \mathbb{R}(x)$, $a = 0$ and the differential polynomial

$$P(Y) := YY''x - (Y')^2x + YY',$$

whose zero set is $\{cx^k : c \in \mathbb{R}, k \in \mathbb{Z}\}$.

In the proof of Theorem 3.4, we shall use the following lemma.

Lemma 3.6. *Let Y be a differential indeterminate (over the trivial differential field $\mathbb{Q}$), and put $Z = Y'/Y \in \mathbb{Q}\langle Y\rangle$. Then for each $n \geq 1$ we have*

$$Y^{(n)}/Y = p_n(Z)$$

for some differential polynomial p_n of order $n-1$ with integral coefficients.

Proof. By induction on n. For $n = 1$ we put $p_n(Z) = Z$. Suppose $Y^{(n)}/Y = p_n(Z)$ where p_n has integral coefficients and order $n-1$. Then

$$\big(Y^{(n)}/Y\big)' = \big(Y^{(n+1)}/Y\big) - (Y'/Y)\cdot\big(Y^{(n)}/Y\big) = p_n(Z)',$$

so

$$Y^{(n+1)}/Y = p_n(Z)' + Z\cdot p_n(Z) = p_{n+1}(Z),$$

where p_{n+1} is of order n and has integral coefficients. □

Proof of Theorem 3.4. It is convenient to establish by induction on n a slightly stronger result:

(H_n) *Let $P(Y) \in K\{Y\}\setminus K$ be of order at most n, and $0 < g \in K$. Then there exists an element f of the subfield of K generated by g, x, and the coefficients of P such that either $P(y) \geq g$ for all $y \geq \mathrm{E}_n(f)$ in each Liouville closed H-field extension of K, or $P(y) \leq -g$ for all $y \geq g$ in each Liouville closed H-field extension of K.*

For $n = 0$ we have

$$P(Y) = a_dY^d + a_{d-1}Y^{d-1} + \cdots + a_0 \qquad (a_i \in K,\ a_d \neq 0,\ d > 0).$$

Then $f := 1 + |a_{d-1}/a_d| + \cdots + |a_0/a_d| + |g/a_d| \in \mathbb{Q}(a_0,\dots,a_d,g)$ has the desired property.

Suppose $n > 0$ and (H_{n-1}) holds. Let $P = \sum_{\boldsymbol{i}} a_{\boldsymbol{i}}Y^{\boldsymbol{i}} \in K\{Y\}\setminus K$ be of order $\leq n$ and of total degree $d > 0$, and $g \in K^{>0}$. Let Q be the homogeneous part of degree d of P, that is, $Q = \sum_{|\boldsymbol{i}|=d} a_{\boldsymbol{i}}Y^{\boldsymbol{i}}$, and write $P = Q + R$, so $R = \sum_{|\boldsymbol{i}|<d} a_{\boldsymbol{i}}Y^{\boldsymbol{i}}$. Consider a multiindex $\boldsymbol{i} \in \mathbb{N}^{n+1}$ of degree $< d$ and an element y in a Liouville closed H-field extension of K, with $|y| \geq \mathrm{E}(x^2)$. As $1 \prec y^{(i)}/y \lll y$ for each $i \geq 0$ (Lemma 1.2), we have $y^{\boldsymbol{i}}/y^{|\boldsymbol{i}|} \lll y$, hence in particular

$$y^{\boldsymbol{i}}/y^d \preceq y^{\boldsymbol{i}}/y^{1+|\boldsymbol{i}|} \prec 1,$$

since $|\boldsymbol{i}| < d$. Thus $|a_{\boldsymbol{i}}y^{\boldsymbol{i}}/y^d| \leq |a_{\boldsymbol{i}}|$ for all terms $a_{\boldsymbol{i}}Y^{\boldsymbol{i}}$ of R, and hence

$$\big|R(y)/y^d\big| \leq \sum_{|\boldsymbol{i}|<d} |a_{\boldsymbol{i}}y^{\boldsymbol{i}}/y^d| \leq \sum_{|\boldsymbol{i}|<d} |a_{\boldsymbol{i}}| =: h. \tag{3.1}$$

Note that h is an element of the subfield of K generated by the coefficients $a_{\boldsymbol{i}}$ of P.

We first consider the case that some $Y^{(i)}$ with $i > 0$ actually occurs in Q. Let $Z = Y'/Y \in K\langle Y\rangle$. Then by Lemma 3.6,

$$Q/Y^d = Q\big(1, Y'/Y, \dots, Y^{(n)}/Y\big) = q(Z)$$

for a differential polynomial $q \in F\{Z\}$ of order $\leq n-1$, where F is the subfield of K generated by the coefficients of Q, and Z is treated as a differential indeterminate; in fact, the order of q is one less than the maximal i such that $Y^{(i)}$ occurs in Q, in particular $q \notin K$. By the inductive assumption, there exists $f \in F(g,h,x)$ such that either

$$q(z) \geq h + g \tag{3.2}$$

for all $z \geq \mathrm{E}_{n-1}(f)$ in each Liouville closed H-field extension of K, or

$$q(z) \leq -h - g$$

for all $z \geq \mathrm{E}_{n-1}(f)$ in each Liouville closed H-field extension of K. We may of course assume that $f \geq x^2$. Suppose the first alternative holds, and let y be an element of a Liouville closed H-field extending K, and

$$\begin{cases} y \geq \mathrm{E}(xf) & \text{if } n = 1, \\ y \geq \mathrm{E}_n(f) & \text{if } n > 1. \end{cases}$$

Then $z = y^\dagger \geq \mathrm{E}_{n-1}(f)$ by Lemma 1.3, hence

$$P(y)/y^d = R(y)/y^d + q(z) \geq g$$

by (3.1) and (3.2), thus

$$P(y) \geq P(y)/y^d \geq g.$$

If the second alternative holds, one concludes similarly that

$$P(y) \leq -g.$$

This finishes the inductive step in case some $Y^{(i)}$ with $i > 0$ actually occurs in Q.

Suppose no $Y^{(i)}$ with $i > 0$ occurs in Q. Then $Q = a_d Y^d$, where $a_d \in K^\times$. Let y be an element of a Liouville closed H-field extension of K such that $y \geq \max\bigl\{\mathrm{E}(x^2), \bigl((g+h)/a_d\bigr)^2\bigr\}$. As before, using Lemma 1.2 one shows that $|a_{\boldsymbol{i}} y^{\boldsymbol{i}}/y^{d-1/2}| \leq |a_{\boldsymbol{i}}|$ for all terms $a_{\boldsymbol{i}} Y^{\boldsymbol{i}}$ of R, hence

$$\bigl|R(y)/y^{d-1/2}\bigr| \leq h.$$

Now $a_d y^{1/2} \geq g+h$ if $a_d > 0$, and $a_d y^{1/2} \leq -g-h$ if $a_d < 0$. Thus if $a_d > 0$, then

$$P(y)/y^{d-1/2} = R(y)/y^{d-1/2} + a_d y^{1/2} \geq g,$$

hence $P(y) \geq g$, and if $a_d < 0$, then

$$P(y)/y^{d-1/2} = R(y)/y^{d-1/2} + a_d y^{1/2} \leq -g,$$

implying that $P(y) \leq -g$. □

Corollary 3.7. *If y in a Liouville closed H-field extension of K satisfies $P(y) = 0$, where $P(Y) \in K\{Y\} \setminus \{0\}$ has order at most n, then $|y| < \mathrm{E}_n(f)$ for some $f \in K$.* □

Remarks.

(1) For $n = 0$ this corollary is well-known. (See [4], Lemma 1.2.11.) For $n = 1$ and $K = \mathbb{R}(x)$ (with x the germ at $+\infty$ of the identity function) the corollary is due to Borel ([5], p. 30). The proof of Theorem 3.4 generalizes the main idea of Borel's argument. In [6], [26] and [29] similar but weaker results are proved for Hardy fields.

(2) For the Hardy field $\mathbb{R}(x)$ Corollary 3.7 is best possible in the following sense: $\mathrm{E}_n(x)$, for $n > 0$, is a zero of a differential polynomial of order n over $\mathbb{R}(x)$, and $\mathrm{E}_n(x) > \mathrm{E}_{n-1}(f)$ for each $f \in \mathbb{R}(x)$, see [7].

(3) For equations of order 1 over Hardy fields more precise results are available: see [15] for the case $K = \mathbb{R}(x)$ and [26] for the case of an arbitrary Hardy field containing x.

(4) The second theorem of the Introduction follows from Theorem 3.4. This is because any Liouville closed H-field turns into one that contains a positive $x \succ 1$ with $x' = 1$ after replacing its derivation ∂ by $a\partial$ for a suitable $a > 0$.

4. Intermediate Value Property for First-Order Differential Polynomials

We begin this section by showing that *linear* differential polynomials of order 1 have the intermediate value property in Liouville closed H-fields. Next we prove an intermediate value property for arbitrary differential polynomials of order 1.

Proposition 4.1. *Let K be a Liouville closed H-field. Then each of the functions*

$$y \mapsto y' \colon K \to K, \quad y \mapsto y^\dagger \colon K^{<0} \to K, \quad y \mapsto y^\dagger \colon K^{>0} \to K$$

is surjective and has the intermediate value property.

Proof. Let $a, b \in K$, $a < b$, and let $s \in K$ lie strictly between a' and b'. We have to find $y \in (a, b)$ with $y' = s$. Now since K is Liouville closed, there exists $z \in K$ such that $z' = s$. Passing from (a, b) to $(a - z, b - z)$, we may assume that $s = 0$. So we have $a'b' < 0$, and we have to find $y \in C \cap (a, b)$. If $a < 0 < b$, we may take $y := 0$. Suppose $0 < a < b$. (The case $a < b < 0$ is similar.) If $b > \mathcal{O}$, then $b' > 0$, and necessarily $a \in \mathcal{O}$, so certainly $C \cap (a, b) \neq \emptyset$. If $b \in \mathcal{O}$ and $b' < 0 < a'$, then $a < y < b$ for $y \in C$ with $y \sim a$. If $b \in \mathcal{O}$ and $a' < 0 < b'$, and $c, d \in C$ are such that $c \sim a$, $d \sim b$, then $c < d$, and for any $y \in C \cap (c, d)$ we have $a < y < b$ as required. Thus $y \mapsto y' \colon K \to K$ has the intermediate value property.

We now prove the intermediate value property for $y \mapsto y^\dagger \colon K^{>0} \to K$. (Since $y^\dagger = (-y)^\dagger$ for $y \in K^\times$, this will also imply the intermediate value property for $y \mapsto y^\dagger \colon K^{<0} \to K$.) Let $0 < a < b$ be in K, and $s \in K$ strictly between $a^\dagger$ and $b^\dagger$. We have to find $y \in (a, b)$ with $y^\dagger = s$. Since K is Liouville closed we can choose $z \in K^{>0}$ with $z^\dagger = s$, by Liouville closedness of K. Passing from (a, b) to $(a/z, b/z)$, we may assume $s = 0$. So $a'b' < 0$, and we have to show $C \cap (a, b) \neq \emptyset$. But this follows from the intermediate value property of $y \mapsto y'$ on K. □

Corollary 4.2. *Let K be a Liouville closed H-field, and $\alpha, \beta \in K$. The differential polynomial function*

$$y \mapsto \alpha y + \beta y' : K \to K$$

has the intermediate value property.

Proof. This is clear from the previous result if $\alpha = 0$ or $\beta = 0$. Let $\alpha, \beta \neq 0$; we can assume $\beta = 1$. Let $a < b$ in K and suppose $s \in K$ lies strictly between $\alpha a + a'$ and $\alpha b + b'$. We have to find y with $a < y < b$ such that $\alpha y + y' = s$. Since K is Liouville closed, there exists $z \in K$ with $\alpha z + z' = s$, so passing from (a, b) to $(a - z, b - z)$, we reduce to the case $s = 0$. If $a < 0 < b$, we can take $y = 0$. Otherwise, either $a < b < 0$ or $0 < a < b$, and by the intermediate value property of $y^\dagger$ on $K^{<0}$ and $K^{>0}$, respectively, it follows that there exists $y \in (a, b)$ with $y^\dagger = -\alpha$, that is, $\alpha y + y' = 0$ as required. □

The main result in this section is an intermediate value property where we allow extensions of H-fields:

Theorem 4.3. *Let K be an H-field, and $F(Y, Z) \in K[Y, Z]$. Let $\phi < \theta$ in K such that $F(\phi, \phi')$ and $F(\theta, \theta')$ are non-zero and of opposite sign in K. Then there is an H-field extension L of K with an element η such that $\phi < \eta < \theta$ and $F(\eta, \eta') = 0$.*

See [10] for the analogue of this result for the category of Hardy fields. In the proof of Theorem 4.3, we need the chain rule from the next subsection.

Composing derivations and semialgebraic functions. Let K be a real closed field equipped with a derivation $a \mapsto a'$. In the following, the term "semialgebraic" is to be taken in the sense of K. (See [4] for basic facts about semialgebraic sets and functions.)

Lemma 4.4. *Let $U \subseteq K^n$ be an open semialgebraic set, $g \colon U \to K$ a semialgebraic function of class C^1, and suppose we have a polynomial $P \in K[X]$, with $X = (X_1, \dots, X_{n+1})$, such that $P\big(u, g(u)\big) = 0$ and $\frac{\partial P}{\partial X_{n+1}}\big(u, g(u)\big) \neq 0$ for all $u \in U$. Then there is a continuous semialgebraic function $\widetilde{g} \colon U \to K$ such that*

$$g(u)' = \widetilde{g}(u) + \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(u) \cdot u_i' \qquad \textit{for all } u = (u_1, \dots, u_n) \in U. \tag{4.1}$$

Proof. Define $\widetilde{g} \colon U \to K$ with $\widetilde{g}(u) = g(u)' - \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(u) \cdot u_i'$. We have to show that $\widetilde{g}$ is continuous and semialgebraic. Put $P_i := \frac{\partial P}{\partial X_i} \in K[X]$ for $i = 1, \dots, n+1$, and write $P = \sum_\alpha c_\alpha X^\alpha$ with coefficients $c_\alpha \in K$. Then we have for $u \in U$

$$P\big(u, g(u)\big)' = \widetilde{P}\big(u, g(u)\big) + \left(\sum_{i=1}^{n} P_i\big(u, g(u)\big) \cdot u_i' \right) + P_{n+1}\big(u, g(u)\big) \cdot g(u)' = 0,$$

where $\widetilde{P} := \sum_\alpha c_\alpha' X^\alpha \in K[X]$. Differentiating the identity $P\big(u, g(u)\big) = 0$ with respect to x_i for $i = 1, \dots, n$ gives

$$P_i\big(u, g(u)\big) + P_{n+1}\big(u, g(u)\big) \cdot \frac{\partial g}{\partial x_i}(u) = 0$$

on U. Substituting this into the preceding identity for $P\big(u,g(u)\big)'$ gives

$$\widetilde{P}\big(u,g(u)\big)+\left(\sum_{i=1}^{n}-P_{n+1}\big(u,g(u)\big)\cdot\frac{\partial g}{\partial x_i}(u)\cdot u_i'\right)+P_{n+1}\big(u,g(u)\big)\cdot g(u)'=0$$

on U, which implies

$$\widetilde{g}(u)=-\widetilde{P}\big(u,g(u)\big)/P_{n+1}\big(u,g(u)\big)$$

on U, so $\widetilde{g}$ is indeed continuous semialgebraic on U. □

Remarks.

(1) The proof shows that if ${}^*K$ is a real closed differential extension field of K, then (4.1) remains valid when we replace U, g and $\widetilde{g}$ by their extensions ${}^*U$, ${}^*g$ and ${}^*\widetilde{g}$ that are defined by the same formulas in the language of ordered rings over K as U, g and $\widetilde{g}$, respectively.

(2) Suppose the derivation on K is continuous with respect to the order topology. (For example, this is the case if K is a real closed H-field, see §3.) Then the hypothesis in the lemma on the existence of the polynomial P vanishing nonsingularly on the graph of g may be dropped. To see this, note that then the function $\widetilde{g}\colon U\to K$ defined by $\widetilde{g}(u)=g(u)'-\sum_{i=1}^{n}\frac{\partial g}{\partial x_i}(u)\cdot u_i'$ is continuous, because of the continuity of the derivation of K. Hence it suffices to find finitely many semialgebraic open subsets of U whose union is dense in U, and on each of which $\widetilde{g}$ is semialgebraic. This allows us to reduce to the case treated in the lemma.

We shall also need the following general extension result.

Lemma 4.5. *Let K be a pre-H-field such that Ψ has no largest element. Let L be an ordered differential field extension of K such that if $f\in L$ and $f>r$ for some $r\in K$ with $r>\mathcal{O}$, then $f'>0$. Then L is a pre-H-field with respect to the valuation with valuation ring*

$$\mathcal{O}_L:=\big\{u\in L:|u|<r \textit{ for each } r\in K \textit{ with } r>\mathcal{O}\big\}.$$

Proof. Let $u\in\mathcal{O}_L$ and $0\neq b\in\mathfrak{m}$. We claim that then $v(u')>v(b^\dagger)$. To see this, let any $r\in K$ with $r>\mathcal{O}$ be given. Then $r-u>r/2>\mathcal{O}$, and similarly $r+u>r/2>\mathcal{O}$, so $r'-u'>0$ and $r'+u'>0$. Hence $|u'|\leq|r'|$, so $v(u')\geq v(r')$. Since the set $\Psi=\big\{v(b^\dagger):0\neq b\in\mathfrak{m}\big\}$ has no largest element, every element of Ψ is bounded from above by an element of $(\mathrm{id}+\psi)\big(\Gamma^{<0}\big)=\big\{v(r'):r\in K, r>\mathcal{O}\big\}$. Hence $v(u')>v(b^\dagger)$ for all $b\in\mathfrak{m}\setminus\{0\}$, as we claimed.

Next, let $f\in L^\times$ with $v(f)\neq 0$. We claim that then $v(f^\dagger)\leq v(r^\dagger)$ for some $r\in K$ with $r>\mathcal{O}$. (Note that the lemma follows from this claim in combination with the previous claim.) We may assume that $f>\mathcal{O}_L$. Then $f>r>\mathcal{O}$ for some $r\in K$. Since Γ has no smallest positive element, we can decrease r if necessary so that $v(f)<v(r)<0$. Then $f/r>\mathcal{O}_L$, hence $(f/r)' > 0$, so $f^\dagger = f'/f > r'/r = r^\dagger > 0$, and thus $v(f^\dagger) \le v(r^\dagger)$, as promised. $\square$

Proof of Theorem 4.3. After passing to the real closure of K we may assume that K is already real closed. We then carry out a reduction to the proposition below exactly as in the setting of Hardy fields, see [10].

Proposition 4.6. *Let K be a real closed H-field, I an interval in K and $f\colon I \to K$ a continuous semialgebraic function. Let $a, b \in I$ with $a < b$ such that $a' - f(a)$ and $b' - f(b)$ are non-zero and of opposite sign. Then there is a real closed H-field extension of K containing an element c with $a < c < b$ such that $c' = f(c)$.*

In the proposition's conclusion, and its proof below, f is extended in the usual way to any real closed field extending K; this extension of f is also denoted by f.

Proof (of Proposition 4.6). By suitably extending K we may assume that Ψ has no largest element. We now first consider the case that $a' < f(a)$ and $b' > f(b)$. Let

$$A := \big\{y \in (a,b) : y' < f(y)\big\}, \qquad B := \big\{y \in (a,b) : y > A\big\}.$$

Then A and B are non-empty. If A has a supremum c in K, then $c' = f(c)$ by continuity, and we are done, by taking $L := K$ as the desired H-field. So we may assume in the following that A has no supremum in K, and thus that B has no infimum in K. Let $K(c)$ be an ordered field extension of K with $A < c < B$, and let L be the real closure of $K(c)$. Equip L with the unique derivation that extends the one on K and satisfies $c' = f(c)$.

Claim. Let $s \in L$ and $s > r$ for some $r \in K$ with $r > \mathcal{O}$. Then $s' > 0$.

Once this claim is established, it follows from Lemma 4.5 that L with the given ordering and derivation, and the valuation with valuation ring

$$\mathcal{O}_L := \big\{u \in L : |u| < r \text{ for each } r \in K \text{ with } r > \mathcal{O}\big\}$$

is a pre-H-field (which hence can be embedded into a real closed H-field as desired).

Proceeding to the proof of the claim, write $s = g(c)$ with $g\colon J \to K$ a semialgebraic function, $J \subseteq K$ an open interval containing c. After decreasing J suitably we may assume that g is of class C^1 and that some polynomial in $K[X_1, X_2]$ vanishes nonsingularly on the graph of g, as in the hypothesis of Lemma 4.4. Hence there is a continous semialgebraic function $\widetilde{g}\colon J \to K$ such that

$$g(y)' = \widetilde{g}(y) + g'(y)y' \qquad \text{for all } y \in J.$$

By the first remark following that lemma,

$$s' = g(c)' = \widetilde{g}(c) + g'(c)c' = \widetilde{g}(c) + g'(c)f(c).$$

From $s = g(c) > r > \mathcal{O}$ we obtain $g(y) > r$ for all $y \in J$, after decreasing J once more if necessary. Hence $g(y)' = \widetilde{g}(y) + g'(y)y' > 0$ for $y \in J$.

Suppose for a contradiction that $s' \leq 0$, so $\widetilde{g}(c) + g'(c)f(c) \leq 0$, and hence $\widetilde{g}(y) + g'(y)f(y) \leq 0$ for $y \in J$, after perhaps decreasing J again. Hence $g'(y)\big(y' - f(y)\big) > 0$ for $y \in J$. We now choose a subinterval J_0 of J containing both elements in A and in B (so c belongs to the natural extension of J_0 in L), such that

(1) if $g'(c) > 0$, then $g'(y) > 0$ for all $y \in J_0$,
(2) if $g'(c) < 0$, then $g'(y) < 0$ for all $y \in J_0$.

(The case $g'(c) = 0$ cannot occur since $g'(y) \neq 0$ for $y \in J$.) In case (1) we obtain $y' - f(y) > 0$ for all $y \in J_0$, contradicting $A \cap J_0 \neq \emptyset$. In case (2) we obtain $y' - f(y) < 0$ for all $y \in J_0$, contradicting $B \cap J_0 \neq \emptyset$.

The case that $a' > f(a)$ and $b' < f(b)$ is treated in the same way, after setting $A := \big\{y \in (a,b) : y' > f(y)\big\}$ and $B := \big\{y \in (a,b) : y > A\big\}$. $\square$

5. The Valuation of Higher Derivatives

Let (Γ, ψ) be an asymptotic couple. We define for each n a map

$$\psi^{(n)}\colon \Gamma_\infty \to \Gamma_\infty : \qquad \psi^{(0)} := \mathrm{id}_{\Gamma_\infty}, \qquad \psi^{(n+1)} := \psi^{(n)} + \psi \circ \psi^{(n)}.$$

So $\psi^{(n)}$ is the n-fold iterate of the map $\mathrm{id} + \psi\colon \Gamma_\infty \to \Gamma_\infty$. An easy induction on n shows that if (Γ, ψ) is the asymptotic couple of a pre-differential-valued field K and $\psi^{(n)}\big(v(a)\big) \neq \infty$, $a \in K$, then $\psi^{(n)}\big(v(a)\big) = v\big(a^{(n)}\big)$, where $a^{(n)}$ is the n-th derivative of a. This is why we write $\psi^{(n)}$—not to be confused with the n-th iterate ψ^n of ψ—and why we derive identities for $\psi^{(n)}(\alpha)$.

Lemma 5.1. *For all $\alpha \in \Gamma$,*

(1) $\psi(\alpha) < \psi^2(\alpha) \ \Rightarrow\ \psi^{(n)}(\alpha) = \alpha + n\psi(\alpha)$, *for all n,*
(2) $\psi(\alpha) > \psi^2(\alpha) \ \Rightarrow\ \psi^{(n)}(\alpha) = \alpha + \psi(\alpha) + (n-1)\psi^2(\alpha)$, *for all $n > 0$,*

Proof. We first show (1). For $n = 0$ and $n = 1$ this holds by definition. Assume (1) holds for a certain $n > 0$. Let $\alpha \in \Gamma^*$ with $\psi(\alpha) < \psi^2(\alpha)$. Then

$$\psi^{(n+1)}(\alpha) = \psi^{(n)}(\alpha) + \psi\big(\psi^{(n)}(\alpha)\big) = \alpha + n\psi(\alpha) + \psi\big(\alpha + n\psi(\alpha)\big).$$

From $\psi(\alpha) < \psi^2(\alpha)$ we get $\psi\big(\alpha + n\psi(\alpha)\big) = \psi(\alpha)$, so

$$\psi^{(n+1)}(\alpha) = \alpha + (n+1)\psi(\alpha).$$

Next we prove by induction on $n > 0$ that (2) holds. The case $n = 1$ is trivial. Assume (2) holds for a certain $n > 0$. Let $\alpha \in \Gamma^*$ with $\psi(\alpha) > \psi^2(\alpha)$. Then

$$\psi(\psi^2(\alpha) - \psi(\alpha)) > \min\big\{\psi^2(\alpha), \psi(\alpha)\big\} = \psi^2(\alpha),$$

hence

$$\psi\big(n\psi^2(\alpha)\big) = \psi\big(\psi^2(\alpha)\big) = \psi\big(\psi^2(\alpha) - \psi(\alpha) + \psi(\alpha)\big) = \psi^2(\alpha).$$

This equality and the preceding inequality imply

$$\psi\big(\psi(\alpha) + (n-1)\psi^2(\alpha)\big) = \psi\big(\psi(\alpha) - \psi^2(\alpha) + n\psi^2(\alpha)\big) = \psi^2(\alpha).$$

Therefore, using the inductive assumption,

$$\begin{aligned}\psi^{(n+1)}(\alpha) &= \psi^{(n)}(\alpha) + \psi\big(\psi^{(n)}(\alpha)\big)\\ &= \alpha + \psi(\alpha) + (n-1)\psi^2(\alpha) + \psi\big(\alpha + \psi(\alpha) + (n-1)\psi^2(\alpha)\big)\\ &= \alpha + \psi(\alpha) + n\psi^2(\alpha).\end{aligned}$$

□

Remark. Recall from [2] that an asymptotic couple (Γ, ψ) is said to be of **H-type** if $0 < \alpha \le \beta \Rightarrow \psi(\alpha) \ge \psi(\beta)$ for all $\alpha, \beta \in \Gamma$. If (Γ, ψ) is of H-type, then

$$\psi(\alpha + \beta) = \min\big\{\psi(\alpha), \psi(\beta)\big\} \qquad \text{for } \alpha, \beta \in \Gamma^{>0}. \tag{5.1}$$

Lemma 5.2. *Let $1 \in \Gamma^{>0}$ be such that $\psi(1) = 1$, and identify $\mathbb{Z}$ with the subgroup $\mathbb{Z} \cdot 1$ of Γ via $k \mapsto k \cdot 1$. Then we have for $\alpha \in \Gamma^*$:*

1. *$\psi(\alpha) < 1 \;\Rightarrow\; \psi(\alpha) < \psi^2(\alpha)$ and $\psi^{(n)}(\alpha) = \alpha + n\psi(\alpha)$ for all n,*
2. *$\psi(\alpha) > 1 \;\Rightarrow\; \psi(\alpha) > \psi^2(\alpha)$ and $\psi^{(n)}(\alpha) = \alpha + \psi(\alpha) + (n-1)$ for all $n > 0$,*
3. *$\psi(\alpha) > 0 \;\Rightarrow\; \psi^{(n)}(\alpha) > \alpha$ for all $n > 0$,*
4. *if (Γ, ψ) is of H-type, $\alpha > 0$ and $\psi(\alpha) = 1$, then $\psi^{(n)}(\alpha) = \alpha + n$ for all n.*

Proof. If $\psi(\alpha) < 1$, then $-1 < -\psi(\alpha)$, hence $0 = -1 + \psi(-1) < -\psi(\alpha) + \psi\big(-\psi(\alpha)\big) = -\psi(\alpha) + \psi^2(\alpha)$, that is, $\psi(\alpha) < \psi^2(\alpha)$. Similarly, one shows that $\psi(\alpha) > 1$ implies $\psi(\alpha) > \psi^2(\alpha)$. Moreover, if $\psi(\alpha) \ge 1$, then

$$\psi\big(\psi(\alpha) - 1\big) = \psi\big(\psi(\alpha) - \psi(1)\big) > \min\big\{\psi(\alpha), \psi(1)\big\} = 1,$$

hence $\psi^2(\alpha) = \psi(\psi(\alpha - 1) + 1) = 1$. So parts (1) and (2) follow from Lemma 5.1. Easy inductions on n prove (3) and (4). (Use (5.1) for (4).) □

Remark. Sometimes we cannot assume that there exists $1 \in \Gamma^{>0}$ with $\psi(1) = 1$ but only that $\psi^{(1)}\big(\Gamma^{>0}\big) \subseteq \Gamma^{>0}$. With this weaker assumption we can still conclude:

$$\big(\alpha \in \Gamma, \psi(\alpha) \le 0\big) \quad \Rightarrow \quad \big(\psi^{(n)}(\alpha) = \alpha + n\psi(\alpha) \text{ for all } n\big).$$

To see this, show that $\psi(\alpha) \le 0$ implies $\psi(\alpha) < \psi^2(\alpha)$, and apply Lemma 5.1.

Lemma 5.3. *Suppose (Γ, ψ) is of H-type and $\psi(1) = 1$, with $1 \in \Gamma^{>0}$. Then for all $\alpha \in \Gamma^*$ and $n > 0$, we have*

$$\begin{aligned}\psi^{(n)}(\alpha) < \alpha \quad &\Longleftrightarrow \quad \psi(\alpha) < 0,\\ \psi^{(n)}(\alpha) = \alpha \quad &\Longleftrightarrow \quad \psi(\alpha) = 0.\end{aligned}$$

Proof. Let $\alpha \in \Gamma^*$, $n > 0$. Suppose $\psi(\alpha) \le 0$. Then $\psi^{(n)}(\alpha) = \alpha + n\psi(\alpha) \le \alpha$, with equality exactly if $\psi(\alpha) = 0$. Conversely, $\psi^{(n)}(\alpha) \le \alpha$ implies $\psi(\alpha) \le 0$ by part (3) of the previous lemma. □

Lemma 5.4. *Let K be a pre-differential-valued field whose asymptotic couple (Γ,ψ) is of H-type, and let $\psi(1)=1$, with $1\in\Gamma^{>0}$. Then*

$$v(y^{\boldsymbol{i}}) = |\boldsymbol{i}|v(y) + (w\boldsymbol{i})\psi\big(v(y)\big)$$

for $\boldsymbol{i}\in\mathbb{N}^{n+1}$ and $y\in K^\times$ with $v(y)\geq 1$.

Proof. For such y we have $\psi(y)\leq\psi(1)=1$. Hence by parts (1) and (4) of Lemma 5.2 we have $\psi^{(k)}\big(v(y)\big)=v(y)+k\psi\big(v(y)\big)=v\big(y^{(k)}\big)$, for all $k\in\mathbb{N}$. Now use the definition of $y^{\boldsymbol{i}}$ as the product of factors $\big(y^{(k)}\big)^{i_k}$. □

6. Simple Zeros of Differential Polynomials

Let K be a pre-differential-valued field with corresponding asymptotic couple (Γ,ψ). Let $P\in K\{Y\}$ be of order n. Taylor expansion around $a\in K$ gives

$$\begin{aligned} P(a+Y) &= \sum_{\boldsymbol{i}} \frac{1}{\boldsymbol{i}!}\frac{\partial^{|\boldsymbol{i}|}P}{\partial Y^{\boldsymbol{i}}}(a)Y^{\boldsymbol{i}} \\ &= P(a) + \sum_{i=0}^{n}\frac{\partial P}{\partial Y^{(i)}}(a)Y^{(i)} + \text{ terms of degree at least 2.}\end{aligned}$$

Definition 6.1. We say that $a\in K$ is a **simple zero** of $P(Y)$ if $P(a)=0$ and $\frac{\partial P}{\partial Y^{(i)}}(a)\neq 0$ for some i.

If Ψ is bounded from below in Γ, then by Kolchin [18] each simple zero of P in K is isolated in the set of all zeros of P in K. *For the rest of this section, we suppose that (Γ,ψ) is of H-type, with an element $1\in\Gamma^{>0}$ such that $\psi(1)=1$, and that Ψ is* not *bounded from below in* Γ. We will show:

Proposition 6.2. *Let $a\in K$ be a simple zero of P, and let m be maximal such that $\frac{\partial P}{\partial Y^{(m)}}(a)\neq 0$. There exists $0\neq\varepsilon\prec 1$ such that for all $y\prec\varepsilon$ in all pre-differential-valued field extensions of K of H-type, and for all $\boldsymbol{i}\in\mathbb{N}^{n+1}$ with $|\boldsymbol{i}|>1$ or $w\boldsymbol{i}<m$:*

$$\frac{\partial P}{\partial Y^{(m)}}(a)y^{(m)} \succ \frac{\partial^{|\boldsymbol{i}|}P}{\partial Y^{\boldsymbol{i}}}(a)y^{\boldsymbol{i}}.$$

(So the term $\frac{\partial P}{\partial Y^{(m)}}(a)y^{(m)}$ of the Taylor expansion of P around a *dominates* the other terms. In particular, a is isolated in the set of zeros of P in K.)

For the proof of the proposition, we first note that after translating by $-a$ we may assume $a=0$. So we can write $P=\sum_{\boldsymbol{i}} a_{\boldsymbol{i}}Y^{\boldsymbol{i}}$, where the sum ranges over all $\boldsymbol{i}\in\mathbb{N}^{n+1}$ with $|\boldsymbol{i}|>0$, $a_{\boldsymbol{i}}\in K$, all but finitely many zero, and $a_{\boldsymbol{i}}\neq 0$ for some $\boldsymbol{i}$ with $|\boldsymbol{i}|=1$. The proposition now follows from:

Lemma 6.3. *Let $\boldsymbol{i},\boldsymbol{j}\in\mathbb{N}^{n+1}$ and $a,b\in K^\times$. The following are equivalent:*

1. *There exists $\varepsilon\prec 1$ in $K^\times$ such that $ay^{\boldsymbol{i}}\succ by^{\boldsymbol{j}}$ for all non-zero $y\prec\varepsilon$ in all pre-differential-valued field extensions of K of H-type.*
2. *One of the following holds:*

(a) $|\boldsymbol{i}| < |\boldsymbol{j}|$, *or*
(b) $|\boldsymbol{i}| = |\boldsymbol{j}|$ *and* $w\boldsymbol{i} > w\boldsymbol{j}$, *or*
(c) $|\boldsymbol{i}| = |\boldsymbol{j}|$, $w\boldsymbol{i} = w\boldsymbol{j}$, *and* $a \succ b$.

This lemma is an immediate consequence of the last lemma in the previous section.

The case of H-fields. Suppose now in addition that K is an H-field. In the next proposition, if I is an interval in K, we also write I for the natural extension of I to an interval in an ordered field extension L of K.

Proposition 6.4. *Suppose that* $a_n := \frac{\partial P}{\partial Y^{(n)}}(a) \neq 0$. *Then there exists an interval I around a in K such that in every H-field extension of K, the map $y \mapsto P(y)$ is strictly increasing on I if $a_n > 0$ and n is even, or $a_n < 0$ and n is odd, and strictly decreasing on I otherwise.*

Proof. By passing from $P(Y)$ to $P(Y+a)$ if necessary, we may assume $a = 0$. Below, let $\boldsymbol{i}$ range over the (finitely many) multiindices in $\mathbb{N}^{n+1}$ with $\partial^{|\boldsymbol{i}|}P/\partial Y^{\boldsymbol{i}} \neq 0$. Let β be an element of K such that

$$\beta \left| \frac{\partial^{|\boldsymbol{i}|}}{\partial Y^{\boldsymbol{i}}}(0) \right| < \left| \frac{\partial P}{\partial Y^{(n)}}(0) \right| \qquad \text{for all } \boldsymbol{i}.$$

By continuity there exists $\varepsilon > 0$ in K such that for all y in all H-field extensions of K with $-2\varepsilon < y < 2\varepsilon$:

$$\beta \left| \frac{\partial^{|\boldsymbol{i}|}}{\partial Y^{\boldsymbol{i}}}(y) \right| < \left| \frac{\partial P}{\partial Y^{(n)}}(y) \right| \qquad \text{for all } \boldsymbol{i}.$$

Decreasing ε if necessary, we may assume in addition that for all those y, if $y \neq 0$, then $0 \neq y^{(i)} \prec 1$ for $i = 0, \dots, n$, so $\operatorname{sign} y^{(n)} = (-1)^n \operatorname{sign} y$, and

$$\beta y^{(n)} \succ y^{\boldsymbol{i}} \qquad \text{for } \boldsymbol{i} \text{ with } |\boldsymbol{i}| > 1 \text{ or } w\boldsymbol{i} < n,$$

by Lemma 6.3. Now let y and z be elements of an H-field extending K with $-\varepsilon < y < z < \varepsilon$. Taylor expansion around y gives

$$P(z) = P(y) + \sum_{i=0}^{n} \frac{\partial P}{\partial Y^{(i)}}(y)(z-y)^{(i)} + \sum_{|\boldsymbol{i}|>1} \frac{1}{\boldsymbol{i}!} \frac{\partial^{|\boldsymbol{i}|}P}{\partial Y^{\boldsymbol{i}}}(y)(z-y)^{\boldsymbol{i}}.$$

By choice of ε and β we have, for all $\boldsymbol{i}$ with $|\boldsymbol{i}| > 1$ or $w\boldsymbol{i} < n$:

$$\frac{\partial^{|\boldsymbol{i}|}P}{\partial Y^{\boldsymbol{i}}}(y)(z-y)^{\boldsymbol{i}} \prec \frac{\partial P}{\partial Y^{(n)}}(y)(z-y)^{(n)}.$$

Hence

$$\operatorname{sign}\big(P(z) - P(y)\big) = \operatorname{sign}\left(\frac{\partial P}{\partial Y^{(n)}}(y)(z-y)^{(n)}\right) = (-1)^n \operatorname{sign} a_n.$$

So $P(y) < P(z)$ if n is even and $a_n > 0$ or if n is odd and $a_n < 0$, and $P(y) > P(z)$ otherwise. $\square$

Remark. Suppose $a_n = 0$. Then there is by Corollary 3.5 an $\varepsilon \in K^{>0}$ such that either $\frac{\partial P}{\partial Y^{(n)}}(y) > 0$ for all y in all H-field extensions of K with $a < y < a + \varepsilon$, or $\frac{\partial P}{\partial Y^{(n)}}(y) < 0$ for all y in all H-field extensions of K with $a < y < a + \varepsilon$. Take such an ε and assume we are in the first case (positive sign) and n is even. By the proposition, each y_0 in any H-field extension with $a < y_0 < a + \varepsilon$ has an interval in that H-field extension around it on which the differential polynomial function $y \mapsto P(y)$ is strictly increasing. Can one choose ε such that this function is even strictly increasing on the entire interval $(a, a + \varepsilon)$ in all H-field extensions of K?

7. Exponential Maps and Powers

In this section we study exponential maps and power functions on H-fields. By an *exponential map* on an ordered field K we mean an isomorphism $K \to K^{>0}$ of the ordered additive group of K onto its ordered multiplicative group of positive elements. (See [21] for general facts on exponential maps.) We first show that any exponential map on the constant field C of a Liouville closed H-field K can be extended to an exponential map on K.

A Hardy field $K \supseteq \mathbb{R}$ is *closed under powers* if $f^c \in K$ for all $f \in K^{>0}$ and $c \in \mathbb{R}$; such $y = f^c$ satisfies the differential equation $y^\dagger = cf^\dagger$. We use this observation to define when an H-field is closed under powers, and how to make the value group of such an H-field into an ordered vector space over the constant field.

Lemma 7.1. *Let K be a Liouville closed H-field.*

(1) *There is an order preserving isomorphism*

$$a \mapsto \exp(a) \colon \mathfrak{m} \to 1 + \mathfrak{m}$$

from the additive group $\mathfrak{m}$ onto the multiplicative group $1 + \mathfrak{m}$ that assigns to each $a \in \mathfrak{m}$ the unique $y = \exp(a) \in 1 + \mathfrak{m}$ such that $y^\dagger = a'$.

(2) *There is an order preserving group isomorphism*

$$K/\mathcal{O} \to K^{>0}/C^{>0}(1 + \mathfrak{m})$$

that assigns to each additive coset $a + \mathcal{O}$ the multiplicative coset $yC^{>0}(1 + \mathfrak{m})$ where y is any element of $K^{>0}$ such that $y^\dagger = a'$.

Proof. For part (1), we first show that for $a \in \mathfrak{m}$, the equation $y^\dagger = a'$ has a unique solution in $1 + \mathfrak{m}$. Take $y_0 \in K^\times$ with $y_0^\dagger = a'$. Then necessarily $y_0 \asymp 1$. Take $c \in C^\times$ with $cy_0 \sim 1$. Then $cy_0 \in 1 + \mathfrak{m}$ is a solution of $y^\dagger = a'$. If $y, z \in 1 + \mathfrak{m}$ and $y^\dagger = z^\dagger = a'$, then $y/z \in (1 + \mathfrak{m}) \cap C = \{1\}$, so $y = z$. Thus $\exp\colon \mathfrak{m} \to 1 + \mathfrak{m}$ is a surjective homomorphism of groups.

Let $0 < a \in \mathfrak{m}$. We have to show that then $\exp(a) > 1$. Note that $a' < 0$ by a remark preceding Lemma 1.4 in [2], so with $\exp(a) = 1 + b$, $b \in \mathfrak{m}$, we have $(b + 1)^\dagger = a' < 0$, and thus $b' < 0$. The derivation being strictly decreasing on $\mathfrak{m}$, we have $b > 0$, as required.

The map defined in part (2) is clearly a surjective group homomorphism. To finish the proof of (2), let $y \in K^{>0}$, $y^\dagger = a'$, $a \in K$, $a > \mathcal{O}$. We have to show that $y \succ 1$. If $y \prec 1$, then $y^\dagger < 0 < a'$, a contradiction. If $y \asymp 1$, then $y^\dagger \asymp y' \prec a'$, a contradiction. Hence $y \succ 1$. □

Remark. Let K be a Liouville closed H-field. Since $\mathcal{O}$ is a C-linear subspace of K we can choose a C-linear subspace A of K that is a direct summand to $\mathcal{O}$: $K = A \oplus \mathcal{O}$. The group homomorphism $y \mapsto y^\dagger : K^{>0} \to K$ is surjective and $K^{>0}$ is divisible and torsion-free, so we can choose a divisible subgroup F of $K^{>0}$ that is mapped injectively onto $A' := \{a' : a \in A\}$ by this homomorphism. The previous lemma tells us that then F is a direct factor in $K^{>0}$ of $C^{>0}(1+\mathfrak{m})$, in particular $K^{>0} = F \cdot C^{>0}(1+\mathfrak{m})$. This gives us an isomorphism

$$\exp_{A,F} \colon A \to F$$

of ordered abelian groups, which sends $a \in A$ to the unique $y \in F$ such that $y^\dagger = a'$. Suppose that in addition there is given an exponential map $\exp_C$ on the ordered field C. We combine these two exponential maps with the isomorphism $\exp\colon \mathfrak{m} \to 1+\mathfrak{m}$ in part (1) of the lemma to obtain an exponential map

$$\exp\colon K = A \oplus C \oplus \mathfrak{m} \to F \cdot C^{>0}(1+\mathfrak{m}) = K^{>0}$$

on the ordered field K: $\exp(a+c+\varepsilon) := \exp_{A,F}(a)\exp(c)\exp(\varepsilon)$ for $a \in A$, $c \in C$ and $\varepsilon \in \mathfrak{m}$. (If $C = \mathbb{R}$, we can of course take for $\exp_C$ the usual exponential function $x \mapsto e^x$.) We then have $\exp(f)^\dagger = f'$ for all $f \in K$; in particular, $\mathrm{E} = \exp$ satisfies (E1)–(E6) from Section 1.

Corollary 7.2. *No Liouville closed H-field is maximally valued.*

Proof. Let K be a Liouville closed H-field, and let A, F, and $\exp_{A,F}\colon A \to F$ be as in the remark above. Then we define an ordered group embedding $s\colon \Gamma \to K$ by

$$s(\gamma) = a \qquad \Longleftrightarrow \qquad v\big(-\exp_{A,F}(a)\big) = \gamma,$$

for $\gamma \in \Gamma$, $a \in K$. This map satisfies $s(\Gamma) \cap \mathcal{O} = \{0\}$ and $K = s(\Gamma) \oplus \mathcal{O}$.

In [20], Theorem 4, it is shown that the existence of such a map is incompatible with K being maximally valued. □

***H*-Fields closed under powers.** Let K be an H-field. We say that K is **closed under powers** if for every $c \in C$ and $f \in K^\times$, the differential equation

$$y^\dagger = cf^\dagger \tag{7.1}$$

has a solution y in $K^\times$. (So if K is Liouville closed, then K is closed under powers.) The ratio of any two solutions to (7.1) in $K^\times$ is a non-zero constant.

In the rest of this section K is an H-field closed under powers.

We extend the map $(k,f)\mapsto f^k\colon \mathbb{Z}\times K^{>0}\to K^{>0}$ to a map

$$(c,f)\mapsto f^c\colon C\times K^{>0}\to K^{>0}$$

such that for each $f\in K^{>0}$ and $c\in C$, the element $y=f^c$ satisfies (7.1). (For what follows it doesn't matter how such a map is chosen.)

We write $f=_C g$, for $f,g\in K$, if $f=cg$ for some $c\in C^{>0}$. With this notation we have the following simple rules, for $f,g\in K^{>0}$ and $c,c_1,c_2\in C$:

(P1) $f=_C g\Rightarrow f^c=_C g^c$.
(P2) $f^{c_1}f^{c_2}=_C f^{c_1+c_2}$;
(P3) $(f^{c_1})^{c_2}=_C f^{c_1c_2}$;
(P4) $(fg)^c=_C f^cg^c$;
(P5) suppose $c>0$; then $f\prec g\iff f^c\prec g^c$; also $f\asymp g\iff f^c\asymp g^c$.

The proofs of (P1)–(P4) are obvious. For (P5) we first use (P4) to reduce to the case $g=1$. So let $0<f\prec 1$; it suffices to show that then $f^c\prec 1$. (For the converse, take $1/c$ instead of c.) We have $(f^c)^\dagger=cf^\dagger<0$, so $f^c\preceq 1$. If $f^c\asymp 1$, then $(f^c)^\dagger\asymp\varepsilon'$ for some infinitesimal ε, hence $(f^c)^\dagger\prec f^\dagger$, a contradiction. Thus $f^c\prec 1$, as required.

We can now characterize comparability in terms of powers as promised in §1. Recall in this connection that for $f,g\in K$ with $f,g>C$ we have

$$\mathrm{Cl}(f)<\mathrm{Cl}(g)\iff f\prec\!\!\!\prec g\iff f^\dagger\prec g^\dagger.$$

Proposition 7.3. *Let $f,g\in K$, $f,g>C$. Then*

$$f\prec\!\!\!\prec g\iff f^c<g \text{ for all } c\in C^{>0}.$$

Proof. Suppose that $f^c<g$ for all $c\in C^{>0}$. Then $f^c\prec f^{c+1}<g$ for all $c\in C^{>0}$, so $f^c\prec g$ for all $c\in C^{>0}$, hence $(f^c)^\dagger=cf^\dagger<g^\dagger$ for all $c\in C^{>0}$, by Lemma 1.4 in [2]. Thus $f^\dagger\prec g^\dagger$. The converse follows by reversing these steps. □

In the remainder of this section we establish a link to the notions and results from [1]. This link will play a role in our further work on asymptotic differential algebra, in collaboration with J. van der Hoeven. We first make the value group Γ into an ordered vector space over the constant field C:

Lemma 7.4. *For $c\in C$, $\gamma=v(f)\in\Gamma$ with $f\in K^{>0}$ and $y\in K^{>0}$ with $y^\dagger=cf^\dagger$, the element $v(y)\in\Gamma$ only depends on (c,γ) (not on the choice of f and y), and is denoted by $c\cdot\gamma$. The scalar multiplication $(c,\gamma)\mapsto c\cdot\gamma:=v(f^c)\colon C\times\Gamma\to\Gamma$ makes Γ into an ordered vector space over the ordered field C.*

Proof. That $v(y)$ depends only on (c,γ) follows from (P5). The second assertion of the lemma then follows easily from (P1)–(P5). □

Next we recall the definition of "Hahn space" from [1]. Let V be an ordered vector space over an ordered field $\boldsymbol{k}$. Then the $\boldsymbol{k}$-archimedean class $[v]_{\boldsymbol{k}}$ of a vector $v \in V$ is its equivalence class under the equivalence relation on V defined by

$$v \sim w \quad :\Longleftrightarrow \quad \exists \lambda \in \boldsymbol{k}^{>1} : \frac{1}{\lambda}|v| \leq |w| \leq \lambda |v|.$$

(If Γ is a divisible ordered abelian group, considered as ordered vector space over $\mathbb{Q}$, then $[\gamma]_{\mathbb{Q}}$ coincides with the archimedean class $[\gamma]$ of $\gamma \in \Gamma$ as defined in [2], §2.) We put $[V]_{\boldsymbol{k}} := \big\{[v]_{\boldsymbol{k}} : v \in V\big\}$ and linearly order $[V]_{\boldsymbol{k}}$ by

$$[v]_{\boldsymbol{k}} < [w]_{\boldsymbol{k}} \quad :\Longleftrightarrow \quad [v]_{\boldsymbol{k}} \neq [w]_{\boldsymbol{k}} \text{ and } |v| < |w|.$$

Then V is said to be a **Hahn space** if for all vectors $v, w \in V^*$

$$[v]_{\boldsymbol{k}} = [w]_{\boldsymbol{k}} \quad \Rightarrow \quad \exists \lambda \in \boldsymbol{k} : [v - \lambda w]_{\boldsymbol{k}} < [w]_{\boldsymbol{k}}.$$

Hahn spaces behave nicely under scalar extension, and satisfy an analogue of the Hahn embedding theorem for ordered abelian groups (see [1], §2).

Proposition 7.5. *The ordered vector space $\Gamma = v(K^\times)$ over C is a Hahn space, and for all $\gamma, \delta \in \Gamma^*$ we have*

$$[\gamma]_C \leq [\delta]_C \quad \Longleftrightarrow \quad \psi(\gamma) \geq \psi(\delta). \tag{7.2}$$

If K has an element $x > C$ with $x' = 1$, then (Γ, ψ) is an H-couple over C, with distinguished positive element $1 := v(x^{-1})$. If in addition K is Liouville closed, then (Γ, ψ) is a closed H-couple. (On H-couples, see [2].)

Proof. We first show (7.2), and then derive the Hahn space property of Γ. Since ψ is decreasing on $\Gamma^{>0}$ (Lemma 2.2 in [2]) and $\psi(c \cdot \gamma) = \psi(\gamma)$ for all $c \in C^\times$ and $\gamma \in \Gamma^*$, the direction from left to right in (7.2) is clear. The converse is an easy consequence of Proposition 7.3.

It now follows quickly that Γ is a Hahn space over C: Let $f, g \in K^{>0}$ with $f, g \not\asymp 1$ and $\big[v(f)\big]_C = \big[v(g)\big]_C$. Then $\psi\big(v(f)\big) = \psi\big(v(g)\big)$, so there exists a constant c such that $f^\dagger \sim c g^\dagger$. Then we have $\psi\big(v(f) - v(g^c)\big) > \psi\big(v(f)\big)$, so by (7.2), $\big[v(f) - c \cdot v(g)\big]_C = \big[v(f) - v(g^c)\big]_C < \big[v(f)\big]_C$, as desired. □

In the next section we need the following lemma which gives the first terms in the "binomial expansion" for powers of elements of $1 + \mathfrak{m}$.

Lemma 7.6. *Let $c \in C$, $\varepsilon \in K$, $\varepsilon \prec 1$. Then $(1+\varepsilon)^c =_C 1 + c\varepsilon + z$ with $z \preceq \varepsilon^2$.*

Proof. For $y \in K^{>0}$ with $y =_C (1+\varepsilon)^c$, we have $y^\dagger = c(1+\varepsilon)^\dagger \sim c\varepsilon'$, hence $y \asymp 1$. Thus we can take $y \in K^{>0}$ with $y =_C (1+\varepsilon)^c$ and $y \sim 1$. Then $y = 1 + \delta$ with $\delta \prec 1$, so $\delta' = y' \sim y^\dagger \sim c\varepsilon'$, hence $\delta \preceq \varepsilon$. Put $z := \delta - c\varepsilon$. Then

$$\begin{aligned} z' &= \delta' - c\varepsilon' = y' - y^\dagger(1+\varepsilon) = y^\dagger\big(y - (1+\varepsilon)\big) \\ &= y^\dagger(\delta - \varepsilon) \preceq c\varepsilon'\varepsilon \preceq (\varepsilon^2)', \end{aligned}$$

and since $z, \varepsilon^2 \prec 1$, this yields $z \preceq \varepsilon^2$. □

8. Adjoining Powers

We continue here our study of powers. The reason we pay so much attention to this issue is our ultimate interest in *existentially closed H-fields*, see Section 14. Such H-fields are closed under powers, so their asymptotic couples carry a "definable" ordered vector space structure as indicated in the last section. Among our conjectures on existentially closed H-fields is that their asymptotic couples carry no further (definable) structure, see the introduction of [1]. Thus we expect the structure coming from powers to be important in any model-theoretic analysis of (existentially closed) H-fields. In this section we generalize the main results on adjoining powers to Hardy fields from [28] to the setting of H-fields.

Power extensions and closure under powers. A **power extension** of a differential field K of characteristic 0 is a differential field extension L of K such that $C_L|C$ is algebraic, and for each $a \in L$ there are $t_1, \dots, t_n \in L^\times$ with $a \in K(t_1, \dots, t_n)$ and for each $i = 1, \dots, n$, either

(1) t_i is algebraic over $K(t_1, \dots, t_{i-1})$, or
(2) $t_i^\dagger = cf^\dagger$ for some $c \in C_L$, $0 \neq f \in K(t_1, \dots, t_{i-1})$.

(So a power extension is in particular a Liouville extension as defined in [2].)

Definition 8.1. A **closure under powers** of an H-field K is an H-field L extending K which is real closed, closed under powers, and such that $L|K$ is a power extension.

Note that if K is an H-field with trivial derivation ($K = C$), then its real closure is, up to isomorphism over K, the unique closure under powers of K.

In this section we prove that every H-field K has (up to isomorphism over K) at least one and at most two closures under powers. The arguments are similar to those used in proving analogous facts for Liouville closures in [2].

Lemmas on power extensions. We need variants of some results from [2]. *In the next three lemmas and accompanying remarks K is a real closed H-field.*

Lemma 8.2. *If K is closed under powers, then K has no proper power extension with the same constants as K.*

(See the proof of part (1) of Lemma 6.3 in [2].)

Lemma 8.3. *Let $r, b \in K^\times$ be such that $r \neq a^\dagger$ for all $a \in K^\times$ and $v(r - b^\dagger) \in (\mathrm{id} + \psi)(\Gamma^{>0})$. Let $L = K(z)$ be a field extension of K with z transcendental over K, and equip L with the unique derivation extending the derivation of K such that $z^\dagger = r$. Then there is a unique pair consisting of a valuation on L and an ordering of L that makes L a pre-H-field extension of K with $z \sim b$. With this valuation and ordering L is an H-field and an immediate extension of K.*

Proof. Set $y := (z/b) - 1$, so $(1+y)^\dagger = r - b^\dagger$. Now apply Lemma 5.2 in [2] and the remark following it to $s := r - b^\dagger$ and $L = K(y)$. $\square$

Remark. With K, b and r as in the lemma, let $E \supseteq K$ be an H-field extension with $C_E = C$ and $z_0 \in E^\times$ such that $z_0^\dagger = r$. Then $v\big((z_0/b)^\dagger\big) = v(r - b^\dagger) \in (\mathrm{id} + \psi)\big(\Gamma^{>0}\big)$, so $z_0 \asymp b$. Hence $cz_0 \sim b$ for some $c \in C$. With $z := cz_0$ this gives an H-subfield $L = K(z)$ of E exactly as in the lemma.

Lemma 8.4. *Let $s, b \in K^\times$ be such that $v(s - a^\dagger) < (\mathrm{id} + \psi)\big(\Gamma^{>0}\big)$ for each $a \in K^\times$, $b > 0$, and $\Psi < v(s - b^\dagger) < (\mathrm{id} + \psi)\big(\Gamma^{>0}\big)$. Let $L = K(y)$ be a field extension of K with y transcendental over K, and equip L with the unique derivation extending the derivation of K such that $y^\dagger = s$. Then $C_L = C$, and the following holds.*

(1) *There is a unique pair consisting of a valuation of L and an ordering on L that makes L a pre-H-field extension of K with $y > 0$ and $y \not\asymp b$. With this valuation and ordering L is an H-field. Letting $z := y/b$, we have $z \notin C$, $0 < |v(z)| < \Gamma^{>0}$, $\Gamma_L = \Gamma \oplus \mathbb{Z}v(z)$, and $\Psi_L = \Psi \cup \big\{v(z^\dagger)\big\}$, with $v(z^\dagger) = v(s - b^\dagger) \notin \Psi$.*

(2) *There is a unique pair consisting of a valuation of L and an ordering on L that makes L a pre-H-field extension of K with $y > 0$ and $y \sim b$. With this valuation and ordering L is an H-field. Letting $z := (y/b) - 1$, we have $z \notin C$, $0 < v(z) < \Gamma^{>0}$, $\Gamma_L = \Gamma \oplus \mathbb{Z}v(z)$, and $\Psi_L = \Psi \cup \big\{v(z^\dagger)\big\}$, with $v(z^\dagger) = v(s - b^\dagger) - v(z) \notin \Psi$.*

Proof. Passing from s to $s - b^\dagger$ and from y to y/b, we may assume that $b = 1$. So $\Psi < v(s) < (\mathrm{id} + \psi)\big(\Gamma^{>0}\big)$. Replacing s by $-s$ and y by $1/y$ if necessary, we may also assume that $s < 0$. By [2] the H-field K has Liouville closures M_1, M_2 such that $s = f'$ for some $f \succ 1$ in M_1 and $s = g'$ for some $g \prec 1$ in M_2.

For part (1), let $y_1 \in M_1^{>0}$ be such that $y_1^\dagger = s$. Then necessarily $y_1 \not\asymp 1$. Equip $L = K(y)$ with the ordering and valuation such that the isomorphism of differential fields $K(y_1) \to K(y)$ which is the identity on K and maps y_1 to y becomes an isomorphism of valued ordered differential fields. Then L is an H-field. For any $\eta \neq 0$ in any H-field extension of K such that $\eta^\dagger = s$ and $\eta \not\asymp 1$, we have $\eta \prec 1$ (since $s < 0$), hence $0 < v(\eta) < \Gamma^{>0}$. Applying this to $y = \eta$ the rest of (1) follows easily.

For (2), let $y_2 \in M_2^{>0}$ be such that $y_2^\dagger = s$. Then necessarily $y_2 \asymp 1$, and passing from y_2 to cy_2 for a certain constant $c > 0$, we may assume $y_2 \sim 1$. Equip $L = K(y)$ with the ordering and valuation such that the isomorphism of differential fields $K(y_2) \to K(y)$ which is the identity on K and maps y_2 to y becomes an isomorphism of valued ordered differential fields. Then L is an H-field. Given any $\eta > 0$ in any H-field extension of K such that $\eta^\dagger = s$ and $\eta \sim 1$, we have $\zeta := \eta - 1 \prec 1$ and $\zeta' = s\eta < 0$, hence $\zeta > 0$ and $0 < v(\zeta) < \Gamma^{>0}$. Applying this to $y := \eta$ the rest of (2) follows easily. $\square$

Remark. With K, s and b as in the hypothesis of the lemma, let $E \supseteq K$ be an H-field extension with $C_E = C$ and $y_0 \in E^{>0}$ such that $y_0^\dagger = s$ and

$y_0 \asymp b$. Let $c \in C^{>0}$ be such that $cy_0 \sim b$. Then $y := cy_0$ is transcendental over K, $y^\dagger = s$, and the H-field $K(y)$ is exactly as described in part (2) of the lemma.

Remarks. Let $s \in K^\times$ be such that $s \neq a^\dagger$ for every $a \in K^\times$, and put

$$S := \{v(s - b^\dagger) : b \in K^{>0}\} \subseteq \Gamma.$$

Then exactly one of the following three cases applies:

(1) $S \cap (\mathrm{id} + \psi)(\Gamma^{>0}) \neq \emptyset$.

(2) $S \cap (\mathrm{id} + \psi)(\Gamma^{>0}) = \emptyset$, and for each $\alpha \in S$ there exists $\gamma \in \Gamma^*$ with $\alpha \leq \psi(\gamma)$.

(3) $S \cap (\mathrm{id} + \psi)(\Gamma^{>0}) = \emptyset$, and there exists a $\beta \in S$ such that

$$\Psi < \beta < (\mathrm{id} + \psi)(\Gamma^{>0}).$$

In case (1) the hypothesis of Lemma 8.3 holds for $r := s$. In case (2) the hypothesis of Lemma 5.3 in [2] is satisfied when $s < 0$. In case (3) we can take $b \in K^{>0}$ such that $\Psi < v(s - b^\dagger) < (\mathrm{id} + \psi)(\Gamma^{>0})$; then the hypothesis of Lemma 8.4 is satisfied.

Constructing a closure under powers. Let K be an H-field. A **tower on K of power extensions** is a strictly increasing chain $(K_\lambda)_{\lambda \leq \mu}$ of H-fields with corresponding asymptotic couples $(\Gamma_\lambda, \psi_\lambda)$, $\Psi_\lambda := \psi_\lambda(\Gamma_\lambda^*)$, and constant fields C_λ, indexed by the ordinals λ less than or equal to some ordinal μ, such that

(1) $K_0 = K$,

(2) if λ is a limit ordinal, $0 < \lambda \leq \mu$, then $K_\lambda = \bigcup_{\kappa < \lambda} K_\kappa$,

(3) for $\lambda < \mu$, either

(a) $K_{\lambda+1}$ is a real closure of K_λ,

or K_λ is already real closed, $K_{\lambda+1} = K_\lambda(y_\lambda)$ with $0 < y_\lambda \notin K_\lambda$ (so y_λ is transcendental over K_λ),

$$y_\lambda^\dagger = s_\lambda \qquad \text{where } 0 \neq s_\lambda = c_\lambda f_\lambda^\dagger,\ c_\lambda \in C_\lambda^\times,\ f_\lambda \in K_\lambda^\times,$$

and one of the following holds:

(b) $s_\lambda \neq a^\dagger$ for all $a \in K_\lambda^\times$, and there exists $b_\lambda \in K_\lambda^{>0}$ with $v(s_\lambda - b_\lambda^\dagger) \in (\mathrm{id} + \psi_\lambda)(\Gamma_\lambda^{>0})$, $y_\lambda \sim b_\lambda$;

(c) $s_\lambda < 0$ and for all $a \in K_\lambda^\times$ there exists $\gamma \in \Gamma_\lambda^*$ with $v(s_\lambda - a^\dagger) \leq \psi_\lambda(\gamma)$;

(d) $v(s_\lambda - a^\dagger) < (\mathrm{id} + \psi_\lambda)(\Gamma_\lambda^{>0})$ for all $a \in K_\lambda^\times$, and there exists $b_\lambda \in K_\lambda^{>0}$ with $\Psi_\lambda < v(s_\lambda - b_\lambda^\dagger) < (\mathrm{id} + \psi_\lambda)(\Gamma_\lambda^{>0})$, and $y_\lambda \not\sim b_\lambda$;

(e) $v(s_\lambda - a^\dagger) < (\mathrm{id} + \psi_\lambda)(\Gamma_\lambda^{>0})$ for all $a \in K_\lambda^\times$, and there exists $b_\lambda \in K_\lambda^{>0}$ with $\Psi_\lambda < v(s_\lambda - b_\lambda^\dagger) < (\mathrm{id} + \psi_\lambda)(\Gamma_\lambda^{>0})$, and $y_\lambda \sim b_\lambda$.

The H-field K_μ is called the **top** of the tower $(K_\lambda)_{\lambda \leq \mu}$. Note that clause (a) corresponds to the last part of §3 in [2], (b) to Lemma 8.3, (c) to Lemma 5.3 in [2], and (d), (e) to Lemma 8.4, (1) and (2), respectively.

Remarks. Let $(K_\lambda)_{\lambda \leq \mu}$ be a tower as above. Then:

(1) K_μ is a power extension of K.
(2) C_μ is a real closure of C if $\mu > 0$.
(3) $\operatorname{card}(K_\mu) = \operatorname{card}(K)$, hence $\mu < \operatorname{card}(K)^+$. (By Lemma 6.1 in [2].)
(4) For $\lambda < \mu$, we have:
    (a) If $K_{\lambda+1}$ is a real closure of K_λ, then $\Gamma_{\lambda+1} = \mathbb{Q}\Gamma_\lambda$.
    (b) If $K_{\lambda+1} = K_\lambda(y_\lambda)$ is as in 3(b), then $\Gamma_{\lambda+1} = \Gamma_\lambda$.
    (c) Suppose $K_{\lambda+1} = K_\lambda(y_\lambda)$ is as in 3(c). Then $\Gamma_{\lambda+1} = \Gamma_\lambda \oplus \mathbb{Z}v(y_\lambda)$ and Ψ_λ is cofinal in $\Psi_{\lambda+1}$. If Ψ_λ has a largest element, then $\Psi_{\lambda+1}$ has the same largest element. If Ψ_λ has no largest element, then $\Gamma_\lambda^{>0}$ is coinitial in $\Gamma_{\lambda+1}^{>0}$. (For the last two claims, use remarks at end of §5 in [2].)
    (d) If $K_{\lambda+1} = K_\lambda(y_\lambda)$ is as in 3(d), then, setting $z_\lambda := y_\lambda/b_\lambda$, we have $z_\lambda \notin C_\lambda$, $\Gamma_{\lambda+1} = \Gamma_\lambda \oplus \mathbb{Z}v(z_\lambda)$, $\Psi_{\lambda+1} = \Psi_\lambda \cup \big\{v(z_\lambda^\dagger)\big\}$, and $\max \Psi_{\lambda+1} = v(z_\lambda^\dagger) = v(s_\lambda - b_\lambda^\dagger) \in \Gamma_\lambda \setminus \Psi_\lambda$.
    (e) If $K_{\lambda+1} = K_\lambda(y_\lambda)$ is as in 3(e), then, with $z_\lambda := (y_\lambda/b_\lambda) - 1$, we have $z_\lambda \notin C_\lambda$, $\Gamma_{\lambda+1} = \Gamma_\lambda \oplus \mathbb{Z}v(z_\lambda)$, $\Psi_{\lambda+1} = \Psi_\lambda \cup \big\{v(z_\lambda^\dagger)\big\}$, and $\max \Psi_{\lambda+1} = v(z_\lambda^\dagger) = v(s_\lambda - b_\lambda^\dagger) - v(z_\lambda) \notin \Psi_\lambda$.
(5) In the situation of (c) we have $\psi_{\lambda+1}\big(v(y_\lambda)\big) = \psi_\lambda\big(v(f_\lambda)\big)$, hence Ψ_λ is coinitial in $\Psi_{\lambda+1}$. It follows easily that the set Ψ is coinitial in Ψ_μ.

By (3) there exists a *maximal* tower $(K_\lambda)_{\lambda\le\mu}$ on K of power extensions, that is, $(K_\lambda)_{\lambda\le\mu}$ is a tower on K of power extensions that cannot be extended to a tower $(K_\lambda)_{\lambda\le\mu+1}$ on K of power extensions. Given such a maximal tower, K_μ is real closed, $K_\mu|K$ is a power extension of K, and, by the Remarks following Lemma 8.4, K_μ is closed under powers, hence K_μ is a closure under powers of K.

Conclusion: *each H-field has a closure under powers.*

At most two closures under powers. Let K be an H-field with real closed constant field $C \neq K$. Take a maximal tower $(K_\lambda)_{\lambda\le\mu}$ on K of power extensions. Its top $L := K_\mu$ is a closure under powers of K. Let (Γ_L, ψ_L) be the asymptotic couple corresponding to L. We distinguish the following two cases:

Case 1. For each $\lambda < \mu$, $K_{\lambda+1}$ is obtained from K_λ as in 3(a), 3(b), or 3(c) above. Then $\Gamma_L = C\Gamma$ and Ψ is cofinal in Ψ_L. In this case L is the unique closure under powers of K, up to isomorphism over K: Let L' be any closure under powers of K. Then we copy the tower (K_λ) inside L', more precisely, we inductively construct H-field embeddings $j_\lambda : K_\lambda \to L'$ for $\lambda \le \mu$ such that j_0 is the natural inclusion $K \to L'$ and $j_{\lambda'}$ extends j_λ whenever $\lambda < \lambda' \le \mu$. (This is possible by the uniqueness parts of Lemmas 8.3 and [2], 5.3, and the remarks following them.) By Lemma 8.2, we have $L' = j_\mu(L)$.

Case 2. There exists $\lambda < \mu$ such that $K_{\lambda+1}$ is obtained from K_λ as in 3(d) or 3(e). Given such λ, the set $\Psi_{\lambda+1}$ has a maximum, and it follows that

$K_{\nu+1}$ is obtained from K_ν as in 3(a), 3(b), or 3(c), whenever $\lambda < \nu < \mu$. In particular, there is only one such λ. By an argument as in the previous paragraph, using Lemma 8.2, the uniqueness parts of Lemmas 8.3, 8.4 and [2], 5.3, and the remarks following them, one easily shows that K has exactly two closures under powers, up to isomorphism over K.

We summarize this discussion:

Proposition 8.5. *Let K be an H-field with real closed constant field $C \neq K$. Then K has at least one and most two closures under powers, up to K-isomorphism.* □

In the rest of this section we show how to detect in K itself whether K has one or two closures under powers.

Power products. Let L be an H-field extension of an H-field K such that L is closed under powers and $C_L = C$. By Lemma 7.4, Γ_L is an ordered vector space over the ordered field C. Let $C\Gamma$ be the C-linear subspace of Γ_L spanned by Γ. A **power product** of $f_1, \dots, f_n \in L^{>0}$ is an $f \in L$ such that $f =_C f_1^{c_1} \cdots f_n^{c_n}$. Note that then $f^\dagger = c_1 f_1^\dagger + \cdots + c_n f_n^\dagger$; in particular $f^\dagger \in K$ if $f_1, \dots, f_n \in K$. For every positive element a of L with $v(a) \in C\Gamma$ there exists a power product f of elements of K such that $a \sim f$.

Lemma 8.6. *Let $K' \supseteq K$ be an H-subfield of L such that $\Gamma_{K'} = C\Gamma$ and $f^c \in K'$ for all $f \in K^{>0}$ and $c \in C$. For every power product $f \in L$ of positive elements of K' there exists a power product $g \in K'$ of positive elements of K and an $\varepsilon \in L$ such that $f = g(1+\varepsilon)$ and $\varepsilon \preceq \delta \prec 1$ for some $\delta \in K'$.*

Proof. Let $f \in L$, $f_1, \dots, f_m \in (K')^{>0}$ and $c_1, \dots, c_m \in C$ such that $f =_C f_1^{c_1} \cdots f_m^{c_m}$. Using $\Gamma_{K'} = C\Gamma$ we have $f_i =_C g_1^{c_{i1}} \cdots g_n^{c_{in}}(1+\varepsilon_i)$ for $i = 1, \dots, m$, where $g_1, \dots, g_n \in K^{>0}$, $c_{ij} \in C$, and $\varepsilon_i \in K'$, $\varepsilon_i \prec 1$, for $i = 1, \dots, m$ and $j = 1, \dots, n$. Hence
$$f =_C g_1^{e_1} \cdots g_n^{e_n} (1+\varepsilon_1)^{c_1} \cdots (1+\varepsilon_m)^{c_m}$$
where $e_j = c_1 c_{1j} + \cdots + c_m c_{mj}$ for $j = 1, \dots, n$. By Lemma 7.6, we also have $(1+\varepsilon_i)^{c_i} =_C 1 + c_i \varepsilon_i + z_i$ with $z_i \in L$ and $v(z_i) \geq 2v(\varepsilon_i)$, so
$$f =_C g_1^{e_1} \cdots g_n^{e_n}(1+\varepsilon) \text{ with } v(\varepsilon) \geq \min_i v(\varepsilon_i).$$
□

When are there two closures under powers? *For the rest of this section K is an H-field with real closed constant field $C \neq K$.* To detect in K itself when K has two closures under powers, we distinguish three mutually exclusive cases:

(1) The set Ψ has a largest element.
(2) There exist $c_1, \dots, c_n \in C$ and $f_1, \dots, f_n \in K^{>0}$ such that
$$\Psi < v(c_1 f_1^\dagger + \cdots + c_n f_n^\dagger) < (\mathrm{id} + \psi)\big(\Gamma^{>0}\big).$$

(3) Neither (1) nor (2) holds.

In this subsection, we say that K **has type** (n)**,** for $n = 1, 2, 3$, if K satisfies condition (n) above. Familiar examples show that types (1) and (3) occur. The following observation will be used in Section 11 to show that type (2) also occurs:

Lemma 8.7. *Suppose that K is real closed, and let $z \in K$ be such that*

$$\Psi < v(z) < (\mathrm{id} + \psi)\big(\Gamma^{>0}\big)$$

and let $f \in K^\times$, $c \in C$ be such that $v(cf^\dagger - b^\dagger) < v(z)$ for all $b \in K^\times$. Let y be an element in a Liouville closure of K with $y > 0$ and $y^\dagger = z - cf^\dagger$. Then $K(y)$ is an H-field with real closed constant field C, and $K(y)$ has type (2).

Proof. Put $s = z - cf^\dagger$. Changing from z to $-z$ and from c to $-c$, if necessary, we may assume that $s < 0$. Then K and s satisfy the hypotheses of Lemma 5.3 in [2]. By the uniqueness part of that lemma and Remark (1) following it, $K(y)$ is an H-field with constant field C, and $\Psi_{K(y)} < v(z) = v(cf^\dagger + y^\dagger) < (\mathrm{id} + \psi_{K(y)})\big(\Gamma^{>0}_{K(y)}\big)$. Hence $K(y)$ has type (2). □

Fix a closure under powers L of K. Let K' be the real closure inside L of its H-subfield $K\big(f^c : f \in K^{>0}, c \in C\big)$. Let (Γ', ψ') be the asymptotic couple of K'; so $C\Gamma \subseteq \Gamma'$. The following lemma describes (Γ', ψ') depending on the type of K.

Lemma 8.8.

(1) *If K has type* (1)*, then $\Gamma' = C\Gamma$ and $\max \Psi' = \max \Psi$.*

(2) *Suppose $f_1, \dots, f_n \in K^{>0}$ and $c_1, \dots, c_n \in C$ are such that $f := f_1^{c_1} \cdots f_n^{c_n} \in K'$ satisfies $\Psi < v(f^\dagger) < (\mathrm{id} + \psi)\big(\Gamma^{>0}\big)$. Then either*
   - (a) *$\Gamma' = C\Gamma$, $\max \Psi' = v(f^\dagger)$, and Ψ is cofinal in $\Psi' \setminus \big\{v(f^\dagger)\big\}$, or*
   - (b) *there exists $z \in K' \setminus C$ such that $0 < v(z) < (C\Gamma)^{>0}$, $\Gamma' = \mathbb{Q}v(z) \oplus C\Gamma$, $\max \Psi' = v(z^\dagger) = v(f^\dagger) - v(z)$, and Ψ is cofinal in $\Psi' \setminus \big\{v(z^\dagger)\big\}$.*

(3) *If K has type* (3)*, then $\Gamma' = C\Gamma$, and $\Gamma^{>0}$ is coinitial in $(\Gamma')^{>0}$.*

Proof. Let $(K_\lambda)_{\lambda \leq \mu}$ be a tower on K of power extensions such that $K_\mu = K'$ and $f_\lambda \in K$ for $\lambda < \mu$, with f_λ as in the definition of "tower of power extensions".

Suppose first that K has type (1) or (3). Towards a contradiction, assume that $K_{\lambda+1}$ is obtained from K_λ as in 3(d) or 3(e), for some (necessarily unique) $\lambda < \mu$. Hence $K_{\nu+1}$ is obtained from K_ν as in 3(a), 3(b) or 3(c), for each ordinal $\nu < \lambda$. By induction on ν it follows that $\Gamma_\nu \subseteq C\Gamma$ for $\nu \leq \lambda$. Hence we can take a power product $g \in K_\lambda$ of positive elements of K such that $g \sim b_\lambda$. Since $\Psi_\lambda < v(s_\lambda - b_\lambda^\dagger) < (\mathrm{id} + \psi_\lambda)\big(\Gamma_\lambda^{>0}\big)$, we have

$$s_\lambda - b_\lambda^\dagger \succ (b_\lambda/g)' \asymp (b_\lambda/g)^\dagger = (s_\lambda - g^\dagger) - (s_\lambda - b_\lambda^\dagger),$$

so $s_\lambda - g^\dagger \sim s_\lambda - b_\lambda^\dagger$. Hence $f = y_\lambda/g$ is a power product of positive elements of K with $\Psi < v(f^\dagger) < (\mathrm{id} + \psi)\big(\Gamma^{>0}\big)$, a contradiction. Thus $K_{\lambda+1}$

is obtained from K_λ as in 3(a), 3(b), or 3(c), for all $\lambda < \mu$. Parts (1) and (3) of the lemma now follow.

Suppose $f \in K'$ is as in (2), so Ψ has no maximum. Towards a contradiction, suppose that for all $\lambda < \mu$, $K_{\lambda+1}$ is obtained from K_λ as in 3(a), 3(b) or 3(c). In view of the properties listed in parts (a), (b) and (c) of the remarks after the definition of towers of power extensions, it follows that $\Gamma^{>0}$ is coinitial in $(\Gamma')^{>0}$. Hence $\Psi' < v(f^\dagger) < (\mathrm{id} + \psi')\big((\Gamma')^{>0}\big)$. But $v(f^\dagger) \in \Psi'$ if $f \not\asymp 1$, and $v(f^\dagger) = v(f') \in (\mathrm{id} + \psi')\big((\Gamma')^{>0}\big)$ if $f \asymp 1$, a contradiction in both cases. Thus $K_{\lambda+1}$ is obtained from K_λ as in 3(d) or 3(e) for a unique $\lambda < \mu$. Note that $\Gamma^{>0}$ is coinitial in $\Gamma_\lambda^{>0}$, and that $\Gamma_\lambda \subseteq C\Gamma$.

Suppose that $K_{\lambda+1}$ is obtained from K_λ as in 3(d). Then $v(f^\dagger)$ and $v(z_\lambda^\dagger)$ are both elements of Γ_λ strictly between Ψ_λ and $(\mathrm{id} + \psi_\lambda)(\Gamma_\lambda^{>0})$, so $v(f^\dagger) = v(z_\lambda^\dagger) = \max \Psi_{\lambda+1} = \max \Psi' < (\mathrm{id} + \psi)\big(\Gamma^{>0}$. We also have $\Gamma' = C\Gamma$. Hence

$$\Psi' = \big\{v(g^\dagger) : g \in K' \text{ is a power product of elements of } K^{>0},\ v(g) \neq 0\big\}.$$

Therefore $\Psi' \subseteq \Gamma$, which in combination with $\Psi' < (\mathrm{id} + \psi)\big(\Gamma^{>0}\big)$ yields that Ψ is cofinal in $\Psi' \setminus \big\{v(f^\dagger)\big\}$, as claimed.

Finally, suppose that $K_{\lambda+1}$ is obtained from K_λ as in 3(e). Put $z := z_\lambda \in K'$, so $\Gamma_{\lambda+1} = \mathbb{Z}v(z) \oplus \Gamma_\lambda$ with $0 < v(z) < \Gamma_\lambda^{>0}$. Then $v(f^\dagger)$ and $v(z^\dagger)$ are both elements of Γ_λ strictly between Ψ_λ and $(\mathrm{id} + \psi_\lambda)(\Gamma_\lambda^{>0})$, hence $v(f^\dagger) = v(z^\dagger)$. Also, if $\lambda < \nu < \mu$, then $K_{\nu+1}$ is obtained from K_ν as in 3(a), 3(b) or 3(c), with $f_\nu \in K$. It follows that

$$\max \Psi' = \max \Psi_{\lambda+1} = v(z^\dagger) = v(f^\dagger) - v(z) \notin \Gamma_\lambda, \quad \Gamma' = \mathbb{Q}v(z) \oplus C\Gamma.$$

In view of $\psi'\big((C\Gamma)^*\big) \subseteq \Gamma$, this yields $0 < v(z) < (C\Gamma)^{>0}$. Thus

$$\Psi' \setminus \big\{v(z^\dagger)\big\} = \psi'\big((C\Gamma)^*\big) \subseteq \Gamma, \quad \Psi' \setminus \big\{v(z^\dagger)\big\} < v(f^\dagger).$$

Hence Ψ is cofinal in $\Psi' \setminus \big\{v(z^\dagger)\big\}$. □

In part (2) of this lemma we have $f \not\asymp 1$ in case (a), and $f \asymp 1$ in case (b). The lemma shows that if K has type (1) or (2), then K' has type (1). Moreover:

Lemma 8.9. *If K has type* (3)*, then K' also has type* (3).

Proof. Suppose K has type (3). Then Ψ' has no largest element, and $\Gamma' = C\Gamma$, by the previous lemma. Towards a contradiction, assume K' has type (2). Take a power product $f \in L$ of positive elements of K' such that $\Psi' < v(f^\dagger) < (\mathrm{id} + \psi')\big((\Gamma')^{>0}\big)$. Lemma 8.6 yields a power product $g \in K'$ of positive elements of K such that $f = g(1+\varepsilon)$ with $\varepsilon \preceq \delta \prec 1$, $\delta \in K'$. Then

$$v(f^\dagger - g^\dagger) = v(\varepsilon') \geq v(\delta') > v(f^\dagger),$$

so $f^\dagger \sim g^\dagger$, and thus $\Psi < v(g^\dagger) < (\mathrm{id} + \psi)\big(\Gamma^{>0}\big)$, a contradiction. □

We are now in a position to prove the main result of this subsection:

Proposition 8.10. *The H-field K has two closures under powers, non-isomorphic over K, if and only if K has type* (2)*.*

Proof. The backward direction follows from the treatment of type (2) in the proof of Lemma 8.8.

Suppose that K is not of type (2). By induction on n we define an increasing sequence

$$K^{(0)} \subseteq K^{(1)} \subseteq \cdots \subseteq K^{(n)} \subseteq \cdots$$

of H-subfields of L as follows: put $K^{(0)} := K$, and assuming that $K^{(n)} \supseteq K$ has already been defined as an H-subfield of L, put

$$K^{(n+1)} \quad := \quad \text{real closure inside } L \text{ of } K^{(n)}\big(f^c : 0 < f \in K^{(n)}, c \in C\big).$$

The union of this sequence is an H-subfield of L which is closed under powers and contains K; thus $L = \bigcup_n K^{(n)}$, by Lemma 8.2. By Lemmas 8.8 and 8.9 it follows inductively that each $K^{(n)}$ has the same type as K. By the proof of Lemma 8.8, L is the only closure under powers of K, up to K-isomorphism. □

Corollary 8.11. *The value group of a closure under powers of K is as follows:*

1. *If K has type* (1)*, then K has only one closure under powers L, up to K-isomorphism, and $\Gamma_L = C\Gamma$, $\max \Psi_L = \max \Psi$.*
2. *If K has type* (2)*, and $c_1, \dots, c_n \in C$ and $f_1, \dots, f_n \in K^{>0}$ are such that*
$$\Psi < \beta := v(c_1 f_1^\dagger + \cdots + c_n f_n^\dagger) < (\mathrm{id} + \psi)\big(\Gamma^{>0}\big),$$
then in one closure under powers L_1 of K we have
$$\Gamma_{L_1} = C\Gamma, \quad \max \Psi_{L_1} = \beta,$$
and Ψ is cofinal in $\Psi_{L_1} \setminus \{\beta\}$, and in another closure under powers L_2 of K there exists $z \in L_2$ such that $0 < v(z) < (C\Gamma)^{>0}$ and
$$\Gamma_{L_2} = C\Gamma \oplus Cv(z), \quad \max \Psi_{L_2} = v(z^\dagger) = \beta - v(z),$$
and Ψ is cofinal in $\Psi_{L_2} \setminus \big\{v(z^\dagger)\big\}$.
3. *If K is of type* (3)*, then K has only one closure under powers L, up to K-isomorphism, and $\Gamma_L = C\Gamma$, $\Gamma^{>0}$ is coinitial in $\Gamma_L^{>0}$.*

Corollary 8.12. *Suppose Ψ is a singleton. Then K has exactly one closure under powers L, up to K-isomorphism, and $\Gamma_L = C\Gamma$, $\Psi_L = \Psi$. (In particular,* $\dim_C \Gamma_L = 1$.*)* □

Remarks. The results in this section are close to those in [26] on Hardy fields. But Hardy fields have only one "Hardy field" closure under powers, and this fact obscures issues that come to light in the setting of H-fields.

Our Lemma 8.9 and the remarks preceding it are analogous to Corollary 1 of the main theorem in [26]. One particular inference ("therefore ...", last two lines on p. 834) in the proof of that Corollary 1 seems problematic; we needed Lemma 8.6 above to get around this.

9. Constant Field Extension

In this section we show that H-fields are well-behaved under constant field extensions.

Let K be a differential-valued field, and L an extension field of K with a subfield $D \supseteq C$ such that K and D are linearly disjoint over C and $L = K(D)$. Then Theorem 3 of [24] says that there is a unique derivation on L that extends the one of K and is trivial on D; this derivation has D as its constant field. There is also a unique valuation of L that extends the valuation of K and is trivial on D; this valuation of L has the same value group as K, and is a differential valuation of L with respect to the derivation of L that extends the one of K and is trivial on D. In the proposition below we consider L as being equipped with this derivation and valuation.

Proposition 9.1. *Let K and D also be equipped with orderings that make K an H-field for the given derivation and valuation of K, and D an ordered field extension of C. Then there is a unique ordering of L extending the orderings of K and D in which the valuation ring of L is convex. With this ordering L is an H-field for the derivation and valuation of L.*

Proof. We first note that as a consequence of the proof of Theorem 3 in [24] each $f \in K[D] \setminus \{0\}$ is of the form $f = \lambda_1 a_1 + \cdots + \lambda_n a_n$ with all $\lambda_i \in D^\times$, and all $a_i \in K^\times$, with $v(a_1) < v(a_i)$ for all $i = 2, \ldots, n$. We may of course also assume here that $a_1 > 0$. For any such expression of f we have $v(f) = v(a_1)$. Next we observe that for $a, b \in K$ we have $v(a) = v(b)$ if and only if a and b have the same C-archimedean class, that is, there exist $\lambda, \mu \in C^{>0}$ such that $\lambda a \leq b \leq \mu a$. It follows that K as ordered vector space over C is a Hahn space in the sense of [1]. (See also §7.) Hence the D-linear isomorphism $K \otimes_C D \cong K[D]$ given by $a \otimes \lambda \mapsto \lambda a$ implies by Proposition 2.2 in [1] that $K[D]$ can be made in a unique way into an ordered vector space over D such that for any $f = \sum_i \lambda_i a_i$ as above (with $a_1 > 0$) we have $f > 0$ if and only if $\lambda_1 > 0$. It is easily checked that this ordering is compatible with multiplication: if $0 < f, g \in K[D]$, then $0 < fg$. This ordering extends uniquely to the fraction field L of $K[D]$ to make it an ordered field. Clearly this ordering on L is the only candidate for meeting the requirements. It does extend the orderings of K and D, and it is an easy exercise to check that $\mathcal{O}_L$ is the convex hull of D in L. The last statement of the proposition is now clear from remark (2) following Lemma 3.1 in [2]. $\square$

10. Completing H-Fields

We recall that any valued field K can be *completed*: it is dense (with respect to the valuation topology) in a valued field extension K^c such that for each valued field extension $K \subseteq L$ with K dense in L there is a unique valued field embedding $L \to K^{\mathrm{c}}$ that is the identity on K. These properties determine K^{c} up to unique valued field isomorphism over K, and K^{c} is called the **completion of** K. We note that $K^{\mathrm{c}}|K$ is an immediate extension. See [23] for these facts.

Lemma 10.1. *Suppose the derivation of the valued differential field K is continuous. Then there is a unique continuous derivation on K^{c} that extends the derivation of K. Moreover, if K is differential-valued, then K^{c} is differential-valued as well.*

Proof. The derivation of K being additive, it is even uniformly continuous (with respect to the uniform structure which K has as an additive topological group). Thus it extends uniquely to a continuous map $K^{\mathrm{c}} \to K^{\mathrm{c}}$, and this map is a derivation.

Let K be differential-valued, and let $a \in K^{\mathrm{c}}$, $v(a) > 0$. In order to prove that K^{c} is differential-valued, it suffices to show that then $v(a') > \Psi$, by remark (1) following Lemma 3.1 and Lemma 3.4 in [2]. Choose $0 \neq b \in \mathfrak{m}$ with $v\big((a-b)'\big) > \Psi$. Since $v(b') > \Psi$, it follows that $v(a') > \Psi$ as required. $\square$

Suppose K is as in the lemma. Consider K^{c} as the valued differential field whose derivation is the unique continuous derivation on K^{c} that extends the one of K. If $K \subseteq L$ is a valued differential field extension such that K is dense in L and the derivation of L is continuous with respect to the valuation topology, then the unique valued field embedding $L \to K^{\mathrm{c}}$ that is the identity on K is a differential field embedding.

Assume moreover that K is equipped with an ordering such that the valuation ring $\mathcal{O}$ of K is a convex subring of K. The ordering on K extends uniquely to an ordering on K^{c} such that the valuation ring of K^{c} is convex. The ordered field K is dense in K^{c} with respect to the order topology, and if $\mathcal{O} \neq K$, then for each ordered field extension $K \subseteq L$ with K dense in L there is a unique ordered field embedding $L \to K^{\mathrm{c}}$ that is the identity on K. If K is real closed, then so is K^{c}. We refer to [23] for proofs of these and some other facts about K^{c}.

For the rest of this section we let K be an H-field. Then its derivation is continuous with respect to the order topology, and extends uniquely to a continuous derivation on K^{c}. It is worth noting that if $K \subseteq L$ is an H-field extension and B is a subset of L such that for each $b \in B$ and $\varepsilon \in K^{>0}$ there is $a \in K$ with $|a-b| < \varepsilon$, then K is dense in the ordered differential subfield $K\langle B\rangle$ of L.

Lemma 10.2. *The ordered differential field K^{c} is an H-field. If K is Liouville closed, so is K^{c}.*

Proof. The first statement follows from Lemma 10.1 and remark (2) after Lemma 3.1 in [2]. Suppose K is Liouville closed; we want to show that K^{c} is Liouville closed. For this, it is enough to show, by the remark preceding the lemma: if $a \in K^{\mathrm{c}}$ and y, z are elements of a Liouville closure of K^{c} with

$y' = a$ and $z \neq 0$, $z^\dagger = a$, then for all $\varepsilon \in K^{>0}$ there exist $y_0, z_0 \in K$ such that $|y_0 - y| < \varepsilon$ and $|z_0 - z| < \varepsilon$. We may suppose $\varepsilon \in \mathfrak{m}$; we find $y_0 \in K$ with $(y_0 - y)' = y_0' - a \prec \varepsilon'$, hence $|y_0 - y| < \varepsilon$. Similarly, assuming that $\varepsilon \prec z$, we choose $z_0 \in K^\times$ such that $(z_0/z)^\dagger = b^\dagger - a \prec (\varepsilon/z)'$. Then $z \asymp z_0$, and by multiplying z_0 by a suitable non-zero constant, we may assume $z \sim z_0$. So $(z_0/z - 1)' = (z_0/z)' \prec (\varepsilon/z)'$. It follows that $z_0/z - 1 \prec \varepsilon/z$ and hence $|z_0 - z| < \varepsilon$ as required. □

Example. Recall (from [12], p. 69) the construction of the field $\mathbb{R}((t))^{\mathrm{E}}$ of *exponential series* as a subfield of the series field $\mathbb{R}((G^{\mathrm{E}}))$. Here G^{E} denotes the ordered abelian subgroup of $(\mathbb{R}((t))^{\mathrm{E}})^{>0}$ consisting of the *E-monomials* of $\mathbb{R}((t))^{\mathrm{E}}$. The valued field $\mathbb{R}((t))^{\mathrm{E}}$ is dense in the maximally valued field $\mathbb{R}((G^{\mathrm{E}}))$, hence $\mathbb{R}((G^{\mathrm{E}}))$ is the completion of $\mathbb{R}((t))^{\mathrm{E}}$. By the results above, the derivation on $\mathbb{R}((t))^{\mathrm{E}}$ extends uniquely to a continuous derivation on $\mathbb{R}((G^{\mathrm{E}}))$. With this derivation and its usual ordering, $\mathbb{R}((G^{\mathrm{E}}))$ is an H-field extension of $\mathbb{R}((t))^{\mathrm{E}}$.

Corollary 10.3. *Given a Liouville closure L of K, the following are equivalent:*

(1) *K is dense in L.*
(2) *Some H-subfield of K^{c} containing K is Liouville closed.*
(3) *K^{c} is Liouville closed.*
(4) *L is an immediate extension of K.*
(5) *$\Gamma_L = \Gamma$.*

Proof. The implications (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) are either clear, or obvious using the preceding results. So it just remains to show (5) ⇒ (1). Let $\Gamma = \Gamma_L$. Then Γ is divisible, since L is real closed. Assume for a contradiction that K is *not* dense in L. By changing the derivation ∂ on L to $a\partial$, for a suitable $a \in K^{>0}$, we reduce to the case that $0 \in \Psi$. Let $f \in L$ and $\varepsilon \in L^{>0}$. We want to show that $K \cap (f - \varepsilon, f + \varepsilon) \neq \emptyset$. Since $\Gamma = \Gamma_L$, $K^{>0}$ is coinitial in $L^{>0}$, so by decreasing ε if necessary, we may assume $\varepsilon \in K^{>0}$. Multiplying f by $1/\varepsilon$, we can further reduce to the case that $\varepsilon = 1$. Let $g, h \in L^\times$ be such that $g^\dagger = f - 1$, $h^\dagger = f + 1$. We claim that $g \prec h$. Otherwise $g \asymp h$ by Lemma 1.4 in [2], hence

$$1 \asymp h^\dagger - g^\dagger = (h/g)^\dagger \asymp (h/g)' \prec 1,$$

a contradiction. So indeed $g \prec h$. Choose $a \in K^\times$ with $g \prec a \prec h$. (Such a exists since $\Gamma = \Gamma_L$ is divisible, hence densely ordered.) Hence $f - 1 = g^\dagger < a^\dagger < h^\dagger = f + 1$, and thus $a^\dagger \in K \cap (f - 1, f + 1)$. □

11. Constructing H-Fields for given H-Couples

Let (V, ψ) be an H-couple over the scalar field $\boldsymbol{k}$. We shall construct an H-field K with constant field $\boldsymbol{k}$ which is closed under powers and has (V, ψ) as its associated H-couple, under the following additional assumption:

($*$) The $\boldsymbol{k}$-vector space V has a basis $(e_i)_{i\in I}$ consisting of positive elements, such that $[e_i]_{\boldsymbol{k}} \neq [e_j]_{\boldsymbol{k}}$ for all $i \neq j$ in I.

By a theorem of Brown [8], assumption ($*$) is satisfied if V is countably generated as $\boldsymbol{k}$-vector space. It is also satisfied if $[V^*]_{\boldsymbol{k}}$ is well-ordered (equivalently, $\Psi := \psi(V^*)$ is reverse well-ordered): take an injective enumeration $\big([e_i]_{\boldsymbol{k}}\big)_{i\in I}$ of $[V^*]_{\boldsymbol{k}}$ such that each e_i is positive; then (e_i) is a basis as in ($*$).

For the remainder of this section (V,ψ) denotes an H-couple over $\boldsymbol{k}$, and $(e_i)_{i\in I}$ a basis of V as in ($*$). We say that e_j **occurs** in the vector $v = \sum_{i\in I} \lambda_i e_i$ (all $\lambda_i \in \boldsymbol{k}$, and $\lambda_i \neq 0$ for only finitely many i) just in case $\lambda_j \neq 0$.

Let t^V be a multiplicative copy of the (additive) ordered abelian group V, ordered such that $v \mapsto t^v \colon V \to t^V$ is an order-reversing isomorphism. We consider formal sums $f = \sum_{v\in V} a_v t^v$ with coefficients a_v in $\boldsymbol{k}$. For such f we define its support as $\operatorname{supp} f := \big\{v \in V : a_v \neq 0\big\}$. Let

$$K := \boldsymbol{k}(\!(t^V)\!) = \big\{f : \operatorname{supp} f \text{ is well-ordered}\big\}$$

be the field of generalized power series with coefficients in $\boldsymbol{k}$ and exponents in V, considered as an ordered valued field with value group V in the usual way (see [12], §1). (In particular, $0 < t := t^1 < \boldsymbol{k}^{>0}$.) We write $v(a)$ for the valuation of $a \in K$. The valuation ring of this valuation is $\mathcal{O} = \big\{f \in K : \operatorname{supp} f \subseteq V^{\geq 0}\big\}$, with maximal ideal $\mathfrak{m} = \big\{f \in K : \operatorname{supp} f \subseteq V^{>0}\big\}$.

If $f_j \in K$ for all $j \in J$ for some (possibly infinite) set J, we say that **the sum $\sum_{j\in J} f_j$ exists (in K)** if the following two conditions are met:

(1) For each $v \in V$ there are only finitely many $j \in J$ with $v \in \operatorname{supp} f_j$.
(2) The union $\bigcup_{j\in J} \operatorname{supp} f_j$ is well-ordered in V.

If these two requirements are satisfied, we can associate with the family $(f_j)_{j\in J}$ a well-defined element $\sum_{j\in J} f_j$ of K.

We introduce a derivation on K by first considering an element t^v, $v \in V$, with $v = \sum_{i\in I} \lambda_i e_i \in V$ ($\lambda_i \in \boldsymbol{k}$ for all $i \in I$, and $\lambda_i = 0$ for all but finitely many i), and setting

$$(t^v)' := -\sum_{i\in I} \lambda_i t^{\psi(e_i)+v}, \quad v \in V.$$

(In particular, $(t^{e_i})'/t^{e_i} = -t^{\psi(e_i)}$ for all $i \in I$. If the distinguished positive element 1 is among the basis elements e_i, then $t' = (t^1)' = -t^2$ and $(t^{-1})' = 1$.) Next, for $f = \sum_{v\in V} a_v t^v \in K$, we put

$$f' := \sum_{v\in V} a_v (t^v)', \tag{11.1}$$

In order for (11.1) to make sense, we have to show:

(1) For each $w \in V$ there are only finitely many $v \in \operatorname{supp} f$ such that $w = \psi(e_i) + v$ for some basis vector e_i occurring in v.
(2) The set of all $w = \psi(e_i) + v$, where $v \in \operatorname{supp} f$ and e_i occurs in v, is well-ordered.

For (1), suppose $w = \psi(e_i)+u = \psi(e_j)+v$ for elements $u < v$ in $\operatorname{supp} f$, with e_i, e_j occurring in u and v, respectively. Then $\psi(e_i) - \psi(e_j) = v - u > 0$, so $[e_i]_{\boldsymbol{k}} < [e_j]_{\boldsymbol{k}}$ and $[v-u]_{\boldsymbol{k}} = \big[\psi(e_i) - \psi(e_j)\big]_{\boldsymbol{k}} < [e_i - e_j]_{\boldsymbol{k}} = [e_j]_{\boldsymbol{k}}$. Hence e_j occurs in u. So if we have a strictly increasing sequence $(v_n)_{n\in\mathbb{N}}$ in $\operatorname{supp} f$ and a sequence $(i_n)_{n\in\mathbb{N}}$ in I such that for all n, e_{i_n} occurs in v_n and $\psi(e_{i_n}) + v_n = \psi(e_{i_{n+1}}) + v_{n+1}$, then all e_{i_n} occur in v_0, which is impossible as only finitely many e_i occur in v_0. This proves (1). For (2), suppose for a contradiction that $(i_n)_{n\in\mathbb{N}}$ is a sequence in I and $(v_n)_{n\in\mathbb{N}}$ a sequence in $\operatorname{supp} f$ such that e_{i_n} occurs in v_n, for all n, and

$$\psi(e_{i_0}) + v_0 > \psi(e_{i_1}) + v_1 > \cdots .$$

Passing to a subsequence and using that $\operatorname{supp} f$ is well-ordered, we reduce to the case that $v_n \le v_{n+1}$ for all n. Hence $0 \le v_n - v_0 < \psi(e_{i_0}) - \psi(e_{i_n})$, so $[v_n - v_0]_{\boldsymbol{k}} \le \big[\psi(e_{i_0}) - \psi(e_{i_n})\big]_{\boldsymbol{k}} < [e_{i_0} - e_{i_n}]_{\boldsymbol{k}} = [e_{i_n}]_{\boldsymbol{k}}$. Thus each e_{i_n} occurs in v_0. This is impossible as the e_{i_n} are distinct. (This is because $0 \le v_{n+1} - v_n < \psi(e_{i_n}) - \psi(e_{i_{n+1}})$ for all n.) This concludes the proof of (2).

Lemma 11.1. *The map $f \mapsto f' \colon K \to K$ is a derivation on K, and makes K into an H-field with constant field $\boldsymbol{k}$ and associated asymptotic couple (V,ψ).*

Proof. It is easy to check that the map is a derivation on K, trivial on $\boldsymbol{k}$.

Let $f = \sum_{u\in V} a_u t^u \in K^\times$ with $v(f) \neq 0$. We claim that $v(f') = v(f) + \psi\big(v(f)\big)$. Every non-zero term $a_u t^u$ in f with $u \neq 0$ contributes $a_u (t^u)'$ to f', and $v\big(a_u(t^u)'\big) = u + \psi(u)$, see above. As $u+\psi(u)$ is strictly increasing in $u \neq 0$, it follows that if $u_0 = v(f) = \min(\operatorname{supp} f)$, then $v(f') = u_0 + \psi(u_0) = v(f) + \psi\big(v(f)\big)$.

Next assume $f = \sum_{u\in V} a_u t^u \in K^\times$ is such that $f' = 0$. After subtracting from f its constant term a_0, the same argument as before shows that then $f = 0$. Thus $\boldsymbol{k}$ is exactly the constant field of the derivation. It is also clear that $\mathcal{O} = \boldsymbol{k} \oplus \mathfrak{m}$. Let $f = \sum_{v\in V} a_v t^v \in K$ with $f > C = \boldsymbol{k}$. We have to show that then $f' > 0$. We have already seen that $v(f') = v_0 + \psi(e_{i_0})$, where v_0 and i_0 are as before, so

$$f' = -a_{v_0} \lambda_{i_0} t^{v_0 + \psi(e_{i_0})}(1+\varepsilon) \quad \text{for some } \varepsilon \in \mathfrak{m}.$$

Since $v(f) < 0$ and $f > 0$, we have $\lambda_{i_0} < 0$ and $a_{v_0} > 0$, hence $f' > 0$. So K is an H-field, with associated asymptotic couple (V,ψ). $\square$

Remark. Let Γ be a subgroup of V such that $\psi(V^*) \subseteq \Gamma$. Then $\boldsymbol{k}(\!(t^\Gamma)\!)$ is an H-subfield of $K = \boldsymbol{k}(\!(t^V)\!)$ with asymptotic couple $(\Gamma, \psi|\Gamma^*)$. Suppose $(\mathrm{id}+\psi)(V^*) = V$. Then $(\mathrm{id}+\psi)(\Gamma^*) = \Gamma$, hence the H-field $\boldsymbol{k}(\!(t^\Gamma)\!)$ is closed under integration. (By Remark (3) after Lemma 5.1 in [2].)

We now show that the H-field K is closed under powers, so that we can then speak of the H-couple corresponding to K.

The proof of the next lemma is straightforward:

Lemma 11.2. *Suppose $\sum_{j\in J} f_j$ exists in K. Then the sum $\sum_{j\in J} f_j'$ also exists in K, and $\left(\sum_{j\in J} f_j\right)' = \sum_{j\in J} f_j'$.* $\square$

For a formal power series $F \in \boldsymbol{k}[[X_1,\dots,X_n]]$, let $\frac{\partial F}{\partial X_i}$ denote the formal partial derivative of F with respect to the variable X_i, $1 \le i \le n$. The previous lemma and Neumann's Lemma (see [12]) imply:

Corollary 11.3. *Let $F \in \boldsymbol{k}[[X_1,\dots,X_n]]$ and $\varepsilon = (\varepsilon_1,\dots,\varepsilon_n) \in \mathfrak{m}\times\cdots\times\mathfrak{m}$. Then*

$$\big(F(\varepsilon)\big)' = \sum_{i=1}^{n} \frac{\partial F}{\partial X_i}(\varepsilon)\varepsilon_i'.$$

$\square$

Let $f \in K^\times$. Write $f = at^v(1+\varepsilon)$, where $a \in \boldsymbol{k}^\times$, $v = v(f)$, and $\varepsilon \in \mathfrak{m}$. For $c \in \boldsymbol{k}$, we consider the formal power series

$$(1+X)^c := \sum_{n=0}^{\infty} \frac{c(c-1)\cdots(c-n+1)}{n!} X^n \in \boldsymbol{k}[[X]].$$

By Neumann's Lemma, the map $\mathfrak{m} \to 1+\mathfrak{m}$ given by

$$\varepsilon \mapsto (1+\varepsilon)^c := \sum_{n=0}^{\infty} \frac{c(c-1)\cdots(c-n+1)}{n!} \varepsilon^n$$

is well-defined. Let $g := t^{cv}(1+\varepsilon)^c \in K$. Since $\big((1+\varepsilon)^c\big)' = c(1+\varepsilon)^{c-1}\varepsilon'$, by the corollary, and $(t^{cv})' = ct^{(c-1)v}(t^v)'$, we get $g'/g = cf'/f$. We have shown:

Proposition 11.4. *The H-field K is closed under powers, and its associated H-couple is (V,ψ).* $\square$

Consider $\boldsymbol{k}(t^V)$, the subfield of $K = \boldsymbol{k}((t^V))$ generated by the (multiplicative) group t^V over $\boldsymbol{k}$. Then $\boldsymbol{k}(t^V)$ carries the induced ordering and valuation. The derivation on $\boldsymbol{k}((t^V))$ maps $\boldsymbol{k}(t^V)$ into itself and thus restricts to a derivation on $\boldsymbol{k}(t^V)$. Since $\boldsymbol{k}((t^V))$ is an immediate extension of $\boldsymbol{k}(t^V)$ with the same constant field $\boldsymbol{k}$, we get:

Corollary 11.5. *The ordered differential field $\boldsymbol{k}(t^V)$ is an H-subfield of K with constant field $\boldsymbol{k}$ and associated asymptotic couple (V,ψ).* $\square$

Let $\boldsymbol{k}'$ be an ordered field extension of $\boldsymbol{k}$. We identify as usual V with a $\boldsymbol{k}$-linear subspace of the $\boldsymbol{k}'$-vector space $V' = V \otimes_{\boldsymbol{k}} \boldsymbol{k}'$. There is a unique linear ordering on V' extending the one on V which makes V' into an ordered vector space over $\boldsymbol{k}'$ such that $[V']_{\boldsymbol{k}'} = [V]_{\boldsymbol{k}}$ (Proposition 2.2 in [1]). With this ordering, V' is a Hahn space over $\boldsymbol{k}'$ which satisfies $(*)$. Moreover, there is a unique extension of ψ to a map $\psi'\colon (V')^* \to V$ such that (V',ψ') is an H-couple over $\boldsymbol{k}'$. (Lemma 3.1 in [1].) Let $K' = \boldsymbol{k}'((t^{V'}))$, equipped with the ordering and derivation defined above as for K; then K' is an H-field closed under powers with H-couple (V',ψ').

Note that the derivation of K' maps the subfield $\boldsymbol{k}'((t^V))$ of K' into itself. Hence $\boldsymbol{k}'((t^V))$ with the derivation and ordering induced from K' is a real closed H-subfield of K' with constant field $\boldsymbol{k}'$ and asymptotic couple (V,ψ). We now apply these remarks to obtain an example of an H-field (with real closed constant field) which has type (2), in the sense of Section 8:

Example. Suppose $w \in V$ satisfies $\Psi < w < (\mathrm{id}+\psi)\big(V^{>0}\big)$. (See [1], (3.3) for an example of an H-couple (V,ψ) with V satisfying $(*)$ and containing an element w with this property.) By Lemma 3.7 in [1] we have

$$\Psi = \Psi' < w < (\mathrm{id}+\psi')\big((V')^{>0}\big).$$

Assume in addition that $\boldsymbol{k}'$ is real closed and $\boldsymbol{k}' \neq \boldsymbol{k}$. Then $V \cap cV = \{0\}$ for all $c \in \boldsymbol{k}' \setminus \boldsymbol{k}$. Now choose $0 \neq f \in \boldsymbol{k}'((t^V))$ with $f \not\asymp 1$ and $c \in \boldsymbol{k}' \setminus \boldsymbol{k}$ arbitrarily, and put $z = t^w \in \boldsymbol{k}'((t^V))$.

Claim. $v(cf^\dagger - b^\dagger) < v(z)$ for all non-zero $b \in \boldsymbol{k}'((t^V))$.

To see this, let $0 \neq b \in \boldsymbol{k}'((t^V))$. Note that $bf^c \not\asymp 1$ in K', since otherwise $0 \neq cv(f) = -v(b) \in V \cap cV$. Since $cf^\dagger - b^\dagger = (bf^c)^\dagger$ in K', we get $v(cf^\dagger - b^\dagger) = v\big((bf^c)^\dagger\big) < v(z)$ as desired. The claim and Lemma 8.7 imply that $\boldsymbol{k}'((t^V))$ has an H-field extension of type (2).

We conclude with an example to be used in the next section:

Example. Let $\mathfrak{L}$ be the multiplicative subgroup of $\mathbb{R}((x^{-1}))^{\mathrm{LE}}$ generated by the real powers ℓ_n^a $(a \in \mathbb{R})$ of the iterated logarithms $\ell_n = \log_n x$ of x, and let $\mathfrak{L}^{\mathbb{Q}}$ be its subgroup generated by the rational powers ℓ_n^a $(a \in \mathbb{Q})$. We equip the real closed field $\mathbb{R}((\mathfrak{L}))$ with the derivation that is trivial on $\mathbb{R}$, sends each real power ℓ_n^a to $a\ell_n^{a-1}\big(\ell_0\ell_1\cdots\ell_{n-1}\big)^{-1}$ (in particular $x' = 1$), and commutes with infinite summation. Note that $\mathbb{R}((\mathfrak{L}^{\mathbb{Q}}))$ is closed under this derivation.

Claim. $\mathbb{R}((\mathfrak{L}))$ and $\mathbb{R}((\mathfrak{L}^{\mathbb{Q}}))$ are H-fields closed under integration.

To prove this we make the $\mathbb{R}$-vector space

$$V = \bigoplus_{n\in\mathbb{N}} \mathbb{R}e_n$$

into an ordered vector space over the ordered field $\mathbb{R}$ such that $e_n > 0$ and $[e_{n+1}]_{\mathbb{R}} < [e_n]_{\mathbb{R}}$ for all n. We define $\psi\colon V^* \to V$ by making it constant on each archimedean class, and setting

$$\psi(e_n) := e_0 + e_1 + \cdots + e_n \qquad \text{for all } n.$$

(Hence $\psi(e_0) = e_0$.) One verifies easily that (V,ψ) is an H-couple over the scalar field $\mathbb{R}$, with distinguished positive element $1 = e_0$. (Cf. Example 3.3 in [1].) The set $\Psi = \{e_0+e_1+\cdots+e_n : n \in \mathbb{N}\}$ does not have a supremum in V, hence $(\mathrm{id}+\psi)(V^*) = V$. (To obtain this last equality, use the fact that if (Γ,ψ) is any asymptotic couple of H-type and $\beta \in \Gamma$, then $\beta \notin (\mathrm{id}+\psi)(\Gamma^*)$ if and only if $\beta = \sup\psi(\Gamma^*)$, see p. 554 of [2].) The basis $(e_n)_{n\in\mathbb{N}}$ of the

$\mathbb{R}$-vector space V satisfies the condition $(*)$ above. The H-field $\mathbb{R}((t^V))$ has constant field $\mathbb{R}$, is closed under powers with associated H-couple (V,ψ), and is closed under integration. Consider the divisible subgroup

$$\Gamma = \bigoplus_{n\in\mathbb{N}} \mathbb{Q}e_n$$

of $V = \bigoplus_n \mathbb{R}e_n$. We have $\Psi \subseteq \Gamma$, so $\mathbb{R}((t^\Gamma))$ is a real closed H-subfield of $\mathbb{R}((t^V))$ closed under integration, see the remark following Lemma 11.1. (Its associated asymptotic couple is $(\Gamma, \psi|\Gamma^*)$.)

Now observe that we have a unique isomorphism $\mathbb{R}((t^V)) \to \mathbb{R}((\mathfrak{L}))$ of ordered differential fields which is the identity on $\mathbb{R}$, sends t^{ae_n} to $1/\ell_n^a$ for all $a \in \mathbb{R}$ and all n, and commutes with infinite summation. This isomorphism maps $\mathbb{R}((t^\Gamma))$ onto $\mathbb{R}((\mathfrak{L}^{\mathbb{Q}}))$.

12. Gaps in H-Fields

In Example 12.7 below we provide the missing details concerning [2], p. 583, *Example*. This section deals with *gap creation*, a troubling phenomenon for the model theory of H-fields. We show how "gap creators" arise as pseudo-limits.

Recall from §6 in [2] that a **gap** in a pre-H-field K is an element γ of its value group Γ such that $\Psi < \gamma < (\mathrm{id}+\psi)\big(\Gamma^{>0}\big)$. It was shown in [2] that if K is an H-field with a gap, then K has exactly two Liouville closures, up to isomorphism over K. We record some other basic facts on gaps:

Lemma 12.1. *Let K be a pre-H-field.*

1. *K has at most one gap.*
2. *If Ψ has a largest element, then K has no gap.*
3. *If every element of K has an anti-derivative in K, then K has no gap.*
4. *If K has no gap, then the smallest H-field $\widehat{K}$ extending K (as defined in Section 4 of [2]) also has no gap.*
5. *Let L be a pre-H-field extension of K such that $\Gamma^{>0}$ is coinitial in $\Gamma_L^{>0}$. Then a gap in K remains a gap in L.*
6. *A gap in K remains a gap in the real closure of K.*
7. *If K is a directed union of pre-H-subfields that have a smallest comparability class, then K has no gap.*

Proof. Parts (1)–(3) are from Sections 2 and 6 in [2]. Part (4) follows from (2) by [2], Corollary 4.5. For (5), note that the set $(\mathrm{id}+\psi)\big(\Gamma^{>0}\big)$ is coinitial in $(\mathrm{id}+\psi_L)\big(\Gamma_L^{>0}\big)$ and Ψ is cofinal in Ψ_L; so if $\gamma \in \Gamma$ satisfies $\Psi < \gamma < (\mathrm{id}+\psi)\big(\Gamma^{>0}\big)$, then $\Psi_L < \gamma < (\mathrm{id}+\psi_L)\big(\Gamma_L^{>0}\big)$. Part (6) follows from (5): if K has a gap, then $[\Gamma^*]$ has no smallest element, hence $\Gamma^{>0}$ is coinitial in $(\mathbb{Q}\Gamma)^{>0}$. To prove (7), reduce to the case that K has a smallest comparability class, that is, Ψ has a largest element; then apply (1). $\square$

Part of the next result was announced at the end of [2].

Corollary 12.2. *No differentially algebraic pre-H-field extension of the Hardy field $\mathbb{R}(x)$ can have a gap. No differentially algebraic pre-H-field extension of the H-field $\mathbb{R}((x^{-1}))^{\mathrm{LE}}$ can have a gap.*

Proof. The H-fields $\mathbb{R}(x)$ and $\mathbb{R}((x^{-1}))^{\mathrm{LE}}$ satisfy the hypothesis of part (7) of the above lemma, and thus the differentially algebraic pre-H-field extensions of these H-fields satisfy this hypothesis as well, by Lemma 2.1. $\square$

Gap creation. The troublesome gaps from [2] arise in a special way, and to study this situation we assume in the rest of this section:

K is a real closed H-field closed under asymptotic integration.

Thus $C \neq K$, K has no gap and Ψ has no maximum.

Let $s \in K$. We say that s **creates a gap over** K if adjoining a "logarithmic antiderivative" of s can introduce a gap, that is, $v(y)$ is a gap in $L = K(y)$, for some element $y \neq 0$ in some H-field extension of K such that $y^\dagger = s$.

Lemma 12.3. *Suppose $s \in K$ creates a gap over K. Then for every $a \in K^\times$ there exists $\gamma \in \Gamma^*$ such that $v(s - a^\dagger) \leq \psi(\gamma)$.*

Proof. Take y and L as in the definition above, and let $a \in K^\times$. Then $y \not\asymp a$, so $v(s - a^\dagger) = v\big((y/a)^\dagger\big) < (\mathrm{id} + \psi)\big(\Gamma^{>0}\big)$, hence $v(s - a^\dagger) \leq \psi(\gamma)$ for some $\gamma \in \Gamma^*$. $\square$

In particular, a Liouville closed H-field has no gap creator. Suppose $s \in K$ creates a gap over K and y is a non-zero element of an H-field extension of K with $y^\dagger = s$. Then we claim that $L = K(y) \supseteq K$ is an H-field extension, $v(y)$ is a gap in L, $C_L = C$, $\Gamma_L = \Gamma \oplus \mathbb{Z}v(y)$, and $[\Gamma_L] = [\Gamma]$. To see this, note first that $s \neq 0$ and $y \notin K$ by the last lemma, so y is transcendental over K. The claim now follows from this last lemma, and the uniqueness properties in Lemma 5.3 of [2], and subsequent *Remarks*, with s and y replaced by $-s$ and $1/y$ if $s > 0$.

We can detect already in K itself whether $s \in K$ creates a gap over K:

Proposition 12.4. *Let $s \in K$. The following are equivalent:*

1. *s creates a gap over K.*
2. *For each non-zero element y in each H-field extension of K with $y^\dagger = s$, we have $\Psi_L < v(y) < (\mathrm{id} + \psi_L)\big(\Gamma_L^{>0}\big)$, where $L = K(y)$.*
3. *For some non-zero element y in some H-field extension of K with $y^\dagger = s$, we have $\Psi < v(y) < (\mathrm{id} + \psi)\big(\Gamma^{>0}\big)$.*
4. *For each $\varepsilon \in K^\times$ with $\varepsilon \prec 1$, we have $\varepsilon'^\dagger < s < \varepsilon^{\dagger\dagger}$.*

Proof. We already saw that (1) $\Rightarrow$ (2). The implication (2) $\Rightarrow$ (3) is obvious. To prove (3) $\Rightarrow$ (4), let y be as in (3). Then $\varepsilon' \prec y \prec \varepsilon^\dagger$ for $\varepsilon \prec 1$ in $K^\times$, hence $\varepsilon'^\dagger < y^\dagger = s < \varepsilon^{\dagger\dagger}$ for such ε, by Lemma 1.4 in [2].

To prove (4) $\Rightarrow$ (1), assume (4). Take some non-zero y in some H-field extension of K with $y^\dagger = s$. Then $\varepsilon' \preceq y \preceq \varepsilon^\dagger$ for all $\varepsilon \prec 1$ in $K^\times$, by Lemma 1.4 in [2]. Since Ψ does not have a maximum, this yields $\Psi < v(y) <$

$(\mathrm{id}+\psi)\big(\Gamma^{>0}\big)$. In particular, $v(y) \notin \Gamma$, hence $\Gamma_L = \Gamma \oplus \mathbb{Z}v(y)$ where $L := K(y)$. The proof of Lemma 4.5 in [1] with $b := v(y)$ then gives $[\Gamma_L] = [\Gamma]$. So $\Psi = \Psi_L$ and $\Gamma^{>0}$ is coinitial in $\Gamma_L^{>0}$, hence $\Psi_L < v(y) < (\mathrm{id}+\psi_L)\big(\Gamma_L^{>0}\big)$, that is, $v(y)$ is a gap in L. $\square$

Corollary 12.5. *Suppose $s \in K$ creates a gap over K.*

(1) *An element $r \in K$ creates a gap over K if and only if $v(s-r) > \Psi$.*
(2) *If $L \supseteq K$ is a real closed H-field without a gap such that $\Gamma^{>0}$ is coinitial in $\Gamma_L^{>0}$, then s creates a gap over L.*

Proof. For (1), suppose first that $r \in K$ creates a gap over K. Let y, z be non-zero elements of a Liouville closure of K such that $y^\dagger = r$ and $z^\dagger = s$. By the implication (1) $\Rightarrow$ (2) in Proposition 12.4, we have $v(\varepsilon') > v(y), v(z) > v(\varepsilon^\dagger)$ for all $\varepsilon \prec 1$ in $K^\times$. In particular, $-v(\varepsilon) = v(\varepsilon^\dagger/\varepsilon') < v(y/z) < v(\varepsilon'/\varepsilon^\dagger) = v(\varepsilon)$ for all such ε, that is, $\Gamma^{<0} < v(y/z) < \Gamma^{>0}$. Since Ψ has no maximum, this yields $v(s-r) = v\big((y/z)^\dagger\big) > \Psi$. Conversely, suppose $r \in K$ satisfies $v(s-r) > \Psi$. By Lemma 12.3, we get $v(s-r) > v(s-a^\dagger)$ for all $a \in K^\times$; hence $s > a^\dagger \iff r > a^\dagger$, and $s < a^\dagger \iff r < a^\dagger$, for all $a \in K^\times$. Using (1) $\iff$ (4) in Proposition 12.4, it follows that r creates a gap over K.

Part (2) follows from the equivalence of (1), (2), and (3) in Proposition 12.4. $\square$

The proof of (4) $\Longrightarrow$ (1) in the last proposition yields that if $s \in K$ and $E \subseteq \mathfrak{m} \setminus \{0\}$ is such that $v(E)$ is coinitial in $\Gamma^{>0}$, then:

$$s \text{ creates a gap over } K \iff \varepsilon'^\dagger < s < \varepsilon^{\dagger\dagger} \text{ for all } \varepsilon \in E.$$

Example. Let $K = \mathbb{R}(\!(x^{-1})\!)^{\mathrm{LE}}$. The sequence (ℓ_n) is coinitial in $K^{>\mathbb{R}}$, so the sequence $(1/\ell_n)$ is cofinal in $\mathfrak{m}^{>0}$. We define the sequences (y_n), (a_n) and (b_n) in K by

$$\begin{aligned}
y_n &= (1/\ell_n)^\dagger = -\frac{1}{\ell_0\ell_1\cdots\ell_n}, \\
a_n &= (1/\ell_n)^{\dagger\dagger} = y_n^\dagger = -\left(\frac{1}{\ell_0} + \frac{1}{\ell_0\ell_1} + \cdots + \frac{1}{\ell_0\ell_1\cdots\ell_n}\right), \\
b_n &= (1/\ell_n)'^\dagger = a_n - \frac{1}{\ell_0\ell_1\cdots\ell_n}.
\end{aligned}$$

There is clearly no $s \in K$ such that $a_n > s > b_n$ for all n. Thus no $s \in K$ creates a gap over K. (This fact also follows from K's Liouville closedness, but the proof just given is instructive in view of the examples below.)

An example of a gap creator. We will now study a specific example of an H-field with a gap creator. Let $\mathbb{R}(\!(\mathfrak{L})\!)$ and $\mathbb{R}(\!(\mathfrak{L}^{\mathbb{Q}})\!)$ be as in the example at the end of Section 11.

Define the sequences (a_n) and (b_n) in $\mathbb{R}(\ell_n : n \in \mathbb{N}) \subseteq \mathbb{R}(\!(\mathfrak{L}^{\mathbb{Q}})\!)$ as in the example above. The sequence $(1/\ell_n)$ is cofinal in $\big\{f \in \mathbb{R}(\!(\mathfrak{L})\!) : 0 < f \prec 1\big\}$.

Put

$$s := -\left(\frac{1}{\ell_0} + \frac{1}{\ell_0\ell_1} + \frac{1}{\ell_0\ell_1\ell_2} + \cdots\right) \in \mathbb{R}(\!(\mathfrak{L}^{\mathbb{Q}})\!).$$

Then $a_n > s > b_n$ for all n, hence s creates a gap over $\mathbb{R}(\!(\mathfrak{L}^{\mathbb{Q}})\!)$ and over $\mathbb{R}(\!(\mathfrak{L})\!)$. Since $\mathbb{R}(\!(\mathfrak{L})\!)$ is closed under powers, Lemma 12.3 yields the following useful fact:

Lemma 12.6. *For every non-zero $a \in \mathbb{R}(\!(\mathfrak{L}^{\mathbb{Q}})\!)$ and every $\lambda \in \mathbb{R}$ there exists $\gamma \in \Gamma^*$ such that $v(s - \lambda a^\dagger) \leq \psi(\gamma)$.* □

We now fill in the missing details of the *Example* in Section 6 of [2]:

Example 12.7. Let E be the real closure inside $\mathbb{R}(\!(\mathfrak{L}^{\mathbb{Q}})\!)$ of its H-subfield $\mathbb{R}(\ell_n : n \in \mathbb{N})$; so $\mathbb{R}(\!(\mathfrak{L}^{\mathbb{Q}})\!)|E$ is an immediate extension of valued fields. Let K be the real closure inside $\mathbb{R}(\!(\mathfrak{L}^{\mathbb{Q}})\!)$ of the H-subfield $E(s, s', s'', \dots)$ of $\mathbb{R}(\!(\mathfrak{L}^{\mathbb{Q}})\!)$ generated by s over E. Clearly K has no gap and is closed under asymptotic integration, since $\mathbb{R}(\!(\mathfrak{L}^{\mathbb{Q}})\!)$ has no gap and $\mathbb{R}(\!(\mathfrak{L}^{\mathbb{Q}})\!)|K$ is immediate. By Proposition 12.4, it follows that s creates a gap over K. Hence, letting $y \neq 0$ be an element of a Liouville closure of K with $y^\dagger = s$, the H-field $K(y)$ has a gap.

The next examples show that gaps can already arise when passing to real closures and closures under powers. Let K be as in the previous example. We need the following consequence of the last lemma:

Lemma 12.8. *Let $a \in K^\times$ and $\mu \in \mathbb{R} \setminus \{1\}$. Then $r = a^\dagger + \mu s$ does not create a gap over K.*

Proof. By Lemma 12.6, there exists $\gamma \in \Gamma^*$ with

$$v(s - r) = v\big((1-\mu)s - a^\dagger\big) = v\big(s - (1-\mu)^{-1}a^\dagger\big) \leq \psi(\gamma).$$

Corollary 12.5, (1) implies that r does not create a gap over K, as claimed. □

Let $\lambda \in \mathbb{R}^{>0}$, and take an element $z > 0$ in a Liouville closure of K with $z^\dagger = \lambda s$. For each $a \in K^\times$ we have $v(z^\dagger - a^\dagger) = v(\lambda s - a^\dagger) = v(s - \lambda^{-1}a^\dagger) \leq \psi(\gamma)$ for some $\gamma \in \Gamma^*$, by Lemma 12.6. In particular $z \notin K$, so z is transcendental over K. By [2], Lemma 5.3 and the *Remarks* following it, $K(z)$ is an H-field with $\Gamma_{K(z)} = \Gamma \oplus \mathbb{Z}v(z)$.

Example 12.9. Suppose that λ is an integer, $\lambda > 1$. Then $K(z)$ has no gap, but its real closure has a gap.

Proof. If $K(z)$ has a gap, then this gap has the form $v(az^k)$ where $a \in K^\times$ and $k \in \mathbb{Z}$, so $(az^k)^\dagger = a^\dagger + k\lambda \cdot s$ creates a gap over K, contradicting Lemma 12.8 (since $k\lambda \neq 1$). So $K(z)$ has no gap. Take u in the real closure L of $K(z)$ with $u^\lambda = z$, so L is also a real closure of $K(u)$. Then $u^\dagger = \lambda^{-1}z^\dagger = s$, hence $K(u)$ has gap $v(u)$, which remains a gap in its real closure L by Lemma 12.1, (6). □

Example 12.10. Suppose that λ is irrational. Then $K(z)$ has only one closure under powers, up to $K(z)$-isomorphism, and $K(z)$ has no gap, but its closure under powers has a gap.

Proof. The same argument as in the previous example shows that $K(z)$ has no gap. By Proposition 8.10, $K(z)$ has a unique closure under powers L, up to isomorphism over $K(z)$. Let $f = z^{1/\lambda}$ in L, so $f^\dagger = \lambda^{-1}z^\dagger = \lambda^{-1}(\lambda s) = s$. By Corollary 12.5, (2), s creates a gap over $K(z)$, so the H-field $K(z,f)$ has gap $v(f)$. By Corollary 8.11, $\Psi_{K(z)}$ is cofinal in Ψ_L. It follows that $v(f)$ remains a gap in L, by Lemma 12.1, (5). $\square$

Gap creators as pseudo-limits. We now indicate how the construction of the gap creator s of $\mathbb{R}(\!(\mathfrak{L}^{\mathbb{Q}})\!)$ extends to other H-fields, via pseudo-convergence. In this subsection we strengthen our earlier assumption on asymptotic integrability by assuming:

K is a real closed closed H-field closed under integration
(so every element of K has an anti-derivative in K).

As in Section 1 we choose for each $f \in K^{>0}$ a "logarithm" $\mathrm{L}(f) \in K$ with $\mathrm{L}(f)' = f^\dagger$. Next we introduce "iterated logarithms" ℓ_λ, for possibly transfinite λ. More precisely, $(\ell_\lambda)_{\lambda<\kappa}$ is a strictly decreasing sequence of elements of $K^{>C}$, indexed by the ordinals less than some limit ordinal κ. We choose this sequence by transfinite recursion as follows: take any element $\ell_0 > C$ in K, and put $\ell_{\lambda+1} := \mathrm{L}(\ell_\lambda)$; if μ is a limit ordinal such that all ℓ_λ with $\lambda < \mu$ have already been chosen, then we choose ℓ_μ to be any element $> C$ such that $\ell_\mu < \ell_\lambda$ for all $\lambda < \mu$, if there is such a ℓ_μ, while if there is no such element, we put $\kappa := \mu$.

We put $e_\lambda := v(1/\ell_\lambda) \in \Gamma^{>0}$, so $v(\ell_\lambda^\dagger) = \psi(e_\lambda)$. By construction of (ℓ_λ) and property (L4) of L in Section 1, the sequence $\big([e_\lambda]\big)_\lambda$ is strictly decreasing and coinitial in $[\Gamma^*]$, and $\big(\psi(e_\lambda)\big)_\lambda$ is strictly increasing and cofinal in Ψ.

From (ℓ_λ) we obtain sequences (y_λ), (a_λ) and (b_λ) in K as follows:

$$y_\lambda := (1/\ell_\lambda)^\dagger, \quad a_\lambda := (1/\ell_\lambda)^{\dagger\dagger} = y_\lambda^\dagger, \quad b_\lambda := (1/\ell_\lambda)'^\dagger = a_\lambda + y_\lambda,$$

for $\lambda < \kappa$. Then $v(y_\lambda) = \psi(e_\lambda)$ for $\lambda < \kappa$, and:

(1) $(1/\ell_\lambda)$ is strictly increasing and cofinal in $\mathfrak{m}^{>0}$,
(2) $\big(v(y_\lambda)\big)$ is strictly increasing and cofinal in Ψ,
(3) (a_λ) is strictly decreasing and coinitial in $\{\varepsilon^{\dagger\dagger} : 0 \neq \varepsilon \prec 1\}$, and
(4) (b_λ) is strictly increasing and cofinal in $\{\varepsilon'^\dagger : 0 \neq \varepsilon \prec 1\}$.

Hence an element $s \in K$ creates a gap over K if and only if $a_\lambda > s > b_\lambda$ for all $\lambda < \kappa$, by the remark following Corollary 12.5.

Proposition 12.11. *For $\lambda < \kappa$, we have $a_{\lambda+1} - a_\lambda = y_{\lambda+1}$, and for $\lambda < \mu < \kappa$, we have $a_\mu - a_\lambda = y_{\lambda+1} + \delta$ with $v(\delta) > v(y_{\lambda+1})$. In particular, (a_λ) is a pseudo-Cauchy sequence. An element $s \in K$ creates a gap over K if and only if it is a pseudo-limit of (a_λ).*

Proof. Note that we have $y_{\lambda+1} = -\ell'_{\lambda+1}/\ell_{\lambda+1} = -(\mathrm{L}(\ell_\lambda))'/\ell_{\lambda+1} = y_\lambda/\ell_{\lambda+1}$, that is, $y_{\lambda+1}/y_\lambda = 1/\ell_{\lambda+1}$. Hence

$$a_{\lambda+1} - a_\lambda = y_{\lambda+1}^\dagger - y_\lambda^\dagger = (y_{\lambda+1}/y_\lambda)^\dagger = (1/\ell_{\lambda+1})^\dagger = y_{\lambda+1}.$$

Let $\lambda < \mu < \kappa$. We have to show that $a_\mu - a_\lambda = y_{\lambda+1} + \delta$ with $v(\delta) > v(y_{\lambda+1})$. We have just shown that for $\mu = \lambda + 1$ we have $\delta = 0$. In the general case we use this special case, and the fact that $a_\mu - a_{\lambda+1} = (y_\mu/y_{\lambda+1})^\dagger$, so

$$v(a_\mu - a_{\lambda+1}) = \psi\big(v(y_\mu/y_{\lambda+1})\big) = \psi\big(\psi(e_\mu) - \psi(e_{\lambda+1})\big) > \psi(e_{\lambda+1}) = v(y_{\lambda+1}),$$

by Proposition 2.3, (1) in [2]. Note that since $b_\lambda = a_\lambda + y_\lambda$ for all $\lambda < \kappa$, it follows that the sequence (b_λ) is also a pseudo-Cauchy sequence in K.

We now show that $s \in K$ creates a gap over K if and only if s is a pseudo-limit of (a_λ). Let y be a non-zero element in a Liouville closure of K such that $y^\dagger = s$, and put $L = K(y)$. Suppose first that s creates a gap over K. So $v(y) \notin (\mathrm{id} + \psi_L)(\Gamma_L^*)$, hence $v(s - a_\mu) = \psi\big(v(y) - \psi(e_\mu)\big) > \psi(e_\mu)$ for all $\mu < \kappa$, by Lemma 2.5 in [2]. So if $\lambda < \mu < \kappa$, then $v(a_\lambda - a_\mu) = \psi(e_{\lambda+1}) \le \psi(e_\mu) < v(s - a_\mu)$, hence

$$v(s - a_\lambda) = v\big((s - a_\mu) + (a_\mu - a_\lambda)\big) < v(s - a_\mu),$$

showing that s is a pseudo-limit of (a_λ). Conversely, suppose s is a pseudo-limit of (a_λ). Then s is also a pseudo-limit of (b_λ): for every $\lambda < \kappa$, we have

$$v(s - a_\lambda) = v(a_\lambda - a_{\lambda+1}) = v(y_{\lambda+1}) > v(y_\lambda),$$

hence $v(s - b_\lambda) = v(s - a_\lambda - y_\lambda) = v(y_\lambda)$. Therefore $a_\lambda > s > b_\lambda$ for all $\lambda < \kappa$; so s creates a gap over K. □

It follows that any *maximally valued* real closed H-field that is closed under asymptotic integration has a gap creator. (Use that such an H-field is closed under integration, by [2], Remark 3 after Lemma 5.1, or [19].) This yields another proof of Corollary 7.2.

We finish this section by showing:

Proposition 12.12. *Suppose that K does not have a gap creator. Let y be an element of an H-field extension L of K such that*

$$\Psi < v(y) < (\mathrm{id} + \psi_L)\big(\Gamma_L^{>0}\big).$$

Then $Q(y) \neq 0$ for all non-zero $Q(Y) \in K\{Y\}$ of order at most 1.

In the proof we shall use:

Lemma 12.13. *Let F be an H-field with divisible value group and let y be a non-zero element of an H-field extension of F such that, with $z = y^\dagger$:*

(1) $v(y) \notin \Gamma_F$,
(2) $\Gamma_{F(z)} = \Gamma_F$, *and*
(3) z *is transcendental over* F.

Then $Q(y) \neq 0$ for all non-zero $Q(Y) \in F\{Y\}$ of order at most 1.

Proof. Suppose for a contradiction that $Q(y)=0$ where

$$Q(Y)=\sum_{i,j} a_{ij}Y^i(Y')^j \in F\{Y\} \qquad (a_{ij}\in F)$$

is of degree d. We introduce a new indeterminate Z and consider the non-zero polynomial $R(Y,Z)\in F[Y,Z]$ given by

$$R(Y,Z)=\sum_{k=0}^{d} a_k(Z)Y^k, \qquad a_k(Z)=\sum_{i+j=k} a_{ij}Z^j \in F[Z].$$

Then $R(y,z)=Q(y)=0$ and the polynomial $R(Y,z)\in F(z)[Y]$ is non-zero. So $v(y)\in\big(\text{divisible hull of }\Gamma_{F(z)}\big)=\Gamma_F$, a contradiction. □

Proof (Proposition 12.12). We claim that $z=y^\dagger$ is a pseudo-limit of the pseudo-Cauchy sequence (a_λ). To see this, let $\mu<\kappa$. Since $v(y)-\psi(e_\mu)\in\Gamma_L^{>0}$ and $v(y)<(\mathrm{id}+\psi_L)(\Gamma_L^{>0})$ we have

$$\begin{aligned} v(z-a_\mu) &= \psi_L\big(v(y)-\psi(e_\mu)\big)\\ &= (\mathrm{id}+\psi_L)\big(v(y)-\psi(e_\mu)\big)+\psi(e_\mu)-v(y)>\psi(e_\mu). \end{aligned}$$

The claim now follows as in the proof of the "only if" direction in the last statement of Proposition 12.11. By that proposition, the pseudo-Cauchy sequence (a_λ) has no pseudo-limit in K. Hence the valued field $K(z)$ is an immediate extension of K ([23], Chapter III, §3, Lemmas 11 and 14). Now apply the lemma. □

Corollary 12.14. *Suppose that K is Liouville closed. Let $P(U)\in K\{U\}$ be a non-zero homogeneous differential polynomial of order at most 2. There exists $a>C$ in K such that $P(u)\neq 0$ for all u in all H-field extensions L of K with $C_L<u<a$.*

Proof. Suppose not. Model-theoretic compactness yields an H-field extension L of K and a $u\in L$ such that $C_L<u<K^{>C}$ and $P(u)=0$. Let $Y=U^\dagger\in K\langle U\rangle$ and $d=\deg P$, hence $P/U^d=Q(Y)$ with $0\neq Q(Y)\in K\{Y\}$ of order ≤ 1, by Lemma 3.6. With $y=u^\dagger\in L$ we have $Q(y)=0$ and $\Psi<v(y)<(\mathrm{id}+\psi_L)\big(\Gamma_L^{>0}\big)$, contradicting Proposition 12.12. □

13. Undecidability

Our long term aim is to describe the elementary theory of the differential field $\mathbb{R}(\!(x^{-1})\!)^{\mathrm{LE}}$ of *logarithmic-exponential series*. This differential field is obtained from the smaller differential field $\mathbb{R}(\!(x^{-1})\!)^{\mathrm{E}}$ of *exponential series* in a very simple way: replace $x=\ell_0$ successively by $\ell_1,\ell_2,\ell_3,\dots$, and take the union. In view of this fact, the next result suggests that $\mathbb{R}(\!(x^{-1})\!)^{\mathrm{LE}}$ may be near the edge of undecidability (or over it).

Theorem 13.1. *The set $\mathbb{Z}\subseteq\mathbb{R}(\!(x^{-1})\!)^{\mathrm{E}}$ of integers is existentially definable (without parameters) in the differential field $\mathbb{R}(\!(x^{-1})\!)^{\mathrm{E}}$ of exponential series.*

Thus, by the negative solution of Hilbert's 10th Problem (see for example [22]), there is no algorithm which, upon input of a differential polynomial $P(Y_1,\dots,Y_n)\in\mathbb{Q}\{Y_1,\dots,Y_n\}$, decides whether there exist $y_1,\dots,y_n\in\mathbb{R}(\!(x^{-1})\!)^{\mathrm{E}}$ such that $P(y_1,\dots,y_n)=0$. In particular, the elementary theory of $\mathbb{R}(\!(x^{-1})\!)^{\mathrm{E}}$ as a differential field is undecidable.

We view $\mathbb{R}(\!(x^{-1})\!)^{\mathrm{E}}$ here as equipped with the derivation $\frac{d}{dx}$, as usual. However, for the proof it is convenient to change variables, and express differential equations in terms of the derivation $\frac{d}{dt}:=-\frac{1}{x^2}\frac{d}{dx}$ where $t=x^{-1}$. Thus we turn $\mathbb{R}(\!(x^{-1})\!)^{\mathrm{E}}=\mathbb{R}(\!(t)\!)^{\mathrm{E}}$ into a differential-valued field extension of the series field $\mathbb{R}(\!(t^{\mathbb{R}})\!)$ as defined in Section 11, and for $f\in\mathbb{R}(\!(t)\!)^{\mathrm{E}}$ we put $f':=\frac{df}{dt}$.

The theorem above extends a similar result due to Grigor′ev and Singer [14] for a certain differential subfield of $\mathbb{R}(\!(t^{\mathbb{R}})\!)$.

We consider the following system of algebraic differential equations

$$(S_\beta)\qquad Y't=\beta Y,\quad Z'Yt+Z''t^2=-Y+t$$

in the indeterminates Y,Z, depending on the parameter $\beta\in\mathbb{R}$. Theorem 13.1 above follows easily from the following more general result:

Proposition 13.2. *Let K be a differential-valued field extension of $\mathbb{R}(\!(t^{\mathbb{R}})\!)$ with constant field $\mathbb{R}$. Suppose $v(K^\times)$ contains no $\gamma>0$ such that $n\gamma<v(t)$ for all n. Then, for $\beta>0$, the system (S_β) has a solution in K if and only if $\beta=1/n$ for some positive n.*

The constants $\beta\in\mathbb{R}$ are singled out in K by the differential equation $U'=0$, so the proposition leads to an existential definition of $\mathbb{Z}\subseteq K$ in the differential field K with t as distinguished element. In particular, the elementary theory $\mathrm{Th}(K)$ of the differential field K (with or without naming t) is undecidable.

The hypothesis of the proposition is satisfied for $K=\mathbb{R}(\!(t)\!)^{\mathrm{E}}$ (with derivation $\frac{d}{dt}$), and this leads to an existential definition of $\mathbb{Z}$ in the differential field $\mathbb{R}(\!(x^{-1})\!)^{\mathrm{E}}$ with derivation $\frac{d}{dx}$ and a name for x. To get such an existential definition without naming x, note that an element of $\mathbb{R}(\!(x^{-1})\!)^{\mathrm{E}}$ has derivative 1 if and only if it equals $x+c$ for some $c\in\mathbb{R}$, and that each such $x+c$ is the image of x under an automorphism of the differential field $\mathbb{R}(\!(x^{-1})\!)^{\mathrm{E}}$, see [12].

The system (S_β) and the proof of the lemma below are from [14], except for the correction of some mathematical typos. Note that if y is an element in a differential field extension K of $\mathbb{R}(t^{\mathbb{R}})$ satisfying $y't=\beta y$, where $\beta\in\mathbb{R}$, then $y=ct^\beta$ for some constant $c\in C$. We will use this fact without further mention.

Lemma 13.3. *The following are equivalent, for $\beta\in\mathbb{R}$:*

1. *The system (S_β) has a solution in the differential subring $\mathbb{R}[t^{\mathbb{Q}}]$ of $\mathbb{R}(\!(t^{\mathbb{R}})\!)$.*
2. *The system (S_β) has a solution in $\mathbb{R}(\!(t^{\mathbb{R}})\!)$.*

(3) *$\beta = 1/n$ for some positive n.*

Proof. The implication (1) $\Rightarrow$ (2) is trivial. Suppose (S_β) has a solution (y,z) with $y,z \in \mathbb{R}((t^{\mathbb{R}}))$. We may assume that $z \neq 0$, since otherwise $y = t$ and $\beta = 1$. Write $z = \sum_{r \geq r_0} a_r t^r$ with $a_r \in \mathbb{R}$, $a_{r_0} \neq 0$. We have $y \neq 0$, since otherwise $z'' = t^{-1}$, contradicting $z' \in \mathbb{R}((t^{\mathbb{R}}))$. So $y = ct^\beta$ for some $c \in \mathbb{R}^\times$. We have $\beta > 0$: if $\beta < 0$, then

$$-ct^\beta + t = z'yt + z''t^2 = a_{r_0} c r_0 t^{r_0+\beta} + \text{terms of order } > r_0 + \beta,$$

which gives a contradiction by distinguishing the cases $r_0 = 0$ and $r_0 \neq 0$; if $\beta = 0$, then

$$-c + t = z'yt + z''t^2 = \sum r a_r (c + r - 1) t^r,$$

and by comparing coefficients one reaches a contradiction.

So $\beta > 0$; we shall assume $\beta \neq \frac{1}{n}$ for all $n \geq 1$, and arrive once again at an impossibility, by showing that then $t^{-\beta}, t^{-2\beta}, \ldots$ all occur in $u := z'$ with non-zero coefficients. Note that $u = \sum_{r \geq r_0 - 1} b_r t^r$ with $b_r = (r+1)a_{r+1}$ for all r, so

$$-y + t = z'yt + z''t^2 = uyt + u't^2 = \sum_r \left(cb_{r-\beta} + rb_r\right) t^{r+1}.$$

Comparing coefficients of t on both sides of $-y + t = \sum_r \left(cb_{r-\beta} + rb_r\right) t^{r+1}$ gives $cb_{-\beta} = 1$, so $b_{-\beta} \neq 0$. Comparing the coefficients of $t^{1-n\beta}$ with $n > 0$ yields

$$cb_{-(n+1)\beta} = n\beta b_{-n\beta},$$

and by induction on n, it follows that $b_{-n\beta} \neq 0$ for all $n > 0$, as promised. This finishes the proof of (2) $\Rightarrow$ (3).

To prove (3) $\Rightarrow$ (1), let $\beta = 1/n$, $n \geq 1$. We claim that (S_β) has a solution (y,z) in $\mathbb{R}[t^{\mathbb{Q}}]$. If $n = 1$, we may take $(y,z) = (t,0)$. Suppose $n > 1$. We claim that there exist $a_1, \ldots, a_{n-1}, c \in \mathbb{R}$ such that (y,z) with

$$y = ct^{\frac{1}{n}}, \quad z = a_1 t^{\frac{1}{n}} + \cdots + a_{n-1} t^{\frac{n-1}{n}} \tag{13.1}$$

is a solution of (S_β). Clearly any y as in (13.1) is a solution of the first equation in (S_β), and (y,z) as in (13.1) is a solution to (S_β) if and only if

$$\begin{aligned}
c &= a_1 \frac{1}{n}\left(1 - \frac{1}{n}\right) \\
ca_1 \frac{1}{n} &= a_2 \frac{2}{n}\left(1 - \frac{2}{n}\right) \\
&\vdots \\
ca_{n-2}\left(\frac{n-2}{n}\right) &= a_{n-1}\left(\frac{n-1}{n}\right)\left(1 - \frac{n-1}{n}\right) \\
ca_{n-1}\left(\frac{n-1}{n}\right) &= 1.
\end{aligned}$$

These equations imply $c^n = \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$. Choosing $c \in \mathbb{R}$ with c^n equal to the number on the right, we can then determine $a_1,\dots,a_{n-1} \in \mathbb{R}$ such that the n equations above are satisfied. $\square$

Proof of Proposition 13.2. We still need three lemmas.

Lemma 13.4. *Let (Γ,ψ) be an asymptotic couple and Δ a non-zero convex subgroup of Γ. The following conditions are equivalent:*

(1) $\psi(\Delta^*) \cap \Delta \neq \emptyset$.
(2) $\psi(\Delta^*) \subseteq \Delta$.
(3) $(\mathrm{id}+\psi)(\Delta^*) \cap \Delta \neq \emptyset$.
(4) $(\mathrm{id}+\psi)(\Delta^*) \subseteq \Delta$.

Proof. Let $\delta \in \Delta^*$ with $\psi(\delta) \in \Delta$. Then we have for $\delta_1 \in \Delta^*$:

$$|\psi(\delta)-\psi(\delta_1)| \leq |\delta-\delta_1| \in \Delta,$$

so $\psi(\delta_1) \in \Delta$. The equivalences now follow easily. $\square$

Below K satisfies the hypotheses of 13.2. Note that the value group $\Delta := \mathbb{R}v(t)$ (with $v(t) > 0$) of $\mathbb{R}(\!(t^{\mathbb{R}})\!)$ is the smallest non-zero convex subgroup of $\Gamma := v(K^\times)$. Accordingly $\psi\big(v(t)\big) = -v(t) \in \Delta$ is the largest element of Ψ, by [25]. So (Γ,ψ) and Δ satisfy the conditions of the last lemma.

Lemma 13.5. *Let $u, y \in K^\times$, $0 < v(y) \in \mathbb{R}v(t)$ and $\gamma = v(u) \neq 0$. Then*

$$v(uyt + u't^2) = \gamma + \psi(\gamma) + 2v(t).$$

Proof. Write $v(y) = \beta v(t)$ with $\beta \in \mathbb{R}^{>0}$. Then

$$v(uyt) = \gamma + (\beta+1)v(t) > \gamma + v(t) \geq \gamma + \psi(\gamma) + 2v(t) = v(u't^2),$$

since $\psi(\gamma) \leq \psi(v(t)) = -v(t)$. $\square$

Lemma 13.6. *There exists an additive subgroup A of K such that $K = \mathbb{R}(\!(t^{\mathbb{R}})\!) \oplus A$ and $v(a) \notin \Delta$ for all $a \in A$.*

Proof. We equip Γ/Δ with the ordering induced by the ordering on Γ, and let $\pi\colon \Gamma \to \Gamma/\Delta$ be the natural map. Consider the valuation $v_\Delta = \pi \circ v\colon K^\times \to \Gamma/\Delta$ on K, with valuation ring

$$\mathcal{O}_\Delta = \big\{a \in K : v(a) \geq \delta \text{ for some } \delta \in \Delta\big\}$$

and maximal ideal

$$\mathfrak{m}_\Delta = \big\{a \in K : v(a) > \delta \text{ for all } \delta \in \Delta\big\}.$$

We equip the residue field $F = \mathcal{O}_\Delta/\mathfrak{m}_\Delta$ of v_Δ with the valuation $v_F\colon F^\times \to \Delta$ such that $v_F(\overline{a}) = v(a)$ for $a \in \mathcal{O}_\Delta \setminus \mathfrak{m}_\Delta$, where $\overline{a}$ denotes the image of a in $F^\times$. The residue field of v_F is $(\mathcal{O}/\mathfrak{m}_\Delta)/(\mathfrak{m}/\mathfrak{m}_\Delta)$, which we identify with the constant field $C = \mathbb{R}$ of K in the usual way. We have $\mathbb{R}(\!(t^{\mathbb{R}})\!) \subseteq \mathcal{O}_\Delta$, and hence we can naturally construe $\mathbb{R}(\!(t^{\mathbb{R}})\!)$ as a valued subfield of F. Since $\mathbb{R}(\!(t^{\mathbb{R}})\!)$ is maximally valued, we have in fact $\mathbb{R}(\!(t^{\mathbb{R}})\!) = F$. Choosing a direct factor B of the additive group $\mathcal{O}_\Delta$ in K, we get $K = \mathbb{R}(\!(t^{\mathbb{R}})\!) \oplus A$ for $A = \mathfrak{m}_\Delta \oplus B$, as required. $\square$

We now can prove Proposition 13.2. If $\beta = 1/n$ for some $n > 0$, then (S_β) has a solution in $\mathbb{R}[t^{\mathbb{Q}}]$, and hence in K, by Lemma 13.3. Conversely, suppose $\beta > 0$ and let (y, z) be a solution to (S_β) in K. Choose an additive subgroup A of K as in Lemma 13.6, and write $z = z_0 + a$ with $z_0 \in \mathbb{R}(\!(t^{\mathbb{R}})\!)$ and $a \in A$. We claim that $a = 0$, so $z = z_0$. Suppose otherwise; then $b = a' \neq 0$ and $byt + b't^2 = u_0yt + u_0't^2 + y - t \in \mathbb{R}(\!(t^{\mathbb{R}})\!)$, where $u_0 = z_0' \in \mathbb{R}(\!(t^{\mathbb{R}})\!)$. We have $v(b) \notin \Delta$ by Lemma 13.4, and by Lemma 13.5:

$$v(b) + \psi\big(v(b)\big) + 2v(t) = v(byt + b't^2) = v(u_0yt + u_0't^2 + y - t) \in \Delta.$$

Thus $v(b) + \psi\big(v(b)\big) \in \Delta$ and hence $v(b) \in \Delta$, again by Lemma 13.4: a contradiction. Hence (y, z) is a solution to (S_β) in $\mathbb{R}(\!(t^{\mathbb{R}})\!)$, and the implication (2) $\Rightarrow$ (3) in Lemma 13.3 yields $1/\beta \in \mathbb{N}$ as required. This concludes the proof of Proposition 13.2. $\square$

This proof yields the construction of an existential formula in the language of differential fields that defines $\mathbb{Z}$ in $\mathbb{R}(\!(x^{-1})\!)^{\mathrm{E}}$. This formula also defines $\mathbb{Z}$ in its completion $\mathbb{R}(\!(G^{\mathrm{E}})\!)$ (see the Example in Section 10).

14. Existentially Closed H-Fields

In this section we sketch some of our longer term goals in the study of H-fields.

An H-field K is said to be **existentially closed** if every algebraic differential equation (in unknowns $Y_1, \dots, Y_n$ with $Y = (Y_1, \dots, Y_n)$)

$$A(Y) = 0 \qquad (A \in K\{Y\})$$

with a solution in *some* H-field extension of K has a solution in K itself. (Here a *solution* in the H-field extension L of K is a tuple $y \in L^n$ such that $A(y) = 0$; a similar convention holds for solutions of the more general systems considered below.) In this definition we can allow *systems*

$$A_1(Y) = \cdots = A_m(Y) = 0 \qquad (A_1, \dots, A_m \in K\{Y\}) \tag{14.1}$$

instead of single equations, since (14.1) can be replaced by the single equation $A_1(Y)^2 + \cdots + A_m(Y)^2 = 0$. We can also add differential inequations and differential inequalities: if K is an existentially closed H-field, then any system (14.1) augmented by finitely many inequations $B(Y) \neq 0$ and inequalities $C(Y) > 0$ ($B, C \in K\{Y\}$) that is solvable in an H-field extension of K is solvable in K. To see this, note that $B(Y) \neq 0$ can be replaced by an equation $B(Y)Z_B = 1$ where Z_B is an extra differential unknown; similarly, $C(Y) > 0$ can be replaced by an equation $C(Y)Z_C^2 = 1$, by Corollary 3.10 in [2]. In view of [2], Theorem 6.11, it follows that every existentially closed H-field is Liouville closed.

We may even add **asymptotic inequalities** of the form $F(Y) \preceq G(Y)$, and of the form $F(Y) \prec G(Y)$, where $F, G \in K\{Y\}$:

Lemma 14.1. *Suppose K is an existentially closed H-field. Then any system of equations* (14.1) *augmented by finitely many inequations, inequalities* *and asymptotic inequalities as above that is solvable in some H-field extension of K is solvable in K.*

Proof. Fix any positive $\varepsilon \prec 1$ in K. Let L be an H-field extension of K. Then we have for all $z \in L$:

$$z \preceq 1 \iff \exists c \in L\big(c' = 0 \ \&\ -c < z < c\big),$$
$$z \succ 1 \iff \exists h, c \in L\big(0 < h < \varepsilon \ \&\ c' = 0 \ \&\ -cz' < h^{\dagger} < cz'\big).$$

Thus any system $(*)$ as in the lemma with asymptotic inequalities can be replaced (using extra unknowns) by a finite system $(**)$ of algebraic differential equations over K, in the sense that $(*)$ is solvable in L if and only if $(**)$ is solvable in L, and $(*)$ is solvable in K if and only if $(**)$ is solvable in K. □

Model-theoretic considerations. An easy model-theoretic construction shows that every H-field can be embedded into an existentially closed H-field. This fact is of no use by itself, but would acquire force in case of positive answers to the questions which motivate our work on H-fields in this paper and its predecessors [1] and [2]:

Is the H-field $\mathbb{R}(\!(x^{-1})\!)^{\mathrm{LE}}$ existentially closed?

Is the class of existentially closed H-fields an elementary class?

Positive answers would have many rewarding consequences for asymptotic differential algebra. The first question is also interesting for various H-subfields of $\mathbb{R}(\!(x^{-1})\!)^{\mathrm{LE}}$ such as the field of acceleration-summable series, and the field of grid-based series, see [13] and [16].

In order to make the second question precise we specify the (first-order) language $\mathcal{L}$ in which we axiomatize the theory of H-fields. Let

$$\mathcal{L} = \big\{0, 1, +, -, \cdot, \partial, <, \preceq\big\}$$

be the language of ordered rings $\big\{0, 1, +, -, \cdot, <\big\}$ augmented by a unary function symbol ∂ and a binary relation symbol $\preceq$. An H-field K is construed as $\mathcal{L}$-structure in the obvious way, with ∂ interpreted as the derivation. The axioms for H-fields in [2] can be given by $\forall\exists$-sentences in $\mathcal{L}$; thus we have a certain set Σ_H of $\forall\exists$-sentences in $\mathcal{L}$ such that the $\mathcal{L}$-structures satisfying Σ_H are exactly the H-fields. The $\mathcal{L}$-substructures of H-fields whose underlying ring is a field are exactly the pre-H-fields, see [2], §4. By the last lemma, the existentially closed H-fields are exactly the existentially closed models of the $\mathcal{L}$-theory of H-fields, as defined in model theory, see [9]. The second question now has the following precise formulation:

Is there a set Σ of $\mathcal{L}$-sentences such that the H-fields satisfying Σ are exactly the existentially closed H-fields?

Such a set Σ would axiomatize a model-complete $\mathcal{L}$-theory, and this theory is then called the *model companion* of the $\mathcal{L}$-theory of H-fields, see [9]. So we want to find a set Σ of elementary properties of existentially closed H-fields such that, conversely, each H-field satisfying Σ is existentially closed.

In this paper we have shown that existentially closed H-fields K have the following elementary properties:

(1) K is Liouville closed;
(2) K has the intermediate value property for first-order differential polynomials: given a first-order differential polynomial $P(Y) \in K\{Y\}$ and elements $\phi < \theta$ in K such that $P(\phi)$ and $P(\theta)$ are non-zero and of opposite sign, there exists $\eta \in K$ with $P(\eta) = 0$ and $\phi < \eta < \theta$. (This is Theorem 4.3.)

Maximal Hardy fields also satisfy (1) and (2), see [26] and [10].

Our best guess is that (1) and an extension of (2) to all $P(Y) \in K\{Y\}$ might yield a set Σ of elementary properties as desired. In the next subsection we discuss this extension of (2).

The intermediate value property. Let K be an H-field K. Given a differential polynomial $P(Y) \in K\{Y\}$ in a single indeterminate Y, we say that $P(Y)$ **has the intermediate value property in** K if for any $\phi < \theta$ in K such that $P(\phi)$ and $P(\theta)$ are non-zero and of opposite sign there exists $\eta \in K$ with $\phi < \eta < \theta$ and $P(\eta) = 0$. We say that K **has the intermediate value property** if every differential polynomial in $K\{Y\}$ has the intermediate value property in K.

Van der Hoeven (in [17]) proved the remarkable fact that the H-field $\mathbb{R}((x^{-1}))^{\mathrm{LE}}$ has the intermediate value property. This fact and a potential analogy with ordered fields suggest that "Liouville closed & intermediate value property" might single out the existentially closed H-fields among H-fields. (One side of this analogy would be the fact that the existentially closed ordered fields are exactly the real closed fields, that is, the ordered fields with the intermediate value property for ordinary one-variable polynomials.)

We note that "Liouville closed" does not imply "intermediate value property":

Example. Let $K = \mathbb{R}(x) \subseteq L = \mathbb{R}((x^{-1}))^{\mathrm{LE}}$ and $P(Y, Y') = xY' + YY' - Y \in K\{Y\}$. Then $P(y, y') < 0$ for all sufficiently small $y > \mathbb{R}$ in L, and $P(y, y') > 0$ for all sufficiently large $y > \mathbb{R}$ in L. (See the Example in Section 2.) Hence P has a zero $y \in L^{>\mathbb{R}}$, and such y satisfies

$$(\log y)' = y^{\dagger} = 1/y - xy'/y^2 = (x/y)',$$

hence $y \cdot (c + \log y) = x$ for some $c \in \mathbb{R}$. As in the proof of [11], Corollary 4.5, it follows that y is transcendental over the Liouville closure of K in L.

Note that real closed H-fields with trivial derivation have the intermediate value property. But even for H-fields with non-trivial derivation, "intermediate value property" does not imply "Liouville closed". This follows from the construction below on H-fields which is useful for other reasons as well.

Residue fields of H-fields under coarsening. Let K be an H-field such that $0 < (\mathrm{id} + \psi)\big(\Gamma^{>0}\big)$. Let Δ be a convex subgroup of Γ with $\psi(\Delta^*) \cap \Delta \neq \emptyset$. Then $\psi(\Delta^*) \subseteq \Delta$ by Lemma 13.4, and $(\Delta, \psi|\Delta^*)$ is an asymptotic couple of H-type. Since $[\psi(\gamma)] < [\gamma]$ for all $\gamma \in \Gamma^{>\Delta}$, we have

$$(\mathrm{id} + \psi)\big(\Gamma^{>\Delta}\big) = (\mathrm{id} + \psi)(\Gamma^*) \cap \Gamma^{>\Delta}. \tag{14.2}$$

As in the proof of Lemma 13.6 we equip Γ/Δ with the unique ordering making the natural homomorphism $\pi\colon \Gamma \to \Gamma/\Delta$ order-preserving, and we consider the valuation

$$v_\Delta = \pi \circ v\colon K^\times \to \Gamma/\Delta$$

on K (a coarsening of v).

Lemma 14.2. *The valuation ring $\mathcal{O}_\Delta$ of v_Δ and its maximal ideal $\mathfrak{m}_\Delta$ are closed under the derivation of K.*

Proof. If $f \in K^\times$ and $v(f) > \Delta$, then $v(f') > \Delta$ by (14.2), showing $\mathfrak{m}_\Delta' \subseteq \mathfrak{m}_\Delta$. Let $f \in \mathcal{O}_\Delta$. Then $v(f) \geq \delta$ where $\delta \in \Delta^*$, so $v(f') \geq \delta + \psi(\delta) \in \Delta$, hence $f' \in \mathcal{O}_\Delta$. $\square$

Thus the derivation $f \mapsto \partial f$ of K induces a derivation $\overline{f} \mapsto \partial\overline{f} := \overline{\partial f}$ (where $\overline{f} := f + \mathfrak{m}_\Delta$ for $f \in \mathcal{O}_\Delta$) on the residue field $F := \mathcal{O}_\Delta/\mathfrak{m}_\Delta$ of v_Δ, turning F into a differential field. We have a field embedding $c \mapsto \overline{c}\colon C \to F$, and we identify C with a subfield of F in this way. The ordering on K induces an ordering on F which makes F an ordered field:

$$\overline{f} > 0 \quad :\Longleftrightarrow \quad f > 0, \qquad \text{for } f \in \mathcal{O}_\Delta \setminus \mathfrak{m}_\Delta.$$

The convex hull of C in F is the valuation ring $\mathcal{O}/\mathfrak{m}_\Delta$ of F, with associated valuation $v_F\colon F^\times \to \Delta$ given by $v_F(\overline{f}) := v(f)$ for $f \in \mathcal{O}_\Delta \setminus \mathfrak{m}_\Delta$. Its residue field is $(\mathcal{O}/\mathfrak{m}_\Delta)/(\mathfrak{m}/\mathfrak{m}_\Delta)$, which we identify as usual with $\mathcal{O}/\mathfrak{m} = \mathrm{res}(K)$.

Lemma 14.3. *The ordered differential field F is an H-field, with constant field C and asymptotic couple $(\Delta, \psi|\Delta^*)$. If K has the intermediate value property, then so does F.*

Proof. To show that the constant field of F is C, let $f \in \mathcal{O}_\Delta$ and $f' \in \mathfrak{m}_\Delta$. If $f \in \mathcal{O}$, take $c \in C$ such that $f - c \in \mathfrak{m}$, so $(f-c)' = f' \in \mathfrak{m}_\Delta$, hence $f - c \in \mathfrak{m}_\Delta$ by (14.2). If $f \notin \mathcal{O}$, then $f \in \mathfrak{m}_\Delta$, again by (14.2). In both cases, $\overline{f} \in C$.

Suppose now that $\overline{g} > C$, where $g \in \mathcal{O}_\Delta$. Then $g > C$ and hence $g' > 0$, since K is an H-field. Moreover, $g \in \mathcal{O}_\Delta \setminus \mathcal{O}$ implies $g' \in \mathcal{O}_\Delta \setminus \mathfrak{m}_\Delta$, by (14.2); hence $\overline{g}' > 0$. The rest now follows easily. $\square$

With $K = \mathbb{R}(\!(x^{-1})\!)^{\mathrm{LE}}$, put

$$\Delta := \big\{\gamma \in \Gamma : |\gamma| \leq nv(x^{-1}) \text{ for some } n\big\}.$$

Then by [17] the H-field F as defined above has the intermediate value property. But F is not Liouville closed, since ψ only takes positive values on the value group Δ of F.

The appearance of gaps. Understanding how gaps can arise seems important in the model theory of H-fields, and in this direction we can ask:

Is there a set of $\mathcal{L}$-sentences whose models are exactly the H-fields none of whose differentially algebraic H-field extensions have a gap?

The following lemma might be useful in answering this question:

Lemma 14.4. *Let K be an H-field closed under asymptotic integration. Then K has a differentially algebraic H-field extension with a gap if and only if there exists an element y of a differentially algebraic H-field extension L of K such that*

(1) $C_L < y < K^{>C}$, *and*
(2) *for every $f \in K\langle y'\rangle$ with $f > C_L$ there exists $a \in K$ such that $C < a < f$.*

Proof. Let M be a differentially algebraic H-field extension of K with gap $v(z)$, $z \in M^{>0}$. Take $y \succ 1$ in some H-field extension of M with $y' = z$. Then $L := M(y)$ is a differentially algebraic H-field extension of K and $C_L < y < M^{>C_M}$; in particular $C_L < y < K^{>C}$. The pre-H-field $K\langle y'\rangle$ has gap $v(y')$; in particular, $K\langle y'\rangle$ does not have a smallest comparability class. Since $K\langle y'\rangle$ is differentially algebraic over K, (2) follows from Lemma 2.1.

Conversely, let y be an element of a differentially algebraic H-field extension L of K satisfying (1) and (2). By (1), we have $\Gamma^{<0} < v(y) < 0$, and by (2), $\Gamma^{<0}$ is cofinal in $\Gamma^{<0}_{K\langle y'\rangle}$. Hence $v(y')$ is a gap in the pre-H-field $K\langle y'\rangle$. Let M be the smallest H-subfield of L containing $K\langle y'\rangle$. By Corollary 4.5, (2) in [2], $\Gamma_M = \Gamma_{K\langle y'\rangle}$. Hence M is a differentially algebraic H-field extension of K with a gap. □

In [3] we give an example of a Liouville closed H-field that has a differentially algebraic H-field extension with a gap. Here we show:

Proposition 14.5. *Let K be an existentially closed H-field.*

(1) *For every non-zero differential polynomial $P(Y) \in K\{Y\}$ there exists an element $a > C$ in K such that $P(Y)$ has no zero y in any H-field extension L of K with $C_L < y < a$.*
(2) *No differentially algebraic H-field extension of K has a gap.*

Proof. For (1), let $P(Y) \in K\{Y\} \setminus \{0\}$. By the Liouville closedness of K and Corollary 2.6, we can take $a > C$ in K such that $P(y) \neq 0$ for all $y \in K$ with $C < y < a$. So if y is an element of an H-field extension L of K with $C_L < y < a$, then $P(y) \neq 0$, by Lemma 14.1 and the equivalence

$$C_L < y < a \iff 1 \prec y \;\&\; 0 < y < a.$$

Part (2) follows from (1) and Lemma 14.4. □

An open question. At this stage our understanding of existentially closed H-fields is rudimentary. Many basic problems remain to be solved. Here is one:

> Is every existentially closed H-field the inductive union of its H-subfields with a smallest comparability class?

As indicated in Section 2, $\mathbb{R}((x^{-1}))^{\mathrm{LE}}$ is such an inductive union.

15. Errata to [2]

At the end of the proof of Lemma 5.3, "First assume $j < 0$" should be "First assume $j > 0$", and the subsequent inequality "$s < d'/d$" should be "$s > d'/d$". The last sentence of this proof "The case $j > 0$ is similar." should be replaced by: "Suppose $j < 0$. Then $v(d) < v(y)$, and we distinguish the cases $v(d) > 0$ (similar to the case $j > 0$), $v(d) = 0$ (where we use $s < 0$ and $v(s) < v(d'/d) = v(a'/a)$), and $v(d) < 0$ (where we use $v(a'/a) = v(d'/d) < v(s)$ and $a'/a > 0$)."

Right after the proof of Lemma 6.3 on p. 581, it is asserted, for any differential field extension $K \subseteq L$: "the subfield of L generated by any collection of intermediate Liouville extension fields is also a differential subfield of L and a Liouville extension of K. Hence there exists a biggest Liouville extension of K contained in L."

This is true with the extra assumption that C_L is algebraic over K. For a counterexample when the extra assumption is omitted, let $K = \mathbb{Q}$ and $L = \mathrm{Li}(\mathbb{R})$ (the Liouville closure of $\mathbb{R}$ as a Hardy field). Then the Hardy fields $K(x)$ and $K(x+\pi)$ are both Liouville extensions of K (with $\mathbb{Q}$ as field of constants), but $K(x, x+\pi) = K(x, \pi)$ is not, since the constant π is not algebraic over $\mathbb{Q}$.

Consequently, one should add to the hypothesis of Lemma 6.4 that C_L is algebraic over C, and in Lemma 6.6 and its proof, the phrase "not contained in $\mathbb{R}$" should be replaced by "properly containing $\mathbb{R}$."

References

1. M. Aschenbrenner, L. van den Dries, *Closed asymptotic couples,* J. Algebra **225** (2000), 309–358.
2. ———, *H-fields and their Liouville extensions,* Math. Z. **242** (2002), 543–588.
3. M. Aschenbrenner, L. van den Dries, J. van der Hoeven, *Differentially algebraic gaps,* preprint (2003).
4. J. Bochnak, M. Coste, M.-F. Roy, *Géométrie Algébrique Réelle,* Ergebnisse Math. **12**, Springer-Verlag, Berlin (1987).
5. É. Borel, *Mémoire sur les séries divergentes,* Ann. Sci. École Norm. Sup. **16** (1899), 9–131.
6. M. Boshernitzan, *New "orders of infinity",* J. Analyse Math. **41** (1982), 130–167.
7. N. Bourbaki, *Fonctions d'une Variable Réelle,* Chapitre V, Appendice *Corps de Hardy. Fonctions* (H)*,* Hermann, Paris (1976).
8. R. Brown, *Valued vector spaces of countable dimension,* Publ. Math. Debrecen **18** (1971), 149–151.
9. C. Chang, H. Keisler, *Model Theory,* 3rd ed., Studies in Logic and the Foundations of Mathematics **73**, North-Holland, Amsterdam-New York (1990).
10. L. van den Dries, *An intermediate value property for first-order differential polynomials,* Quaderni di Matematica **6** (2000), 95–105.

11. L. van den Dries, D. Marker, A. Macintyre, *Logarithmic-exponential power series,* J. London Math. Soc. **56** (1997), 417–434.
12. ———, *Logarithmic-exponential series,* Ann. Pure Appl. Logic **111** (2001), 61–113.
13. J. Écalle, *Introduction aux Fonctions Analysables et Preuve Constructive de la Conjecture de Dulac,* Actualités Mathématiques, Hermann, Paris (1992).
14. D. Grigor′ev, M. Singer, *Solving ordinary differential equations in terms of series with real exponents,* Trans. Amer. Math. Soc. **327** (1991), no. 1, 329–351.
15. G. Hardy, *Some results concerning the behaviour at infinity of a real and continuous solution of an algebraic differential equation of the first order,* Proc. London Math. Soc. **10** (1912), 451–468.
16. J. van der Hoeven, *Asymptotique Automatique,* Thèse, École Polytechnique, Paris (1997).
17. ———, *A differential intermediate value theorem,* Prépublications d'Orsay 2000-50, Université Paris-Sud, Paris (2000).
18. E. Kolchin, *Rational approximation to solutions of algebraic differential equations,* Proc. Amer. Math. Soc. **10** (1959), 238–244.
19. F.-V. Kuhlmann, *Maps on ultrametric spaces, Hensel's Lemma, and differential equations over valued fields,* Comm. Algebra, to appear.
20. F.-V. Kuhlmann, S. Kuhlmann, S. Shelah, *Exponentiation in power series fields,* Proc. Amer. Math. Soc. **125** (1997), 3177–3183.
21. S. Kuhlmann, *Ordered Exponential Fields,* Fields Institute Monographs **12**, American Mathematical Society, Providence, RI (2000).
22. Y. Matiyasevich, *Hilbert's Tenth Problem,* Foundations of Computing Series, MIT Press, Cambridge, MA, 1993.
23. S. Prieß-Crampe, *Angeordnete Strukturen: Gruppen, Körper, projektive Ebenen,* Ergebnisse Math. **98**, Springer-Verlag, Berlin (1983).
24. M. Rosenlicht, *Differential valuations,* Pacific J. Math. **86** (1980), 301–319.
25. ———, *On the value group of a differential valuation,* II, Amer. J. Math. **103** (1981), 977–996.
26. ———, *Hardy fields,* J. Math. Analysis and Appl. **93** (1983), 297–311.
27. ———, *The rank of a Hardy field,* Trans. Amer. Math. Soc. **280** (1983), 659–671.
28. ———, *Rank change on adjoining real powers to Hardy fields,* Trans. Amer. Math. Soc. **284** (1984), 829–836.
29. M. Singer, *Asymptotic behavior of solutions of differential equations and Hardy fields: preliminary report,* unpublished.

Department of Mathematics, University of California at Berkeley, Evans Hall, Berkeley, CA 94720, U.S.A.

E-mail address: `maschenb@math.berkeley.edu`

Current address: Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, 851 S. Morgan St., Chicago, IL 60607, U.S.A.

Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 W. Green St., Urbana, IL 61801, U.S.A.

E-mail address: `vddries@math.uiuc.edu`