ASYMPTOTIC DIFFERENTIAL ALGEBRA

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Abstract. We believe there is room for a subject named as in the title of this paper. Motivating examples are Hardy fields and fields of transseries. Assuming no previous knowledge of these notions, we introduce both, state some of their basic properties, and explain connections to o-minimal structures. We describe a common algebraic framework for these examples: *the category of H-fields*. This unified setting leads to a better understanding of Hardy fields and transseries from an algebraic and model-theoretic perspective.

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Introduction

In taking asymptotic expansions *à la Poincaré* we deliberately neglect *transfinitely* small terms. For example, with $f(x) := \frac{1}{1-x^{-1}} + x^{-\log x}$, we have

$$f(x) \sim 1 + \frac{1}{x} + \frac{1}{x^2} + \cdots \quad (x \to +\infty),$$

so we lose any information about the transfinitely small term $x^{-\log x}$ in passing to the asymptotic expansion of f in powers of x^{-1}. *Hardy fields* and *transseries* both provide a kind of remedy by taking into account *orders of growth* different from $\dots, x^{-2}, x^{-1}, 1, x, x^2, \dots$.

Hardy fields were preceded by du Bois-Reymond's *Infinitärcalcül* [9]. Hardy [30] made sense of [9], and focused on *logarithmic-exponential functions* (*LE-functions* for short). These are the real-valued functions in one variable defined on neighborhoods of $+\infty$ that are obtained from constants and the identity function by algebraic operations, exponentiation and taking logarithms. The asymptotic behavior of non-oscillating real-valued solutions of algebraic differential equations can often be described in terms of LE-functions (Borel [10], Lindelöf [44], Hardy [29]). See also [30] for a list of references to the literature on "orders of infinity" prior to 1910. Hardy proved the fundamental fact that the germs at $+\infty$ of LE-functions

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make up an ordered differential field: every LE-function has ultimately constant sign, is ultimately differentiable, and its derivative is again an LE-function. Bourbaki [14] took this result as the defining feature of a *Hardy field*. (See Section 1 for a precise definition of this notion.) The theory of Hardy fields has grown considerably due to the efforts of Rosenlicht [59]–[63], Boshernitzan [11], [12], [13], Shackell [67] and others. Recently, Hardy fields have shown up in model theory and its applications to real analytic geometry, via *o-minimal structures on the real field*.

Transseries can be seen as *formal* counterparts to (germs of) functions in Hardy fields. The key example of a field of transseries is the field $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ of *logarithmic-exponential series* (*LE-series* for short) over $\mathbb{R}$. It extends the field of Laurent series $\mathbb{R}[[x^{\mathbb{Z}}]]$ to an ordered differential field equipped with a natural exponential function, which agrees with the usual exponential on $\mathbb{R}$. (See Section 2.) It was introduced independently by Écalle [25] (under the name "trigèbre $\mathbb{R}[[[x]]]$ des transséries") in his work on Dulac's Problem, and by the model-theorists Dahn and Göring [15] in connection with Tarski's problem on real exponentiation. The subject of transseries has been further developed by van den Dries, Macintyre and Marker [22], [23], van der Hoeven [32]–[35], and Schmeling [65]. (The notion of a *field of transseries* is first axiomatized formally in [65], (2.2.1).) The papers [32] and [65] construct many other fields of transseries that strictly extend $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$, and [32] even considers a complex analogue of $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$. These newer fields of transseries are also beginning to play a natural role in asymptotic differential algebra. In this paper we focus on the field $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ of LE-series. Some aspects of transseries can be found already in the *generalized power series* of Hahn [27].

We have indicated two approaches to the asymptotic behavior of real-valued functions: Hardy fields and transseries. How are they related and what do they have in common? Many Hardy fields, for example the field of germs of Hardy's LE-functions, can be embedded into $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ as ordered differential fields. Such an embedding associates to an element of the Hardy field a series expansion (often divergent) in logarithmic-exponential monomials. *Asymptotic differential algebra* should help in constructing such embeddings.

With this aim in mind we introduced in [4] the purely algebraic notion of *H-field.* (In Section 3 we give the precise definition.) Each Hardy field $K \supseteq \mathbb{R}$ is an H-field, as is every ordered differential subfield $K \supseteq \mathbb{R}$ of $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$. Every H-field K carries a *valuation* $v\colon K \setminus \{0\} \to \Gamma$, taking values in an ordered abelian group Γ. Thus H-fields are amenable to the methods of valuation theory, a well-developed chapter of algebra (see [38] or [53]). In [2], [3] we explored the structure induced by the derivation on the *value group* Γ, continuing Rosenlicht [56], [58]. A basic fact is that the valuation $v(f')$ of the derivative f' of a non-zero element f of an H-field K only depends on $v(f)$, if $v(f) \neq 0$. Consequently, the logarithmic derivative on K yields a function $\psi\colon \Gamma \setminus \{0\} \to \Gamma$ via $\psi\big(v(f)\big) = v(f'/f)$, for $f \in K \setminus \{0\}$ with $v(f) \neq 0$. The pair (Γ, ψ) is called the *asymptotic couple* of K (Rosenlicht's terminology). We investigated abstract asymptotic couples, that is, pairs (Γ, ψ) where Γ is an ordered abelian group and $\psi\colon \Gamma \setminus \{0\} \to \Gamma$ a function subject to certain axioms satisfied by asymptotic couples coming from H-fields. Our results in [3] yield an elimination theory for the asymptotic relations in the H-field $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ that can be expressed in terms of its associated asymptotic couple. See Section 3 for an exposition of some of our results about H-fields and asymptotic couples. Finally, in Section 4 we touch upon the subject of algebraic differential equations over H-fields. This is a vast topic, and at present our understanding of it is only rudimentary. But some promising first steps have been made already, which we report here.

Ultimately we are interested in the *model-theoretic* properties of Hardy fields and fields of transseries. In this paper we will avoid model-theoretic language altogether, in order to keep the exposition self-contained. We prefer to direct the reader to [52] for a leisurely introduction to model theory (aimed at geometers) and to surveys of two subjects in which model theory has been particularly successful: real algebraic geometry [8] and differential algebra [47]. Among other things, model-theory suggests what a *universal domain* in these subjects should be and what it is good for, in analogy with A. Weil's universal domains in algebraic geometry. A key problem in isolating the universal domains for asymptotic differential algebra is to characterize in some useful way the algebraic differential equations that can be solved in H-fields. We refer to [5], Section 14 for some questions and speculations in this direction. Our hope is that the field $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ of logarithmic-exponential series will turn out to be such a universal domain. (See also Section 4.)

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Preliminaries. Throughout the paper, we let m and n range over the set of natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$. We put $S^0 := \{0\}$ (a one-element set), for any set S. For a field K, we put $K^\times := K \setminus \{0\}$. Below we recall some basic notions and facts from algebra that will be used freely in the sequel. The reader may skip this part and come back to it for clarification whenever necessary. More detailed information can be found in [36] (on differential algebra) and [53] (on ordered abelian groups).

Differential algebra. A **differential ring** is a commutative ring R (with 1) equipped with a **derivation**, that is, a map $a \mapsto a' \colon R \to R$ satisfying $(a+b)' = a' + b'$ and the Leibniz rule $(a \cdot b)' = a' \cdot b + a \cdot b'$, for all $a, b \in R$. A **differential field** is a differential ring whose underlying ring is a field. If K is a differential field, we denote by C_K (or C, if no confusion is possible) the **field of constants** of K: $C_K = \{c \in K : c' = 0\}$. If f is a non-zero element of a differential field, we put $f^\dagger := f'/f$, the **logarithmic derivative** of f.

Let R be a differential ring and let $Y, Y', \dots, Y^{(n)}, \dots$ be distinct indeterminates. The derivation on R extends uniquely to a derivation on the polynomial ring $R\{Y\} := R[Y, Y', \dots]$ such that $(Y^{(n)})' = Y^{(n+1)}$ for all n. The differential ring $R\{Y\}$ is called the ring of **differential polynomials** in Y with coefficients in R. If R is a domain, then so is $R\{Y\}$. Inductively we put $R\{Y_1, \dots, Y_n\} := R\{Y_1, \dots, Y_{n-1}\}\{Y_n\}$ for distinct differential interderminates $Y_1, \dots, Y_n$, $n > 0$. If K is a differential field, then the fraction field of $K\{Y\}$ is denoted by $K\langle Y\rangle$ and is called the field of **differential rational functions** in Y with coefficients in K. Given $P(Y) \in R\{Y\}$ the least n such that $P(Y) \in R[Y, Y', \dots, Y^{(n)}]$ is called the **order** of P. For $\boldsymbol{i} = (i_0, \dots, i_n) \in \mathbb{N}^{n+1}$ we put $|\boldsymbol{i}| := i_0 + i_1 + \cdots + i_n$ (the **degree of $\boldsymbol{i}$**), and we set $Y^{\boldsymbol{i}} := Y^{i_0}(Y')^{i_1} \cdots (Y^{(n)})^{i_n}$. Every differential polynomial $P(Y) \in R\{Y\}$ of order at most n can be written in the form

$$P(Y) = \sum_{\boldsymbol{i}} a_{\boldsymbol{i}} Y^{\boldsymbol{i}},$$

where the sum is understood to range over all $\boldsymbol{i} \in \mathbb{N}^{n+1}$, and $a_{\boldsymbol{i}} \in R$ for every $\boldsymbol{i}$, with $a_{\boldsymbol{i}} \neq 0$ for only finitely many $\boldsymbol{i}$. For $P \neq 0$ the (total) **degree** of P is the largest natural number d such that $d = |\boldsymbol{i}|$ for some $\boldsymbol{i} \in \mathbb{N}^{n+1}$ with $a_{\boldsymbol{i}} \neq 0$. If $P(Y) \in R\{Y\}$ is a differential polynomial and $y \in R$, we obtain an element $P(y)$ of R by substituting $y, y', \dots$ for $Y, Y', \dots$, respectively. Hence P gives rise to a **differential polynomial function** $R \to R \colon y \mapsto P(y)$.

Let L be a differential field extension of the differential field K. Then $K\langle y\rangle := K(y, y', y'', \dots)$ denotes the differential subfield of L generated over K by an element $y \in L$. Likewise, $K\langle y_1, \dots, y_n\rangle$ denotes the differential subfield of L generated over K by elements $y_1, \dots, y_n \in L$. We say that $y \in L$ is **differentially algebraic over** K if $P(y) = 0$ for some non-zero $P(Y) \in K\{Y\}$. The extension $L|K$ is called **differentially algebraic** if every element of L is differentially algebraic over K.

Ordered abelian groups. An **ordered abelian group** is an abelian group Γ (here written additively) together with a total ordering $\leqslant$ of Γ such that

$$\alpha \leqslant \beta \Rightarrow \alpha + \gamma \leqslant \beta + \gamma \qquad \text{for all } \alpha, \beta, \gamma \in \Gamma.$$

Throughout we let Γ be an ordered abelian group. Put $\Gamma^* := \Gamma \setminus \{0\}$, and

$$S^{>\alpha} := \{\gamma \in S : \gamma > \alpha\}, \qquad S^{<\alpha} := \{\gamma \in S : \gamma < \alpha\},$$

for $S \subseteq \Gamma$ and $\alpha \in \Gamma$. (Similarly for $\geqslant$ and $\leqslant$ in place of $>$ and $<$, respectively.) We say that $S \subseteq \Gamma$ is **convex** in Γ if for all $\alpha, \beta, \gamma \in \Gamma$,

$$\alpha < \gamma < \beta \;\&\; \alpha, \beta \in S \quad \Rightarrow \quad \gamma \in S.$$

We define an equivalence relation $\sim$ on Γ by

$$\alpha \sim \beta \quad :\Longleftrightarrow \qquad |\alpha| \leqslant m|\beta| \text{ and } |\beta| \leqslant n|\alpha| \text{ for some } m, n > 0.$$

Here as usual $|\alpha| = \max\{\alpha, -\alpha\}$ for $\alpha \in \Gamma$. The equivalence class of an element $\alpha \in \Gamma$ is written as $[\alpha]$, and is called its **archimedean class.** By $[\Gamma]$ we denote the set of archimedean classes of Γ, and we set $[\Gamma^*] := [\Gamma] \setminus \big\{[0]\big\}$. If $[\Gamma^*]$ is finite, we call the number of elements of $[\Gamma^*]$ the **rank** of Γ; otherwise, we say that Γ has infinite rank. Thus Γ has rank 1 if and only if Γ is isomorphic to an ordered non-trivial subgroup of the ordered additive group $\mathbb{R}$ of real numbers. An example of an ordered abelian group of rank n is $\mathbb{Z}^n$ ordered lexicographically, that is, for $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$: $a > 0 \iff$ there exists $i \in \{1, \dots, n\}$ such that $a_1 = \dots = a_{i-1} = 0$ and $a_i > 0$. We linearly order $[\Gamma]$ by setting

$$\begin{aligned} [\alpha] < [\beta] \;&:\Longleftrightarrow\; n|\alpha| < |\beta| \text{ for all } n \\ &\iff\; [\alpha] \neq [\beta] \text{ and } |\alpha| < |\beta|. \end{aligned}$$

We call Γ **divisible** if for every $\alpha \in \Gamma$ and every $n \neq 0$ there exists $\beta \in \Gamma$ with $n\beta = \alpha$; we denote this unique β by $\frac{1}{n}\alpha$. Thus if Γ is divisible, then it has a natural structure as vector space over the field $\mathbb{Q}$ of rational numbers. We consider Γ as a subgroup of the abelian group $\mathbb{Q}\Gamma = \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma$ by means of the embedding $\alpha \mapsto 1 \otimes \alpha$. We equip $\mathbb{Q}\Gamma$ with the unique linear ordering that extends the one on Γ and makes $\mathbb{Q}\Gamma$ into a divisible ordered abelian group.

1. Hardy Fields

After giving the definition and some basic properties, we discuss the dominance relations and the valuation associated to a Hardy field. We then state several extension theorems. Finally we show how Hardy fields arise in the subject of o-minimality.

Definitions and basic properties. Given a property $P(x)$, with x ranging over real numbers, we say that $P(x)$ *holds ultimately* (or *ultimately* $P(x)$) if there exists $x_0 \in \mathbb{R}$ such that $P(x)$ holds for all $x > x_0$. We define an equivalence relation on the collection of real-valued functions defined on subsets of $\mathbb{R}$ that contain an interval of the form $(a, +\infty)$ $(a \in \mathbb{R})$, by declaring f and g equivalent if ultimately $f(x) = g(x)$; we denote the equivalence class of such a function f by $\overline{f}$, and call it the **germ** of f (at $+\infty$). Adding and multiplying functions in this collection respects the equivalence relation, so we can add and multiply germs by $\overline{f} + \overline{g} = \overline{f+g}$ and $\overline{f} \cdot \overline{g} = \overline{f \cdot g}$, making the set of germs into a commutative ring $\mathcal{G}$. If f is ultimately differentiable, we define $\overline{f}\,' := \overline{f'}$. From now on we omit the bar and use the same letter for a function and its germ. In particular, we consider the field $\mathbb{R}$ of real numbers as a subring of $\mathcal{G}$, by identifying $r \in \mathbb{R}$ with the germ of the constant function with value r. Given $p \in \mathbb{N} \cup \{\infty\}$ we let $\mathcal{G}_p$ denote the subring of $\mathcal{G}$ consisting of the germs of functions that are ultimately of class C^p, and we put $\mathcal{G}_{(\infty)} := \bigcap_{p \in \mathbb{N}} \mathcal{G}_p$. Thus $\mathcal{G}_{(\infty)}$ is a differential ring with respect to the operations on germs indicated above, with $\mathcal{G}_\infty$ as a (proper) differential subring.

Definition 1.1. (N. Bourbaki, 1961 [14].) A subring K of $\mathcal{G}_1$ is called a **Hardy field** if K is a field, and $f' \in K$ for all $f \in K$.

Example. The field $\mathbb{R}(x)$ of rational functions is a Hardy field, where x denotes the germ of the identity function on $\mathbb{R}$ (so $x' = 1$). (More interesting examples are given below.)

Let K be a Hardy field. Clearly K is a differential subring of $\mathcal{G}_{(\infty)}$. Moreover, for non-zero $f \in K$ there is $g \in K$ with $f \cdot g = 1$, so ultimately $f(x) \neq 0$, hence either ultimately $f(x) < 0$ or ultimately $f(x) > 0$ (by ultimate continuity of f). We make K into an *ordered field* by declaring $f > 0$ (for $f \in K$) if ultimately $f(x) > 0$. Given $f \in K$ we also have $f' \in K$, so either $f' < 0$, or $f' = 0$, or $f' > 0$, and accordingly, f is either ultimately strictly decreasing, or ultimately constant, or ultimately strictly increasing, hence the limit $\lim\limits_{x \to +\infty} f(x)$ always exists, as an element of the extended real line $\mathbb{R}_\infty := \mathbb{R} \cup \{\pm\infty\}$.

Dominance relations. Every Hardy field is an *ordered differential field,* that is, an ordered field with a derivation on the field. (The constant field of a Hardy field is a subfield of $\mathbb{R}$.) Every ordered differential field K (with constant field C) comes equipped with a *dominance* relation: For $f, g \in K$ define

$$
\begin{aligned}
&(1.1) \qquad & f \preccurlyeq g \quad &\Longleftrightarrow \quad |f| \leqslant c|g| \text{ for some } c \in C^{>0}, \\
& & f \prec g \quad &\Longleftrightarrow \quad |f| \leqslant c|g| \text{ for all } c \in C^{>0}.
\end{aligned}
$$

(Note that $f \prec g$ if and only if $f \preccurlyeq g$ and $g \not\preccurlyeq f$.) Here we use Hardy's notations—with those of Bachmann and Landau one would write $f = O(g)$ for $f \preccurlyeq g$ and $f = o(g)$ instead of $f \prec g$. More generally:

Definition 1.2. Let K be an ordered field. A **dominance relation** on K is a binary relation $\preccurlyeq$ on K such that for all $f, g, h \in K$:

(D1) $f \preccurlyeq f$,
(D2) $f \preccurlyeq g$ and $g \preccurlyeq h \Rightarrow f \preccurlyeq h$,
(D3) $f \preccurlyeq g$ or $g \preccurlyeq f$,
(D4) $f \preccurlyeq g \Rightarrow fh \preccurlyeq gh$,
(D5) $f \preccurlyeq h$ and $g \preccurlyeq h \Rightarrow f - g \preccurlyeq h$.
(D6) $0 \leqslant f \leqslant g \Rightarrow f \preccurlyeq g$,

The relation $\preccurlyeq$ on an ordered differential field K defined in (1.1) is a dominance relation, which we call the **natural** dominance relation of K. (In Section 3 we have to consider more general dominance relations on ordered differential fields.) If $\preccurlyeq$ is a dominance relation on an ordered field K and L an ordered subfield of K, then the restriction of $\preccurlyeq$ to L is a dominance relation on L.

Let K be an ordered field and $\preccurlyeq$ a dominance relation on K. We define

$$f \prec g \quad \Longleftrightarrow \quad f \preccurlyeq g \text{ and } g \not\preccurlyeq f.$$

We shall also write $f \preccurlyeq g$ as $g \succcurlyeq f$ and $f \prec g$ as $g \succ f$. We call an element f of K **bounded** if $f \preccurlyeq 1$, **infinitesimal** if $f \prec 1$, and **infinite** if $f \succ 1$. The set of bounded elements of K and the set of infinitesimal elements of K are convex in K by (D6). We define an equivalence relation $\asymp$ on K as follows:

$$f \asymp g \qquad \Longleftrightarrow \qquad f \preccurlyeq g \text{ and } g \preccurlyeq f.$$

(If $f \asymp g$, we say that f and g are **asymptotic.**) By (D1)–(D4), the equivalence classes $v(f)$, where $f \in K^\times = K \setminus \{0\}$, are the elements of an ordered abelian group $\Gamma = v(K^\times)$; the group operation and the ordering are given by

$$v(f) + v(g) = v(f \cdot g) \quad \text{and} \quad v(f) \geqslant v(g) \Longleftrightarrow f \preccurlyeq g,$$

respectively. We also introduce a symbol $\infty \notin \Gamma$, and we put $\Gamma_\infty := \Gamma \cup \{\infty\}$ and $v(0) := \infty$. We extend addition to Γ_∞ by setting $\alpha + \infty = \infty + \alpha = \infty$ for all $\alpha \in \Gamma_\infty$, and extend the ordering of Γ to a linear ordering on Γ_∞ by setting $\gamma < \infty$ for $\gamma \in \Gamma$. The implications (D4)–(D6) translate into the following rules for the map $v\colon f \mapsto v(f), K \to \Gamma_\infty$: For $f, g \in K$,

(V1) $v(f \cdot g) = v(f) + v(g)$;
(V2) $v(f + g) \geqslant \min\bigl\{v(f), v(g)\bigr\}$;
(V3) $0 \leqslant f \leqslant g \Rightarrow v(f) \geqslant v(g)$.

The properties (V1) and (V2) express that v is a *valuation* (in the sense of Krull [53]) on K with *value group* Γ. (From (V2) it follows that $v(f + g) = v(g)$ if $v(f) > v(g)$.) Conversely, any valuation $v\colon K \to \Gamma_\infty$ satisfying (V3) gives rise to a dominance relation on K by $f \preccurlyeq g \Longleftrightarrow v(f) \geqslant v(g)$ for $f, g \in K$.

For non-zero $f, g \in K$ we define $f \sim g \;:\Longleftrightarrow\; f - g \prec g$. It is easy to see that $\sim$ is an equivalence relation on $K^\times$, with $f \sim g \Rightarrow f \asymp g$.

The valuation of a Hardy field. The natural dominance relation $\preccurlyeq$ of a Hardy field and the derived asymptotic relations $\prec$, $\asymp$,$\sim$ introduced above allow useful reinterpretations in terms of limits as $x \to +\infty$:

Lemma 1.3. *Let K be a Hardy field. For $f, g \in K$, $g \neq 0$, we have:*

(1) $f \preccurlyeq g \Longleftrightarrow \lim\limits_{x \to +\infty} \frac{f(x)}{g(x)} \in \mathbb{R}$,

(2) $f \prec g \iff \lim_{x\to+\infty} \frac{f(x)}{g(x)} = 0$,
(3) $f \asymp g \iff \lim_{x\to+\infty} \frac{f(x)}{g(x)} \in \mathbb{R}^\times$,
(4) $f \sim g \iff \lim_{x\to+\infty} \frac{f(x)}{g(x)} = 1$.

So for instance, $f \succ 1$ if and only if $\lim_{x\to+\infty} |f(x)| = +\infty$.

Remark. According to Rosenlicht ([59], p. 303), this valuation on a Hardy field was already implicit in du Bois-Reymond's paper [9], but went unnoticed until the work of Lightstone and Robinson [43] in the 1970s.

Example. Suppose $K = \mathbb{R}(x)$. Then $a_0+a_1x+\cdots+a_nx^n \sim a_nx^n$ for $a_0, a_1, \dots, a_n \in \mathbb{R}$, $a_n \neq 0$. It follows that $\Gamma = \mathbb{Z}v(x)$ with $v(x) < 0 = v(1)$, and the valuation is given by $v(g/h) = (\deg g - \deg h)v(x)$, for $g, h \in \mathbb{R}[x]$, $g, h \neq 0$.

Here are some properties of Hardy fields pertaining to the interaction between the asymptotic relations and the derivation. Recall that $f^\dagger$ denotes the logarithmic derivative $f^\dagger = f'/f = (\log|f|)'$ of $f \in K^\times$.

Proposition 1.4. *Let K be a Hardy field and $f, g \in K^\times$.*
(1) *If $f \succ 1$, then $f^\dagger > 0$.*
(2) *If $f \preccurlyeq 1$, then $f - c \prec 1$ for some $c \in \mathbb{R}$.*
(3) *If $f \preccurlyeq 1$, $g \not\asymp 1$, then $f' \prec g^\dagger$.*
(4) *If $f, g \not\asymp 1$, then $f \preccurlyeq g \iff f' \preccurlyeq g'$.*
(5) *If $f \preccurlyeq 1$, then $f' \prec 1$.*

Proof. Part (1) is clear: if f is positive infinite, say, then f is ultimately strictly increasing, hence its derivative is ultimately positive. Part (2) follows from the fact that every bounded element in a Hardy field has a limit in $\mathbb{R}$. For (3), we may assume that $f \asymp 1$; otherwise we replace f by $f + 1$. Now using Lemma 1.3, we see that $\lim_{x\to+\infty} f(x) \in \mathbb{R}^\times$ and $\lim_{x\to+\infty} g(x) \in \{0, \pm\infty\}$. Hence by l'Hospital's rule we have

$$f = fg/g \sim (fg' + f'g)/g' = f + f'g/g'.$$

So $f'g/g' \prec f \asymp 1$ and therefore $f' \prec g'/g = g^\dagger$. The proof of (4) is also essentially by l'Hospital's rule and Lemma 1.3 above. Finally, for (5) we use the fact (see Theorem 1.9 below) that any Hardy field can be enlarged to a Hardy field containing the germ x of the identity function. We can therefore assume $x \in K$. Then $f \preccurlyeq 1$ yields $f \prec x$, and hence $f' \prec x' = 1$ by part (4). $\square$

In particular, by (4) it follows that if $f \in K^\times$, $f \not\asymp 1$, then the valuation $v(f')$ of the derivative of f only depends on the valuation $v(f)$ of f, not on f itself.

Comparability and rank. Let K be a Hardy field. Elements f, g of K with $f, g \succ 1$ are called **comparable** if $|f| < |g|^n$ and $|g| < |f|^n$ for some n. Comparability is an equivalence relation among infinite elements of K, and we speak of the **comparability class** $\mathrm{Cl}(f)$ of an infinite element f of K. We linearly order the set of comparability classes of K by $\mathrm{Cl}(f) < \mathrm{Cl}(g) \iff |f|^n < |g|$ for all n. Note that $f, g \succ 1$ are comparable if and only if $v(f), v(g)$ lie in the same archimedean class of Γ, that is, if $\big[v(f)\big] = \big[v(g)\big]$ in $[\Gamma^*]$. The map $\mathrm{Cl}(f) \mapsto \big[v(f)\big]$ is an order-reversing bijection between the set of comparability classes of infinite elements of K and the set $[\Gamma^*]$ of non-zero archimedean classes of Γ. Hence the rank of the value group Γ

of K agrees with the number of comparability classes of K; we call this common number the **rank** of K. For example, a Hardy field K has rank 0 if and only if $K \subseteq \mathbb{R}$. The Hardy field $\mathbb{R}(x)$ has rank 1. The following proposition is due to Rosenlicht [60]:

Proposition 1.5. *Let $K \subseteq L$ be an extension of Hardy fields and $t_1, \dots, t_n \in L^\times$ algebraically dependent over K. There are integers $a_1, \dots, a_n \in \mathbb{Z}$, not all zero, such that $a_1 v(t_1) + \dots + a_n v(t_n) \in \Gamma = v(K^\times)$. In particular, if r is the transcendence degree of L over K, then there are at most r comparability classes of L that do not have representatives in K.*

In practice, many Hardy fields have finite rank: If $K = \mathbb{R}(f, f', f'', \dots)$ is a Hardy field and f satisfies an *algebraic differential equation* over $\mathbb{R}$, that is, $P(f) = 0$ for some non-zero differential polynomial $P(Y) \in \mathbb{R}\{Y\}$, then K has finite rank $\leqslant \operatorname{order}(P)$.

Asymptotic analysis in Hardy fields. The properties of the dominance relation in Proposition 1.4 above show that Hardy fields are very convenient for doing asymptotic analysis: if the germ of a function f lives in a Hardy field, this yields a lot of information about the growth of f. Here is one example, again due to Rosenlicht [62]. (Many more examples can be found in [60]–[63].)

Theorem 1.6. *Let K be a Hardy field of finite rank r and f a positive infinite element of K. Then there is an integer s with $|s| \leqslant r$ having the following property: For every $n > r$ there exists an infinitesimal ε in a Hardy field extension of K of finite rank $\leqslant r + 2n - s + 1$ such that*

$$f = \exp_n\bigl(\log_{n-s}(x) \cdot (1 + \varepsilon)\bigr).$$

Here and below $\exp_n g$ and $\log_n h$ denote $\exp \cdots \exp g$ and $\log \cdots \log h$ (n times), respectively, for elements g, h of a Hardy field, with $h > \exp_{n-1}(0)$ if $n > 0$. (By convention $\exp_0 g = g$, $\log_0 h = h$.) Note that since ε lies again in a Hardy field of finite rank, a similar estimate exists for ε, giving rise to a *nested asymptotic expansion* of f. For improvements of this theorem and its algorithmic aspects see [67] and [32].

On the other hand, it may be difficult to verify that a given germ $f \in \mathcal{G}_{(\infty)}$ lies in a Hardy field: this requires that for every differential polynomial $P(Y) \in \mathbb{R}\{Y\}$ the sign of a representative of the germ $P(f) \in \mathcal{G}_{(\infty)}$ is ultimately constant. As an example how this can be done, we give here the proof by Salvy and Shackell [64] of a theorem on inverses of germs in Hardy fields.

First some remarks on composition. If $f, g \in \mathcal{G}_{(\infty)}$ and $g(t) \to +\infty$ as $t \to +\infty$, then $f \circ g \in \mathcal{G}_{(\infty)}$ is by definition the germ in $\mathcal{G}_{(\infty)}$ such that ultimately $(f \circ g)(t) = f(g(t))$. If $f \in \mathcal{G}_{(\infty)}$ is ultimately strictly increasing, and $f(t) \to +\infty$ as $t \to +\infty$, then there is a unique $g \in \mathcal{G}_{(\infty)}$ such that $g(t) \to +\infty$ as $t \to +\infty$ and $f \circ g = x$; this g is then also ultimately strictly increasing, and is called the *inverse* of f.

Theorem 1.7. *Let f be a positive infinite element in a Hardy field K (so f is ultimately strictly increasing, and $f(t) \to +\infty$ as $t \to +\infty$). Then the inverse of f also belongs to a Hardy field.*

We do not know if this inverse always lies in some Hardy field extension of K.

Proof. Let g be the inverse of f. Using $g' = \frac{1}{f'\circ g}$, induction on $n \geqslant 1$ yields

$$g^{(n)} = R_n\big(f'\circ g, f''\circ g, \dots, f^{(n)}\circ g\big)/(f'\circ g)^{N(n)}$$

with $R_n(X_1,\dots,X_n) \in \mathbb{R}[X_1,\dots,X_n]$ and $0 < N(n) \in \mathbb{N}$. Let $P(Y) \in \mathbb{R}\{Y\}$ be a differential polynomial over $\mathbb{R}$ such that (a representative of) $P(g) \in \mathcal{G}_{(\infty)}$ has arbitrarily large real zeros. We show that then $P(g) = 0$ in $\mathcal{G}_{(\infty)}$. The identities above for the $g^{(n)}$ yield $P(g) = R(f)\circ g$ where $R(Z) \in \frac{1}{(Z')^N}\mathbb{R}(x)\{Z\}$ is a differential rational function over $\mathbb{R}(x)$. Hence $R(f)$ is an element of the Hardy field $\mathbb{R}(x, f, f', \dots)$ and (a representative of it) has arbitrarily large real zeros, so $R(f) = 0$, and thus $P(g) = 0$. □

Next we discuss some tools to construct new Hardy fields from old ones.

Extension theorems. Until the end of Theorem 1.12 we let K denote a Hardy field. Here are some theorems that allow one to enlarge K by solutions of certain algebraic differential equations.

Algebraic equations. An ordered field F is said to be *real closed* if it has the intermediate value property for polynomials in one variable, that is, given $P(Y) \in F[Y]$ and $a < b$ in F such that $P(a)$ and $P(b)$ are non-zero and of opposite sign, there exists $y \in F$ with $P(y) = 0$. (For example, the field $\mathbb{R}$ of real numbers is real closed.) Every ordered field F has a *real closure,* that is, a real closed ordered field extension of F which is algebraic over F; such a real closure is unique up to isomorphism of ordered fields over F, so we can speak of *the* real closure of F, denoted by F^{rc}. (See [8] or [51] for this and other basic facts about ordered fields.)

Theorem 1.8. *The set of all germs $y \in \mathcal{G}_0$ that satisfy a polynomial equation $P(y) = 0$ for non-zero $P(Y) \in K[Y]$ is a Hardy field, and is a real closure of the ordered field K.*

This theorem is due to A. Robinson [54]. An earlier proof for $K \subseteq \mathcal{G}_\infty$ by Sjödin [71] has a gap (see the first sentence of the proof of Lemma 1 on p. 219). An efficient proof is in Rosenlicht [59].

First-order equations. After algebraic equations, the simplest algebraic differential equations are the ones of order 1. The following theorem is due to M. Singer [70], [59], with less general versions by Hardy [29] and Marić [46].

Theorem 1.9. *Let $F(Y), G(Y) \in K[Y]$ and $y \in \mathcal{G}_1$ be such that*

$$G(y) \neq 0 \text{ and } y'G(y) = F(y) \quad (\text{in } \mathcal{G}_1).$$

Then the ring of germs $K[y]$ is an integral domain with fraction field $K(y) \subseteq \mathcal{G}_1$, and $K(y)$ is a Hardy field.

Call a Hardy field **Liouville closed** if it is real closed and contains with each element f also an antiderivative $\int f$ and its exponential $\exp f$. The last two theorems immediately imply:

Corollary 1.10. (Bourbaki [14].) *There exists a smallest Hardy field* $\mathrm{Li}(K)$*, called the* **Liouville closure** *of K, which contains $\mathbb{R}$ and is Liouville closed.* □

In particular, every Hardy field K extends to a smallest Hardy field $L \supseteq \mathbb{R}$ which is **closed under powers,** that is, $f^c \in L$ for all $f \in L^{>0}$ and $c \in \mathbb{R}$. If K has finite rank, then so does L, see [61]. Hardy fields of finite rank which contain $\mathbb{R}$ and are closed under powers were termed **Rosenlicht fields** in [67]. They play a role in the asymptotics of Hardy field solutions to algebraic differential equations with constant coefficients; see [67], [69].

Let $P(Y) \in K\{Y\}$ be a differential polynomial of order 1. In [59], Theorem 3, Rosenlicht gives a necessary and sufficient condition on a germ $y \in \mathcal{G}_1$ with $P(y) = 0$ to be such that there exists a Hardy field extension of K containing y. This allows one to show that all solutions in $\mathcal{G}_1$ of an equation like

$$(Y')^2 + 3YY' + Y^2 = 1$$

lie in a Hardy field. ([59], p. 303.) A related fact is the following *intermediate value property* for first-order differential polynomials in [20]:

Theorem 1.11. *Suppose $P(a)$ and $P(b)$ are non-zero and of opposite sign in K, where $a, b \in K$ and $a < b$. Then there exists an element y in a Hardy field extension of K such that $a < y < b$ and $P(y) = 0$.*

There are examples for K, P, a and b satisfying the hypothesis of the theorem and an ultimately analytic germ y such that $a < y < b$ and $P(y) = 0$, but y does *not* lie in any Hardy field (see [20], Remark 3).

Higher-order equations. Here, our knowledge is rudimentary compared to the case of equations of order 1. The next theorem is in Boshernitzan [12] (Theorem 17.7) and Rosenlicht [63], and concerns linear differential equations of order 2. Note first that an equation of the form

$$u'' + au' + bu = 0$$

with a, b in a Hardy field K can be transformed into an equation of the form

$$4y'' - fy = 0 \tag{1.2}$$

by a change of variables $u = gy$, where $f = a^2 - 4b$ and $g = e^{-\frac{1}{2}\int a}$ is a non-zero solution to the equation $2g' + ag = 0$ in a Hardy field extension of K (which can be taken of finite rank, provided K is of finite rank, Proposition 1.5). Henceforth we may restrict attention to linear differential equations of the form (1.2).

Theorem 1.12. *Suppose that K has finite rank, and let $f \in K$. Then the equation* (1.2) *has a non-trivial solution in a Hardy field extension of K of finite rank if and only if*

$$f > -\left(\frac{1}{(\ell_0)^2} + \frac{1}{(\ell_0\ell_1)^2} + \frac{1}{(\ell_0\ell_1\ell_2)^2} + \cdots + \frac{1}{(\ell_0\ell_1\cdots\ell_n)^2}\right) \tag{1.3}$$

for some n, where $\ell_n := \log_n x$.

The proof of the "only if" direction uses valuation theory, see [63], Theorem 3, part (3). The statement there is not quite correct: after "has a nonzero solution in some Hardy field" add "each infinitely increasing element of which is $> \ell_n$ for some n." (This is connected with the *gap problem* discussed in Section 4 below.) For the converse, one passes to the first-order Ricatti equation

$$2z' + z^2 = f \tag{1.4}$$

associated to (1.2) which is satisfied by $z = 2y^\dagger$ whenever y is a non-zero solution to (1.2). An **oscillating solution** to (1.4) is a non-zero germ $z \in \mathcal{G}_1$ satisfying (1.4) for which there exists a sequence $\{x_n\}$ of real numbers with $x_n \to +\infty$ as $n \to +\infty$ and $z(x_n) = 0$ for all sufficiently large n. We say that f **generates oscillations** if (1.4) has an oscillating solution. In this case, *every* non-zero solution of (1.4) in $\mathcal{G}_1$ is oscillating. Also, if f satisfies the inequalities (1.3) for all n, then f does not generate oscillations ([31], p. 325), and every solution $z \in \mathcal{G}_1$ to (1.4) lives in a Hardy field extension of K. By finding a non-trivial solution to the first-order equation $2y' - zy = 0$ in a bigger Hardy field we obtain the desired non-trivial solution y to our original equation (1.2). With some effort, it is possible to describe the asymptotic expansions at $+\infty$ of two linearly independent solutions to (1.2), see [63].

Hardy fields via o-minimal structures. An important natural source for Hardy fields are o-minimal structures on the field $\mathbb{R}$ of real numbers.

Definition 1.13. A **structure** $\mathcal{S}$ on the field $\mathbb{R}$ is a family $(\mathcal{S}_n)_{n\in\mathbb{N}}$ where each $\mathcal{S}_n$ is a collection of subsets of $\mathbb{R}^n$, such that

(1) $\mathbb{R}^n \in \mathcal{S}_n$, and if $A, B \in \mathcal{S}_n$, then $A \cup B$ and $\mathbb{R}^n \setminus A$ belong to $\mathcal{S}_n$;
(2) $\Delta := \big\{(x,y) \in \mathbb{R}^2 : x = y\big\} \in \mathcal{S}_2$;
(3) the graphs of addition and multiplication on $\mathbb{R}$ belong to $\mathcal{S}_3$;
(4) if $A \in \mathcal{S}_n$, then $A \times \mathbb{R} \in \mathcal{S}_{n+1}$ and $\mathbb{R} \times A \in \mathcal{S}_{1+n}$;
(5) if $A \in \mathcal{S}_{n+1}$, then $\pi(A) \in \mathcal{S}_n$, where $\pi\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection onto the first n coordinates: $\pi(x_1, \dots, x_{n+1}) = (x_1, \dots, x_n)$.

It is because of condition (3) that we are dealing with a structure *on the field* $\mathbb{R}$, and not just with a structure *on the set* $\mathbb{R}$. It is easy to see that for any structure $\mathcal{S}$ on the field $\mathbb{R}$, the ordering

$$\big\{(x,y) \in \mathbb{R}^2 : x < y\big\}$$

of the real line belongs to $\mathcal{S}_2$. Usually, such a structure is generated from a collection $\mathcal{A}$ of subsets of the cartesian spaces $\mathbb{R}^n$ (for various n) which contains the graphs of $+$ and $\times$ by adding to $\mathcal{A}$ the equality relation Δ and closing off under union, complement, cartesian products with $\mathbb{R}$ and projections. In this way we obtain $\mathcal{S} = \mathbb{R}_{\mathcal{A}}$, the smallest structure on the field $\mathbb{R}$ containing $\mathcal{A}$. The sets in a structure $\mathcal{S}$ on the field $\mathbb{R}$ are traditionally called **definable** when $\mathcal{S}$ is clear from context. If $\mathcal{S}$ is given in the form $\mathcal{S} = \mathbb{R}_{\mathcal{A}}$, we also say that a set $A \in \mathcal{S}$ is **definable from** $\mathcal{A}$.

Let $\mathcal{S}$ be a structure on the field $\mathbb{R}$. If every singleton $\{r\}$ with $r \in \mathbb{R}$ is definable, then every interval is definable. Here and below, *interval* always refers to an open interval (a,b) with $a < b$ in $\mathbb{R}_\infty = \mathbb{R} \cup \{\pm\infty\}$. It is often a routine matter to check that a geometric construction of finitary nature, when applied to definable sets, again produces definable sets. For example, the interior and the closure of a definable subset of $\mathbb{R}^n$ are also definable. (We refer to [18] for proofs of these and other basic facts about structures.) A map $f\colon A \to \mathbb{R}^n$ with $A \subseteq \mathbb{R}^m$ is said to be *definable*, if its graph is a definable subset of $\mathbb{R}^{m+n}$.

Example. Let **alg** be the collection whose elements are the singletons $\{r\}$ with $r \in \mathbb{R}$ and the graphs of addition and multiplication (as subsets of $\mathbb{R}^3$). Then $\mathbb{R}_{\mathbf{alg}}$ is the smallest structure on the field $\mathbb{R}$ in which all singletons are definable. The definable subsets of $\mathbb{R}^n$ are exactly the *semialgebraic* subsets of $\mathbb{R}^n$, that is, boolean combinations of sets of the form $\big\{x \in \mathbb{R}^n : f(x) > 0\big\}$ where f is a polynomial with real coefficients in n indeterminates. (That every semialgebraic subset is definable is easy to show— that they are exactly the definable sets is the content of the Tarski-Seidenberg Theorem [66], [72].)

Semialgebraic sets have many remarkable finiteness properties, see [8]. For example, every semialgebraic set has only finitely many connected components, each of which is again semialgebraic. In general, however, the sets definable from a given collection $\mathcal{A}$ may be much more complex than the sets in $\mathcal{A}$, even pathological. (For instance, if we add to **alg** from the previous example also the set $\mathbb{Z}$ of integers, then each Borel subset of each $\mathbb{R}^n$ becomes definable; see [18], p. 16.) One is usually interested in structures whose sets have *tame* topological properties (similar to the semialgebraic sets). A simple condition which ensures this is the o-minimality axiom:

Definition 1.14. A structure $\mathcal{S}$ on the field $\mathbb{R}$ is called **o-minimal** (abbreviating *order-minimal*) if the sets in $\mathcal{S}_1$ are exactly the subsets of $\mathbb{R}$ which have only finitely many connected components (i.e., are finite unions of intervals and points).

This definition of o-minimality concerns only the definable subsets of $\mathbb{R}$, but it yields the kind of finiteness properties of subsets of higher cartesian power $\mathbb{R}^n$ that are familiar from semialgebraic geometry. For example, every set in an o-minimal structure on the field $\mathbb{R}$ has only finitely many connected components, each of which also belongs to the same o-minimal structure. We refer to [18] for a development of this kind of tame topology. Of relevance here is the following result describing the *one-variable* definable functions:

Theorem 1.15. (Smooth Monotonicity Theorem, [16].) *Let $\mathcal{S}$ be an o-minimal structure on the field $\mathbb{R}$, and let $f\colon I \to \mathbb{R}$ be a definable function on an interval $I = (a, b)$, where $a, b \in \mathbb{R}_\infty = \mathbb{R} \cup \{\pm\infty\}$, $a < b$. Given a positive integer p, there are real numbers $a_1, \dots, a_k$ with $a = a_0 < a_1 < \cdots < a_k < a_{k+1} = b$ such that for each $i = 0, \dots, k$ the restriction $f|(a_i, a_{i+1})$ is of class C^p, and either constant, or strictly increasing, or strictly decreasing.*

(For all presently known o-minimal structures on the field $\mathbb{R}$, the Smooth Monotonicity Theorem holds even with $p = \infty$. There are, however, o-minimal structures on $\mathbb{R}$ containing definable functions which are not piecewise *analytic*, see Example (2) below.)

The class of semialgebraic sets is clearly o-minimal. In the last 20 years there have been many constructions of o-minimal structures on the field $\mathbb{R}$ that strictly extend $\mathbb{R}_{\mathbf{alg}}$, with interesting consequences also in geometry. We shall only mention a few such constructions, referring the reader to [19] for more examples and further details concerning Examples (1) and (3) below:

Example 1. Let **an** be the collection which consists of **alg** as well as the graphs of all *analytic* functions f on I^n (for various n), that is, $f = g|I^n$ for some real analytic function g on an open neighborhood of I^n; here and in the next example $I = [-1, 1]$. The subsets of $\mathbb{R}^n$ definable from **an** are exactly the sets that are subanalytic in the projective space $\mathbb{P}^n(\mathbb{R})$, as was shown in [17] using Gabrielov's *theorem of the complement* [26]. Equivalently, a set $S \subseteq \mathbb{R}^n$ is definable from **an** if and only if $\tau(S)$ is the image under the projection map $I^{n+k} \to I^n$ of the zero set of an analytic function on I^{n+k}, for some k. Here $\tau\colon \mathbb{R}^n \to I^n$ is the map $(x_1, \dots, x_n) \mapsto \big(x_1/\sqrt{1 + x_1^2}, \dots, x_n/\sqrt{1 + x_n^2}\big)$. It is a result of Łojasiewicz [45] that zerosets of analytic functions on cubes I^N have only finitely many connected components; it follows that $\mathbb{R}_{\mathbf{an}}$ is o-minimal.

Example 2. A remarkable recent generalization of Example (1) is due to Rolin, Speissegger and Wilkie [55]. Fix a sequence $M = (M_0, M_1, \dots)$ of real numbers with $1 \leqslant M_0 \leqslant M_1 \leqslant \cdots$ which satisfies

$$\sum_{i=0}^{\infty} \frac{M_i}{M_{i+1}} = +\infty \quad \text{and} \quad \left(\frac{M_i}{i!}\right)^2 \leqslant \frac{M_{i-1}}{(i-1)!} \cdot \frac{M_{i+1}}{(i+1)!} \text{ for all } i > 0.$$

Let $\mathcal{C}_n(M)$ be the class of all functions $f\colon I^n \to \mathbb{R}$ with $f = g|I^n$ for some C^∞-function g on an open neighborhood U of I^n such that for some $A \in \mathbb{R}^{>0}$,

$$|g^{(\boldsymbol{i})}(x)| \leqslant A^{|\boldsymbol{i}|+1} M_{|\boldsymbol{i}|} \qquad \text{for all } x \in U \text{ and } \boldsymbol{i} \in \mathbb{N}^n.$$

We call $\mathcal{C}_n(M)$ the **Denjoy-Carleman class on I^n associated to M.** It is well-known that $\mathcal{C}_n(M)$ is a *quasi-analytic class,* that is, the Taylor series of any $f \in \mathcal{C}_n(M)$ at any point of $(-1,1)^n$ uniquely determines f among all functions in $\mathcal{C}_n(M)$. Put $M^{(j)} := (M_j, M_{j+1}, \dots)$ for $j \in \mathbb{N}$, and let $\mathcal{C}(M)$ be the collection consisting of **alg** as well as the graphs of all functions f that belong to $\mathcal{C}_n(M^{(j)})$ for some n and j. The main result of [55] is that $\mathbb{R}_{\mathcal{C}(M)}$ is an o-minimal structure on the real field. If $M_i = i!$ for all $i \geqslant 0$, then $\mathbb{R}_{\mathcal{C}(M)} = \mathbb{R}_{\mathbf{an}}$.

Example 3. Let **alg**, exp be the collection consisting of **alg** and the graph of the usual exponential function on $\mathbb{R}$. By a remarkable theorem of Wilkie [73], the sets definable from **alg**, exp are exactly the **subexponential sets**: An **exponential set** in $\mathbb{R}^n$ is a set of the form

$$\bigl\{x \in \mathbb{R}^n : P(x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n}) = 0\bigr\},$$

where P is a polynomial with real coefficients in $2n$ indeterminates, and a **subexponential set** in $\mathbb{R}^n$ is the image of an exponential set in $\mathbb{R}^{n+k}$ (for some k) under the projection $\mathbb{R}^{n+k} \to \mathbb{R}^n$ onto the first n coordinates. By Khovanskii [37] the exponential sets, and hence the subexponential sets, have only finitely many connected components. Thus $\mathbb{R}_{\mathbf{alg},\exp}$ is o-minimal. By adapting Wilkie's methods, one may also show that $\mathbb{R}_{\mathbf{an},\exp}$ is o-minimal, where **an**, exp consists of **an** together with the graph of the exponential function [24].

An important consequence of the Smooth Monotonicity Theorem is the following:

Corollary 1.16. *Let $\mathcal{S}$ be an o-minimal structure on $\mathbb{R}$. The germs of definable real valued functions on half-lines $(a, +\infty)$, $a \in \mathbb{R}$, form a Hardy field, which we denote by $H(\mathcal{S})$.*

For some o-minimal structures on $\mathbb{R}$, the associated Hardy fields can be described explicitly:

Examples. A *Puiseux series* in x^{-1} with real coefficients is a formal series

$$(1.5) \quad f = \sum_{i=k}^{\infty} a_i x^{-i/d} \qquad \text{with } d, k \in \mathbb{Z},\ d > 0, \text{ and } a_i \in \mathbb{R} \text{ for all } i \in \mathbb{Z},\ i \geqslant k.$$

For $d = 1$ we have a (formal) *Laurent series* in x^{-1} with coefficients in $\mathbb{R}$. We add and multiply Puiseux series in the natural way, and with these operations the set $\mathrm{P}(\mathbb{R})$ of Puiseux series is a field containing the set $\mathbb{R}[[x^{\mathbb{Z}}]] = \mathbb{R}((x^{-1}))$ of Laurent series in x^{-1} as a subfield. We make $\mathrm{P}(\mathbb{R})$ into an ordered differential field by means of the ordering given by

$$f > 0 \qquad :\Longleftrightarrow \qquad a_k > 0,$$

for f as in (1.5) with $a_k \neq 0$, and the derivation

$$f = \sum_{i=k}^{\infty} a_i x^{-i/d} \mapsto f' := \sum_{i=k}^{\infty} a_i(-i/d)\, x^{-(i+d)/d}.$$

The ordered differential field $H(\mathbb{R}_{\mathbf{alg}})$ is isomorphic to the ordered differential subfield of $\mathrm{P}(\mathbb{R})$ consisting of all Puiseux series that are algebraic over the field of rational functions $\mathbb{R}(x)$. The ordered differential field $H(\mathbb{R}_{\mathbf{an}})$ is isomorphic to the ordered differential subfield of $\mathrm{P}(\mathbb{R})$ consisting of the real Puiseux series in x^{-1} that converge for all sufficiently large values of x. In both cases, the isomorphism from series field to Hardy field is given by summing the convergent series for sufficiently large real values of x. The Hardy field $H(\mathbb{R}_{\mathbf{alg},\exp})$ contains Hardy's field of germs of logarithmic-exponential functions (as an ordered differential subfield).

While o-minimal structures on the field $\mathbb{R}$ yield interesting examples of Hardy fields, conversely, Hardy field theory also has striking applications to o-minimality. An example, is the following dichotomy found by C. Miller [48].

Theorem 1.17. *Let $\mathcal{S}$ be an o-minimal structure on the field $\mathbb{R}$. Then either the exponential function is definable, or for each each $f \in H(\mathcal{S})$ there exists n such that $|f| < x^n$.*

The proof of this theorem, besides special properties of $H(\mathcal{S})$ (closure under composition) uses the following observation on Hardy fields due to Rosenlicht [60] (Proposition 6): if a germ f belongs to a Hardy field $K \supseteq \mathbb{R}$ and $f > x^n$ for all n, then there is $g \in K$ such that $g \sim \log f$.

If the second alternative in the theorem above holds, we say that $\mathcal{S}$ is **polynomially bounded.** (For example, $\mathbb{R}_{\mathbf{an}}$ and $R_{\mathcal{C}(M)}$ are polynomially bounded, whereas $\mathbb{R}_{\mathbf{alg},\exp}$ clearly is not.) If $\mathcal{S}$ is a polynomially bounded o-minimal structure on the field $\mathbb{R}$, then the value group of $H(\mathcal{S})$ is naturally isomorphic to the ordered additive group of a subfield of $\mathbb{R}$, see [48]. In particular, $H(\mathcal{S})$ has rank 1.

2. The Field of Logarithmic-Exponential Series

In this section, we first introduce fields of transseries with monomials from an ordered abelian group. An important example is the field of logarithmic-exponential series. We discuss some of its basic properties and outline its construction.

Transseries fields. Let $\mathfrak{M}$ be an ordered abelian group, written *multiplicatively,* with identity 1. We refer to the elements of $\mathfrak{M}$ as **monomials**, write the ordering on $\mathfrak{M}$ as $\preccurlyeq$, and put $\mathfrak{m} \prec \mathfrak{n}$ if $\mathfrak{m} \preccurlyeq \mathfrak{n}$ and $\mathfrak{m} \neq \mathfrak{n}$, for $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}$. Let C be a field. (Usually, $C = \mathbb{R}$.) A **transseries** with coefficients in C and monomials from $\mathfrak{M}$ is a mapping $f\colon \mathfrak{M} \to C$ whose **support**

$$\operatorname{supp} f := \bigl\{\mathfrak{m} \in \mathfrak{M} : f(\mathfrak{m}) \neq 0\bigr\}$$

is **noetherian** (or **anti-well-ordered**), that is, there exists no infinite sequence $\mathfrak{m}_1, \mathfrak{m}_2, \dots$ of monomials in $\operatorname{supp} f$ with $\mathfrak{m}_1 \prec \mathfrak{m}_2 \prec \cdots$. We put $f_{\mathfrak{m}} = f(\mathfrak{m})$, and we usually write f as a formal sum

$$f = \sum_{\mathfrak{m}\in\mathfrak{M}} f_{\mathfrak{m}}\mathfrak{m}.$$

We denote the set of transseries with coefficients in C and monomials from $\mathfrak{M}$ by $C[[\mathfrak{M}]]$. It was first noted by Hahn [27] that $C[[\mathfrak{M}]]$ is a field with respect to the natural addition and multiplication of transseries:

$$f+g = \sum_{\mathfrak{m}\in\mathfrak{M}} (f_{\mathfrak{m}}+g_{\mathfrak{m}})\,\mathfrak{m}, \qquad f\cdot g = \sum_{\mathfrak{m}\in\mathfrak{M}} \left(\sum_{\mathfrak{u}\cdot\mathfrak{v}=\mathfrak{m}} f_{\mathfrak{u}}\cdot g_{\mathfrak{v}}\right)\mathfrak{m}.$$

We call $C[[\mathfrak{M}]]$ the **transseries field** with coefficients in C and monomials from $\mathfrak{M}$. It contains C as a subfield, identifying $c\in C$ with the series $f\in C[[\mathfrak{M}]]$ such that $f_1 = c$ and $f_{\mathfrak{m}} = 0$ for $\mathfrak{m}\neq 1$. Given $f\in C[[\mathfrak{M}]]$ we call $f_1\in C$ the **constant term** of f.

Example. Let R be an ordered subgroup of the ordered additive group $\mathbb{R}$, and let $\mathfrak{M} = x^R$ be a multiplicative copy of R, with order-preserving isomorphism $r\mapsto x^r\colon R\to x^R$. Then $C[[\mathfrak{M}]] = C[[x^R]]$ is a transseries field with coefficients in C and monomials of the form x^r, $r\in R$. Taking $R=\mathbb{Z}$ we obtain the field of formal Laurent series in descending powers of x with coefficients in C. Moreover we have $\mathrm{P}(\mathbb{R}) = \bigcup_{n>0}\mathbb{R}[[x^{\frac{1}{n}\mathbb{Z}}]]$ (inside $\mathbb{R}[[x^{\mathbb{Q}}]]$).

The support of any non-zero transseries $f\in C[[\mathfrak{M}]]$, being noetherian, has a maximal element (with respect to $\preccurlyeq$), called the **dominant monomial**

$$\mathfrak{d}(f) = \max\operatorname{supp} f$$

of f. We also set $\mathfrak{d}(0) := 0$, and extend $\preccurlyeq$ to a linear ordering on $\{0\}\cup\mathfrak{M}$ by declaring $0\preccurlyeq\mathfrak{m}$ for $\mathfrak{m}\in\mathfrak{M}$. This linear ordering is extended to a binary relation on $C[[\mathfrak{M}]]$ by

$$f\preccurlyeq g \qquad :\Longleftrightarrow \qquad \mathfrak{d}(f)\preccurlyeq\mathfrak{d}(g).$$

Every transseries $f\in C[[\mathfrak{M}]]$ can be decomposed as

$$f = f^{\uparrow} + f_1 + f^{\downarrow}$$

where

$$f^{\uparrow} = \sum_{\mathfrak{m}\succ 1} f_{\mathfrak{m}}\mathfrak{m} \qquad (\textbf{infinite part} \text{ of } f),$$

$$f^{\downarrow} = \sum_{\mathfrak{m}\prec 1} f_{\mathfrak{m}}\mathfrak{m} \qquad (\textbf{infinitesimal part} \text{ of } f).$$

This gives rise to a decomposition of $C[[\mathfrak{M}]]$ into a direct sum of C-vector spaces:

$$C[[\mathfrak{M}]] = C[[\mathfrak{M}]]^{\uparrow}\oplus C\oplus C[[\mathfrak{M}]]^{\downarrow}$$

where

$$C[[\mathfrak{M}]]^{\uparrow} := \big\{f\in C[[\mathfrak{M}]] : \mathfrak{m}\succ 1 \text{ for all } \mathfrak{m}\in\operatorname{supp} f\big\}$$

$$C[[\mathfrak{M}]]^{\downarrow} := \big\{f\in C[[\mathfrak{M}]] : \mathfrak{m}\prec 1 \text{ for all } \mathfrak{m}\in\operatorname{supp} f\big\}.$$

(These notations are from [32].)

The field $C[[\mathfrak{M}]]$ comes with a natural notion of summation of (possibly infinite) families of elements of $C[[\mathfrak{M}]]$: Such a family $(f_i)_{i\in I}$ is called **noetherian** if the union $\bigcup_{i\in I}\operatorname{supp} f_i$ is noetherian, and given $\mathfrak{m}\in\mathfrak{M}$ there are only finitely many

$i \in I$ with $\mathfrak{m} \in \operatorname{supp} f_i$; in this case we define $\sum_{i\in I} f_i$ to be the element of $C[[\mathfrak{M}]]$ such that $\big(\sum_{i\in I} f_i\big)(\mathfrak{m}) = \sum_{i\in I} f_i(\mathfrak{m})$ for all $\mathfrak{m} \in \mathfrak{M}$.

Ordering of transseries. Suppose that C is an ordered field. Then we make $C[[\mathfrak{M}]]$ into an ordered field extension of C as follows: for $0 \neq f \in C[[\mathfrak{M}]]$ define

$$f > 0 \quad :\Longleftrightarrow \quad f_{\mathfrak{d}(f)} > 0.$$

The ordered field $C[[\mathfrak{M}]]$ is real closed if and only if C is real closed and $\mathfrak{M}$ is divisible (see, e.g., [51], §8). Note that for $f, g \in C[[\mathfrak{M}]]$,

$$f \preccurlyeq g \qquad \Longleftrightarrow \qquad |f| \leqslant c|g| \text{ for some } c \in C^{>0},$$

and that $\preccurlyeq$ is a dominance relation on the ordered field $C[[\mathfrak{M}]]$. The map

$$\mathfrak{m} \mapsto v(\mathfrak{m}) \colon \mathfrak{M} \to \Gamma$$

is an isomorphism from the multiplicatively written group $\mathfrak{M}$ of monomials of $C[[\mathfrak{M}]]$ onto the additively written value group Γ of $C[[\mathfrak{M}]]$. This group isomorphism is *order-reversing:*

$$\mathfrak{m} \preccurlyeq \mathfrak{n} \qquad \Longleftrightarrow \qquad v(\mathfrak{m}) \geqslant v(\mathfrak{n}).$$

Suppose $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are subgroups of $\mathfrak{M}$ with $\mathfrak{M}_1$ convex in $\mathfrak{M}$, $\mathfrak{M} = \mathfrak{M}_1 \cdot \mathfrak{M}_2$, and $\mathfrak{M}_1 \cap \mathfrak{M}_2 = \{1\}$. Then we have an ordered field isomorphism

$$f = \sum_{\mathfrak{m}\in\mathfrak{M}} f_{\mathfrak{m}}\mathfrak{m} \mapsto \sum_{\mathfrak{m}_2\in\mathfrak{M}_2} \left(\sum_{\mathfrak{m}_1\in\mathfrak{M}_1} f_{\mathfrak{m}_1\mathfrak{m}_2}\mathfrak{m}_1 \right) \mathfrak{m}_2 \quad : \quad C[[\mathfrak{M}]] \to C[[\mathfrak{M}_1]][[\mathfrak{M}_2]]$$

which is the identity on C. We identify $C[[\mathfrak{M}]]$ and $C[[\mathfrak{M}_1]][[\mathfrak{M}_2]]$ via this isomorphism whenever convenient.

Logarithmic-exponential series. Alling [1] (and Laugwitz [42] in a special case) extended analytic functions to the field $\mathbb{R}[[\mathfrak{M}]]$ as follows. Every analytic function $f \colon I \to \mathbb{R}$, where $I = (a, b) \subseteq \mathbb{R}$ is an interval ($a, b \in \mathbb{R}_\infty$, $a < b$) extends naturally to $\widehat{f} \colon \widehat{I} \to \mathbb{R}[[\mathfrak{M}]]$, where

$$\widehat{I} := \big\{ g \in \mathbb{R}[[\mathfrak{M}]] : a < g < b \big\} = \big\{ c + \varepsilon : c \in I, \varepsilon \in \mathbb{R}[[\mathfrak{M}]]^{\downarrow} \big\},$$

$$\widehat{f}(c + \varepsilon) := \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} \varepsilon^n \qquad \text{for } c \in I \text{ and } \varepsilon \in \mathbb{R}[[\mathfrak{M}]]^{\downarrow}.$$

The infinite sum on the right-hand side makes sense in $\mathbb{R}[[\mathfrak{M}]]$ since the family $\big(\frac{f^{(n)}(c)}{n!}\varepsilon^n\big)_{n\in\mathbb{N}}$ is noetherian; see [21]. For example, the real exponential function $c \mapsto e^c \colon \mathbb{R} \to \mathbb{R}^{>0}$ extends to the function

$$c + \varepsilon \mapsto e^{c+\varepsilon} := e^c \cdot \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \qquad (c \in \mathbb{R} \text{ and } \varepsilon \in \mathbb{R}[[\mathfrak{M}]]^{\downarrow}) \tag{2.1}$$

from $\widehat{\mathbb{R}} = \mathbb{R} \oplus \mathbb{R}[[\mathfrak{M}]]^{\downarrow}$ to $\mathbb{R}[[\mathfrak{M}]]^{>0}$. Likewise, every analytic function $U \to \mathbb{R}$, where $U \subseteq \mathbb{R}^n$ is open, extends naturally to an $\mathbb{R}[[\mathfrak{M}]]$-valued function with domain the set of all points in $\mathbb{R}[[\mathfrak{M}]]^n$ at infinitesimal distance to a point in U. It follows that the analytic functions on cubes $[-1, 1]^N$ extend naturally to the corresponding cubes over $\mathbb{R}[[\mathfrak{M}]]$; this leads to a better understanding of $\mathbb{R}_{\mathbf{an}}$, see [21].

One drawback is that $\mathbb{R}[[\mathfrak{M}]]$ does not support a reasonable (total) exponential function if $\mathfrak{M} \neq \{1\}$. For example, since $x > \mathbb{R}$ in $\mathbb{R}[[x^{\mathbb{Z}}]]$, we must expect any such operation on $\mathbb{R}[[x^{\mathbb{Z}}]]$ to satisfy $\exp x > x^n$ for all n, which is clearly impossible.

The next theorem, due to F.-V. Kuhlmann, S. Kuhlmann and S. Shelah [40] (see also [41]) is a general negative result of this kind. Here and below, an **exponential function** on an ordered field K is an isomorphism $f \mapsto \exp(f)$ between the ordered additive group of K and the ordered multiplicative group $K^{>0}$ of positive elements of K. (In this case we often write e^f instead of $\exp(f)$, and the inverse of exp is usually denoted by $\log\colon K^{>0} \to K$.)

Theorem 2.1. *If $\mathfrak{M} \neq \{1\}$, then there does not exist an exponential function on the ordered field $\mathbb{R}[[\mathfrak{M}]]$.*

Nonetheless, we can extend $x^{\mathbb{Z}}$ canonically to a large ordered multiplicative group $\mathfrak{M}^{\mathrm{LE}}$ of so-called **LE-monomials**, and $\mathbb{R}[[x^{\mathbb{Z}}]]$ to a real closed *subfield* $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ of the transseries field $\mathbb{R}[[\mathfrak{M}^{\mathrm{LE}}]]$, such that $\mathfrak{M}^{\mathrm{LE}} \subseteq \left(\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}\right)^{>0}$ inside $\mathbb{R}[[\mathfrak{M}^{\mathrm{LE}}]]$, and such that the usual exponential function on $\mathbb{R}$ extends canonically to an exponential function on $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$. We call $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ the **field of logarithmic-exponential series** (or **LE-series**). (In [23] the notation $\mathbb{R}(\!(x^{-1})\!)^{\mathrm{LE}}$ was used.) The elements of $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ are infinite series of LE-monomials arranged from left to right in decreasing order and multiplied by real coefficients. Typical example:

$$\underbrace{e^{e^x} + \sqrt{2}e^x - \log x}_{\text{infinite part}} + 42 + \underbrace{x^{-1} + x^{-2} + \cdots + e^{-x} + e^{-x^2} + \cdots}_{\text{infinitesimal part}}.$$

We construct $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ at the end of this section. Here are some of its properties (from [23]):

Differentiation. The field of LE-series has a natural *derivation* $f \mapsto f'$ respecting infinite summation, with constant field $\mathbb{R}$, $x' = 1$ and $(e^f)' = f' \cdot e^f$. For example,

$$(e^{-x} + e^{-x^2} + e^{-x^3} + \cdots)' = -(e^{-x} + 2xe^{-x^2} + 3x^2e^{-x^3} + \cdots).$$

Integration. Every LE-series f has an *antiderivative*, that is, an LE-series $g \in \mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ with $g' = f$. Hence $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ has a natural *integration operator* $f \mapsto \int f$ associating to f its unique antiderivative in $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ with constant term zero. For example,

$$\int \frac{e^x}{x} = \sum_{n=0}^{\infty} n!x^{-1-n}e^x.$$

The operator $\int$ also commutes with infinite summation.

Composition. Given LE-series f, g with $g > \mathbb{R}$ it is possible to *substitute g for x in f*, in this way forming the composite $f \circ g = f(g) \in \mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$. For example, if $f = e^{e^x} + \sqrt{2}e^x - \log x$ and $g = x + \log x$, then

$$\begin{aligned} f(g) &= e^{xe^x} + \sqrt{2}xe^x - \log(x + \log x) \\ &= e^{xe^x} + \sqrt{2}xe^x - \left(\log x + \log\left(1 + \frac{\log x}{x}\right)\right) \\ &= e^{xe^x} + \sqrt{2}xe^x - \log x + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(\frac{\log x}{x}\right)^n \end{aligned}$$

Put $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}_{\infty} := \left\{f \in \mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}} : f > \mathbb{R}\right\}$, the set of positive infinite elements of $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$. Substituting any fixed $g \in \mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}_{\infty}$ is an ordered field embedding

$$f \mapsto f \circ g \,:\, \mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}} \to \mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$$

obeying the chain rule $(f \circ g)' = (f' \circ g) \cdot g'$. Also, $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}_{\infty}$ is a group under the composition operation $\circ$, with identity element x.

Cofinality of iterated exponentials. The sequence

$$\ell_0 := x, \ell_1 := \log x, \dots, \ell_n := \log\log \cdots \log x \ (n \text{ times}), \dots$$

is *coinitial* in $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}_{\infty}$: for every $f \in \mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}_{\infty}$ there exists n such that $\ell_n < f$. Dually, the sequence of iterated exponentials

$$e_0 := x, e_1 := \exp x, \dots, e_n := \exp\exp \cdots \exp x \ (n \text{ times}), \dots$$

is *cofinal* in $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$: for every $f \in \mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ there exists n with $f < e_n$.

Although the construction of $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ is *purely algebraic,* many LE-series do have an analytic origin: they often arise as asymptotic expansions of real-valued functions at $+\infty$ in terms of LE-monomials.

Example. The Stirling expansion for Euler's Γ-function: as $x \to +\infty$,

$$\log \Gamma(x) \sim \left(x - \frac{1}{2}\right) \log x - x + \frac{1}{2}\log(2\pi) + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)} x^{1-2k},$$

where $B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, \dots$ are the Bernoulli numbers with positive even index. The series on the right hand side diverges for all $x > 0$.

A profound analytic source for LE-series are the *analyzable functions* encountered in the work of Écalle [25] on Hilbert's 16th problem about limit cycles of planar polynomial vector fields. A smaller natural class of LE-series in [25] emerges also in [21] as part of a characterization of the functions definable in $\mathbb{R}_{\mathbf{an},\exp}$:

Theorem 2.2. *There exists an embedding*

$$\mathrm{e}\colon H(\mathbb{R}_{\mathbf{an},\exp}) \to \mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$$

of ordered differential fields which is the identity on $\mathbb{R}$ *and sends the germ* x *to the element* x *of* $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$.

The series in the image of e can be seen as *convergent* LE-series; in [25] they are called "transséries convergentes." They arise as asymptotic expansions, for $x \to +\infty$, of solutions $y \in \mathcal{G}_1$ to implicit equations like $P(x, y, \log x, e^x, e^y, e^{y^2}) = 0$, where P is a polynomial with real coefficients in six indeterminates.

The embedding e in Theorem 2.2 may be regarded as a *formal expansion operator* which associates to a germ $f \in H(\mathbb{R}_{\mathbf{an},\exp})$ an asymptotic expansion in terms of LE-monomials. It respects the algebraic operations and the ordering on $H(\mathbb{R}_{\mathbf{an},\exp})$ and $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$, respectively, but also the analytic and exponential structure on these two fields. This has several interesting applications, of which we mention two.

First, since $e_0, e_1, \dots$ is cofinal in $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$, every germ $f \in H(\mathbb{R}_{\mathbf{an},\exp})$ is bounded by an iterate of the exponential function: the o-minimal structure $\mathbb{R}_{\mathbf{an},\exp}$ on $\mathbb{R}$ is **exponentially bounded**. It is unknown whether there exist o-minimal structures on the field $\mathbb{R}$ that are not exponentially bounded. (By [13] there do exist Hardy fields with germs f such that $f > \exp_n x$ for all n.)

A second application concerns a question from [28]: is there a positive infinite LE-function whose compositional inverse is not asymptotic to an LE-function? Hardy suggested the LE-function $(\log x)(\log\log x)$ as a counterexample. Shackell [68] answered the question positively by showing that the compositional inverse of $(\log\log x)(\log\log\log x)$ is not asymptotic to an LE-function; his techniques did not apply to $(\log x)(\log\log x)$. The embedding e is used in [22] to confirm Hardy's suggestion that the compositional inverse of $(\log x)(\log\log x)$ is not asymptotic to an LE-function. (Another proof of this suggestion is by van der Hoeven [32].)

The derivation of $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$. We equip the ordered differential field of LE-series with the restriction of the dominance relation on $\mathbb{R}[[\mathfrak{M}^{\mathrm{LE}}]]$ to $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$, also denoted by $\preccurlyeq$. The rules of Proposition 1.4, which relate the derivation of a Hardy field with its ordering and dominance relation, remain true for the field of LE-series:

Proposition 2.3. *Let $0 \neq f, g \in \mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$.*

(1) *If $f \succ 1$, then $f^{\dagger} > 0$.*
(2) *If $f \preccurlyeq 1$, then $f - c \prec 1$ for some $c \in \mathbb{R}$.*
(3) *If $f \preccurlyeq 1$, $g \not\asymp 1$, then $f' \prec g^{\dagger}$.*
(4) *If $f, g \not\asymp 1$, then $f \preccurlyeq g \iff f' \preccurlyeq g'$.*
(5) *If $f \preccurlyeq 1$, then $f' \prec 1$.*

For a proof, see [23], Propositions 4.1 and 4.3. Many more asymptotic-differential properties of Hardy fields are valid for the field of LE-series. As an example, here is an analogue of Theorem 1.12:

Theorem 2.4. *Let $f \in \mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$. Then the equation*

$$4y'' - fy = 0$$

has a non-trivial solution in $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ if and only if

$$f > -\left(\frac{1}{(\ell_0)^2} + \frac{1}{(\ell_0\ell_1)^2} + \frac{1}{(\ell_0\ell_1\ell_2)^2} + \cdots + \frac{1}{(\ell_0\ell_1\cdots\ell_n)^2}\right)$$

for some n.

This theorem was stated in [23]; for a proof in the right setting, see [7].

Construction of $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$. The field of LE-series is obtained from the field

$$K_0 := \mathbb{R}[[\mathfrak{M}_0]], \quad \text{with } \mathfrak{M}_0 := x^{\mathbb{R}},$$

by an inductive procedure of exponentiation and taking logarithms. Since there is no reasonable way to define $\exp f$ as an element of K_0 for $f \in K_0^{\uparrow}$, we enlarge $K_0 = \mathbb{R}[[\mathfrak{M}_0]]$ to a bigger series field $K_1 = \mathbb{R}[[\mathfrak{M}_1]]$ such that $\exp f \in K_1$ for all $f \in K_0^{\uparrow}$: take a multiplicative copy $\exp(K_0^{\uparrow})$ of the ordered additive subgroup $K_0^{\uparrow}$ of K_0, with order-preserving isomorphism

$$f \mapsto \exp f \colon K_0^{\uparrow} \to \exp(K_0^{\uparrow}),$$

and form the direct product of multiplicative groups

$$\mathfrak{M}_1 := \exp\bigl(K_0^{\uparrow}\bigr) \cdot \mathfrak{M}_0.$$

Order $\mathfrak{M}_1$ lexicographically: for $f \in K_0^{\uparrow}$, $\mathfrak{m} \in \mathfrak{M}_0$, put

$$\exp(f) \cdot \mathfrak{m} \succcurlyeq 1 \quad \iff \quad f > 0 \text{ or } (f = 0 \text{ and } \mathfrak{m} \succcurlyeq 1 \text{ in } \mathfrak{M}_0).$$

The natural identification of $\mathfrak{M}_0$ with an ordered subgroup of $\mathfrak{M}_1$ makes K_0 an ordered subfield of K_1. Define $\exp g \in K_1^{>0}$ for $g \in K_0$ by

$$\exp(f + c + \varepsilon) := \exp(f) \cdot e^{c+\varepsilon} \qquad (f \in K_0^{\uparrow}, c \in \mathbb{R}, \varepsilon \in K_0^{\downarrow}),$$

with $e^{c+\varepsilon}$ as in (2.1). Now K_1 has the same defect as K_0: there is no reasonable way to define $\exp f$ as an element of K_1 for $f \in K_1$ with $\mathfrak{d}(f) \succ \mathfrak{M}_0$. In order to add the exponentials of such elements to K_1, enlarge K_1 to a field K_2 just as K_0 was enlarged to K_1. More generally, consider a tuple $(K, A, B, \exp)$ where

(1) K is an ordered field;
(2) A and B are additive subgroups of K with $K = A \oplus B$ and B convex in K;
(3) $\exp\colon B \to K^{>0}$ is a strictly increasing homomorphism.

We call such a tuple a **pre-exponential ordered field.** So $(K_0, A_0, B_0, \exp_0)$ with $A_0 := K_0^{\uparrow}$, $B_0 := \mathbb{R} \oplus K_0^{\downarrow}$ and $\exp_0\colon B_0 \to K_0^{>0}$ given by (2.1) (for $\mathfrak{M} = x^{\mathbb{R}}$) is a pre-exponential ordered field. Given a pre-exponential ordered field $(K, A, B, \exp)$, define a pre-exponential ordered field $(K', A', B', \exp')$ as follows: Take a multiplicative copy $\exp(A)$ of the ordered additive group A with order-preserving isomorphism $\exp_A\colon A \to \exp(A)$, and put

$$K' := K[[\exp(A)]], \quad A' := (K')^{\uparrow}, \quad B' := K \oplus (K')^{\downarrow},$$

and we define $\exp'\colon B' \to (K')^{>0}$ by

$$\exp'(a + b + \varepsilon) := \exp_A(a) \cdot \exp(b) \cdot \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}$$

for $a \in A$, $b \in B$, $\varepsilon \in (K')^{\downarrow}$. Note that then $\exp'$ is defined on the whole field K and extends exp. Moreover, if $K = \mathbb{R}[[\mathfrak{M}]]$ for some multiplicative ordered abelian group $\mathfrak{M}$, then $K' = \mathbb{R}[[\mathfrak{M}']]$ where $\mathfrak{M}' = \exp(A) \cdot \mathfrak{M}$, ordered lexicographically. Inductively, set

$$(K_{n+1}, A_{n+1}, B_{n+1}, \exp_{n+1}) := (K_n', A_n', B_n', \exp_n').$$

Then $K_n = \mathbb{R}[[\mathfrak{M}_n]]$ with $\mathfrak{M}_n = x^{\mathbb{R}} \cdot \exp_n(A_{n-1} \oplus \cdots \oplus A_0)$, and we put

$$\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{E}} := \bigcup_n \mathbb{R}[[\mathfrak{M}_n]], \qquad \mathfrak{M}^{\mathrm{E}} := \bigcup_n \mathfrak{M}_n,$$

the field of **exponential series**, and the group of **exponential monomials**, respectively. Thus $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{E}} \subseteq \mathbb{R}[[\mathfrak{M}^{\mathrm{E}}]]$, as ordered field. Let

$$\exp\colon \mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{E}} \to \big(\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{E}}\big)^{>0}$$

be the common extension of all the $\exp_n$. This map is a strictly increasing group homomorphism, but is not surjective since $x \neq \exp(f)$ for every $f \in \mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{E}}$. We now indicate how to remove this defect by enlarging $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{E}}$.

Take distinct symbols $\ell_0, \ell_1, \ell_2, \dots$ with $\ell_0 = x$. Replace, for each n, the formal variable x in $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{E}}$ by ℓ_n, turning the ordered field $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{E}}$ into an isomorphic copy $\mathbb{R}[[\ell_n^{\mathbb{R}}]]^{\mathrm{E}}$, with the function exp on $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{E}}$ turning into a function on $\mathbb{R}[[\ell_n^{\mathbb{R}}]]^{\mathrm{E}}$ also denoted by exp, and $\mathfrak{M}^{\mathrm{E}} \subseteq \big(\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{E}}\big)^{>0}$ turning into an ordered subgroup $\mathfrak{M}(n)$ of $\big(\mathbb{R}[[\ell_n^{\mathbb{R}}]]^{\mathrm{E}}\big)^{>0}$.

One can show there is a unique ordered field embedding $\mathbb{R}[[\ell_n^{\mathbb{R}}]]^{\mathrm{E}} \to \mathbb{R}[[\ell_{n+1}^{\mathbb{R}}]]^{\mathrm{E}}$ that sends ℓ_n to $\exp(\ell_{n+1})$, and that respects infinite summation and exp. (It maps $\mathfrak{M}(n)$ into $\mathfrak{M}(n+1)$.) Identify $\mathbb{R}[[\ell_n^{\mathbb{R}}]]^{\mathrm{E}}$ with its image in $\mathbb{R}[[\ell_{n+1}^{\mathbb{R}}]]^{\mathrm{E}}$ under this embedding; so $\ell_n = \exp(\ell_{n+1})$ and $\mathfrak{M}(n) \subseteq \mathfrak{M}(n+1)$. Finally, put

$$\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}} := \bigcup_n \mathbb{R}[[\ell_n^{\mathbb{R}}]]^{\mathrm{E}}, \qquad \mathfrak{M}^{\mathrm{LE}} := \bigcup_n \mathfrak{M}(n),$$

so $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}} \subseteq \mathbb{R}[[\mathfrak{M}^{\mathrm{LE}}]]$; let $\exp\colon \mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}} \to \big(\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}\big)^{>0}$ be the common extension of the functions exp on $\mathbb{R}[[\ell_n^{\mathbb{R}}]]^{\mathrm{E}}$. For more details, see [23].

3. H-Fields and Asymptotic Couples

Motivated by the similarities between Hardy fields and the field of LE-series (as ordered differential fields), we introduced in [4] the class of *H-fields:*

Definition 3.1. An **H-field** is an ordered differential field K whose natural dominance relation $\preccurlyeq$ satisfies the following two conditions, for all $f \in K$:

(H1) If $f \succ 1$, then $f^\dagger > 0$.
(H2) If $f \preccurlyeq 1$, then $f - c \prec 1$ for some $c \in C$.

Every Hardy field $K \supseteq \mathbb{R}$ is an H-field, as is every ordered differential subfield $K \supseteq \mathbb{R}$ of the field of LE-series, by parts (1) and (2) of Proposition 1.4 and 2.3, respectively. Part (3) of these propositions turns out to be a formal consequence of the H-field axioms:

Lemma 3.2. *Let K be an H-field, and $f, g \in K^\times$. If $f \preccurlyeq 1$, $g \not\asymp 1$, then $f' \prec g^\dagger$.*

Proof. By (H2) there exists $c \in C$ with $f - c \prec 1$; replacing f by $f - c$ if necessary we may assume $f \prec 1$. If $g \succ 1$ we replace g by $1/g \prec 1$, noting that $(1/g)^\dagger = -g^\dagger \asymp g^\dagger$. So we may assume $g \prec 1$. Then, if $g < 0$ we replace g by $-g$; hence we may assume $0 < g \prec 1$. By (H1) applied to $1/g$ in place of f it follows that $g' < 0$. Let now $c > 0$ in C. Then $(c + f)/g > C$ and $(c - f)/g > C$, hence taking derivatives in the last two relations gives $f'g - (c + f)g' > 0$ and $-f'g - (c - f)g' > 0$, by (H1). Dividing by $g' < 0$ gives
$$-c + f < f'g/g' < c + f.$$
This holds for all positive $c \in C$, so $f' \prec g'/g = g^\dagger$. □

An H-field K is said to be **Liouville closed** if K is real closed and for each $f \in K$ there exist $y, z \in K^\times$ such that $y' = f$ and $z^\dagger = f$. Every Liouville closed Hardy field (as defined in Section 1) containing $\mathbb{R}$ is a Liouville closed H-field. The field $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ of LE-series, or more generally every ordered differential subfield of $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ which contains $\mathbb{R}$ and is closed under exp and $\int$, is a Liouville closed H-field.

Pre-H-fields. If L is an H-field and K an ordered differential subfield of L, then the restriction of the natural dominance relation of L to K does not necessarily agree with the natural dominance relation of K, and even if it does, K need not be an H-field. However, K with this restricted dominance relation is a *pre-H-field* in the following sense:

Definition 3.3. A **pre-H-field** is an ordered differential field K with a dominance relation $\preccurlyeq$ on K such that for all $f, g \in K$:

(PH1) if $f \preccurlyeq 1$ and $0 \neq g \prec 1$, then $f' \prec g^\dagger$;
(PH2) if $f \succ 1$, then $f^\dagger > 0$.

Every H-field with its natural dominance relation is a pre-H-field, and every Hardy field (not necessarily extending $\mathbb{R}$) with its natural dominance relation is a pre-H-field.

Remarks. Let K be a pre-H-field (H-field, Liouville closed H-field) and $a \in K^{>0}$. Then K with its derivation ∂ replaced by $a\partial$ is again a pre-H-field (H-field, Liouville closed H-field, respectively). If K is a Hardy field and $a = x \in K^{>0}$, such a change of derivation amounts to passing from K to the Hardy field

$$K \circ e^x := \{ f \circ e^x : f \in K \},$$

in other words, a *change of the independent variable.* Likewise, if K is an ordered differential subfield of the H-field $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ containing x, then K with its derivation ∂ replaced by $x\partial$ is naturally isomorphic to the ordered differential subfield

$$K \circ e^x := \{ f \circ e^x : f \in K \}$$

of $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$. In [32] the notations $f{\uparrow} := f \circ e^x$ and $f{\uparrow}^n := f{\uparrow}{\uparrow}\cdots{\uparrow}$ (n times) for $f \in \mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ are used. For every $f \in \mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ there exists n such that $f{\uparrow}^n \in \mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{E}}$. This fact is immediate from the construction of $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$, and often allows one to reduce questions about arbitrary LE-series to the simpler case of exponential series.

Let K and L be pre-H-fields, with dominance relations $\preccurlyeq$ and $\preccurlyeq_L$, respectively. An embedding $\varphi\colon K \to L$ of ordered differential fields is an **embedding of pre-H-fields** if $f \preccurlyeq g \iff \varphi(f) \preccurlyeq_L \varphi(g)$, for all $f, g \in K$. If $K \subseteq L$ as sets and the natural inclusion $K \to L$ is an embedding of pre-H-fields, we say that L is a **pre-H-field extension** of K, and if in addition L is an H-field, we call L an **H-field extension** of K. *Does every pre-H-field K have an H-field extension?* In other words, does every pre-H-field arise by taking an ordered differential subfield of an H-field and restricting the natural dominance relation to this subfield, as indicated before Definition 3.3? In [4] (Corollary 4.6) we gave a positive answer to this question:

Theorem 3.4. *Let K be a pre-H-field. Then K has an H-field extension $\widehat{K}$ such that any embedding of K into an H-field L extends uniquely to an embedding from $\widehat{K}$ into L.*

An H-field $\widehat{K}$ as in the theorem is necessarily unique, up to unique isomorphism (of pre-H-fields) over K.

The theorem is proved as follows: Suppose K is a pre-H-field, but *not* an H-field. This is witnessed by a bounded $r \in K$ such that $r' \neq \varepsilon'$ for all infinitesimal $\varepsilon \in K$. Thus for K to extend to an H-field, there must exist an element y in some H-field extension of K such that $y' = r'$, $y \prec 1$, and $K(y)$ with its induced ordering, derivation and dominance relation is a pre-H-field. In order to construct an H-field extension of K, we consider the field extension $L = K(y)$ of K, with y transcendental over K, and we extend the derivation and the dominance relation of K to a derivation and dominance relation on $K(y)$ such that $y' = r'$ and $y \prec 1$. The key fact shown in the proof of Theorem 3.4 is that (under mild assumptions on K) this can be done in a unique way such that $K(y)$ remains a pre-H-field. See [4], Section 4 for details.

The asymptotic couple of a pre-H-field. Part (4) of Proposition 1.4 and 2.3 hold in every pre-H-field, that is: If f and g are non-zero elements of a pre-H-field K with $f, g \not\asymp 1$, then by [57], proof of Corollary 1,

$$f \preccurlyeq g \qquad \Longleftrightarrow \qquad f' \preccurlyeq g'.$$

In particular, the valuation $v(f')$ of f' is uniquely determined by the valuation $v(f)$ of $f \in K^\times$, provided $f \not\asymp 1$. Hence the derivation of K induces a map $\psi\colon \Gamma^* = \Gamma \setminus \{0\} \to \Gamma$ given by

$$\psi\big(v(f)\big) = v(f') - v(f) = v(f^\dagger) \quad \text{for } f \in K^\times \text{ with } v(f) \neq 0.$$

We also put $\psi(0) := \infty \in \Gamma_\infty$. Following Rosenlicht [56], [58] we call the pair (Γ, ψ) the **asymptotic couple** of K. This invariant of a pre-H-field encodes key features of the interaction between the derivation and the dominance relation:

Lemma 3.5. *Let $\alpha, \beta \in \Gamma$. Then*

1. $\psi(\alpha + \beta) \geqslant \min\big\{\psi(\alpha), \psi(\beta)\big\}$ *(so ψ is a* valuation *on Γ);*
2. $\psi(r\alpha) = \psi(\alpha)$ *for $r \in \mathbb{Z} \setminus \{0\}$;*
3. $\psi(\alpha) < \psi(\beta) + |\beta|$ *for $\alpha \neq 0$;*
4. *If $0 < \alpha \leqslant \beta$, then $\psi(\alpha) \geqslant \psi(\beta)$.*

Proof. Part (1) follows from property (V2) of the valuation v and the logarithmic derivative identity $(fg)^\dagger = f^\dagger + g^\dagger$, valid for any non-zero $f, g \in K$. This identity also implies $(f^n)^\dagger = nf^\dagger$ for all $f \in K^\times$ and all n, from which (2) follows. Part (3) is just a reformulation of axiom (PH1) for pre-H-fields. For a proof of (4) see Lemma 2.2 in [4]. $\square$

With id denoting the identity function on Γ, we have

$$\begin{aligned} \Psi_K := \psi(\Gamma^*) = \psi\big(\Gamma^{>0}\big) &= \big\{v(f^\dagger) : 0 \neq f \prec 1\big\}, \\ (\mathrm{id} + \psi)(\Gamma^*) &= \big\{v(f') : 0 \neq f \not\asymp 1\big\}, \\ (\mathrm{id} + \psi)\big(\Gamma^{>0}\big) &= \big\{v(f') : 0 \neq f \prec 1\big\}. \end{aligned}$$

Let K be an H-field. If $f \in K^\times$ and $v(f) \in (\mathrm{id} + \psi)(\Gamma^*)$, then f is **asymptotically integrable** in K in the sense that there exists $g \in K$ with $g' \sim f$. We say that K is **closed under asymptotic integration** if every $f \in K^\times$ is asymptotically integrable.

Examples. If $K = \mathbb{R}(x)$, then $\Gamma = \mathbb{Z}v(x)$ with $v(x) < 0$, so $\psi(\alpha) = v(x^{-1}) = -v(x)$ for all $\alpha \in \Gamma^*$. Next, consider the Hardy field $K = \mathbb{R}(x, e^x)$. Then

$$\Gamma = \mathbb{Z}v(e^{-x}) \oplus \mathbb{Z}v(x^{-1}), \text{ ordered lexicographically,}$$

i.e., for $r, s \in \mathbb{Z}$: $rv(e^{-x}) + sv(x^{-1}) > 0 \iff$ either $r > 0$, or $r = 0$ and $s > 0$. We can describe $\psi\colon \Gamma^* \to \Gamma$ by

$$\psi(\alpha) = \begin{cases} 0 & \text{if } r \neq 0 \\ v(x^{-1}) & \text{if } r = 0 \end{cases}$$

for $\alpha = rv(e^{-x}) + sv(x^{-1})$ with $r, s \in \mathbb{Z}$, not both zero. An element $f \in K^\times$ is asymptotically integrable in K if and only if $f \not\asymp x^{-1}$.

Asymptotic couples. Abstractly, Rosenlicht [58] defined an **asymptotic couple** to be a pair (Γ, ψ) where Γ is an ordered abelian group and $\psi\colon \Gamma^* = \Gamma \setminus \{0\} \to \Gamma$ is a function such that Lemma 3.5, (1)–(3) hold for all $\alpha, \beta \in \Gamma$, where ψ is extended to all of Γ by setting $\psi(0) := \infty$. In a series of papers [56], [57], [58] he studied algebraic properties of asymptotic couples, focusing on the case where the ordered abelian group Γ has finite rank. Our papers [3] and [2] continue this work, but with another focus.

H-asymptotic couples. The map ψ of an asymptotic couple coming from a pre-H-field is decreasing on $\Gamma^{>0}$, as stated in Lemma 3.5, (4). We say that an asymptotic couple (Γ, ψ) is **of H-type**, or an **H-asymptotic couple** for short, if (4) in Lemma 3.5 holds. We will refer to (1), (2), (3), (4) of Lemma 3.5 as axioms (1), (2), (3), (4), respectively, for H-asymptotic couples. Below we deal mainly with H-asymptotic couples whose underlying ordered abelian group is divisible. We can always put ourselves in this situation: If (Γ, ψ) is an asymptotic couple of H-type, then there is a unique function $\psi' \colon (\mathbb{Q}\Gamma)^* \to \Gamma$ such that $(\mathbb{Q}\Gamma, \psi')$ is an asymptotic couple of H-type and $\psi'|\Gamma^* = \psi$. ([4], Lemma 2.14.)

Properties of the map ψ. Below, (Γ, ψ) is an H-asymptotic couple, and we put $\Psi := \psi(\Gamma^*)$. By axioms (2) and (4) of H-asymptotic couples, the function ψ is constant on archimedean classes of Γ. (Therefore, if Γ has finite rank, then Ψ is finite.) Moreover, ψ is *contracting*:

$$\alpha, \beta \in \Gamma^*, \alpha \neq \beta \implies \big[\psi(\alpha) - \psi(\beta)\big] < [\alpha - \beta].$$

Hence the map $x \mapsto x + \psi(x) \colon \Gamma^* \to \Gamma$ is *strictly increasing.* We refer to [2] for proofs of these and the following facts, due to Rosenlicht [58]: The set $(\mathrm{id} + \psi)\big(\Gamma^{>0}\big)$ is closed upward, the set $(\mathrm{id} + \psi)\big(\Gamma^{<0}\big)$ is closed downward, and

$$(-\mathrm{id} + \psi)\big(\Gamma^{>0}\big) = (\mathrm{id} + \psi)\big(\Gamma^{<0}\big) = \big\{\alpha \in \Gamma : \alpha < \psi(x) \text{ for some } x \in \Gamma^*\big\}.$$

Moreover, three mutually exclusive alternatives arise:

(A1) $\mathrm{id} + \psi \colon \Gamma^* \to \Gamma$ is surjective;
(A2) Ψ has a largest element;
(A3) there is an element $\gamma \in \Gamma$ such that $\Psi < \gamma < (\mathrm{id} + \psi)\big(\Gamma^{>0}\big)$.

There can only be one γ as in (A3).

Examples. The asymptotic couple associated to a Liouville closed H-field satisfies (A1). In fact, the asymptotic couple of an H-field K satisfies (A1) if and only if K is closed under asymptotic integration. If Γ has finite non-zero rank, then (Γ, ψ) satisfies (A2). If $\Gamma = \{0\}$, then (A3) holds. For an example of an H-field whose asymptotic couple (Γ, ψ) satisfies (A3) and $\Gamma \neq \{0\}$, see Section 4 below.

Figure 1 shows the qualitative behavior of the functions ψ and $\mathrm{id} + \psi$.

Comparability in pre-H-fields. If (Γ, ψ) is the asymptotic couple associated to a Hardy field K, then we have an order-reversing bijection $[\gamma] \mapsto \psi(\gamma)$ from the set $[\Gamma^*]$ of non-zero archimedean classes of Γ onto the set $\Psi = \psi(\Gamma^*)$. Hence

$$\mathrm{Cl}(f) \mapsto \psi\big(v(f)\big) = v(f^\dagger) \qquad (\text{where } 1 \prec f \in K)$$

is an order-reversing bijection between the set of comparability classes of K and the set Ψ, see [60]. The last assertion is true for all pre-H-fields, if we use the correct generalization of *comparability class*:

Definition 3.6. Let K be a pre-H-field and $1 \prec f, g \in K$. We set

(1) $f \preceq\!\!\!\prec g :\iff f^\dagger \preccurlyeq g^\dagger \iff \psi\big(v(f)\big) \geqslant \psi\big(v(g)\big)$,
(2) $f \asymp\!\!\!\asymp g :\iff f^\dagger \asymp g^\dagger \iff \psi\big(v(f)\big) = \psi\big(v(g)\big)$,
(3) $f \prec\!\!\!\prec g :\iff f^\dagger \prec g^\dagger \iff \psi\big(v(f)\big) > \psi\big(v(g)\big)$.

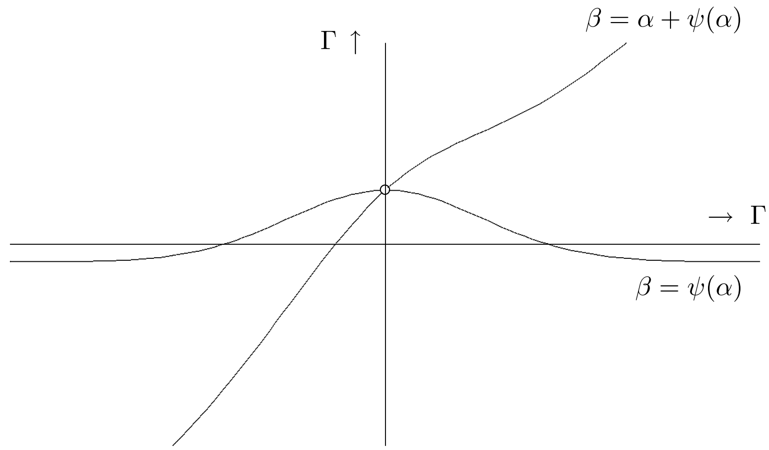

FIGURE 1

We say that f is **flatter** than g if $f \prec\!\!\prec g$, and that f and g are **comparable** if $f \asymp\!\!\asymp g$. Comparability is an equivalence relation on $\{f \in K : f \succ 1\}$. The corresponding equivalence class of $f \succ 1$ is called its **comparability class**, and written as $\operatorname{Cl}(f)$. We linearly order the set of comparability classes by setting

$$\operatorname{Cl}(f) \leqslant \operatorname{Cl}(g) \qquad :\Longleftrightarrow \qquad f \preceq\!\!\preceq g.$$

We then have an order reversing bijection

$$\operatorname{Cl}(f) \mapsto \psi\big(v(f)\big) = v(f^\dagger)$$

from the set of comparability classes onto the subset Ψ of Γ.

Example. In $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ we have

$$\cdots \prec\!\!\prec \ell_{n+1} \prec\!\!\prec \ell_n \prec\!\!\prec \cdots \prec\!\!\prec \ell_1 \prec\!\!\prec \ell_0 = e_0 \prec\!\!\prec e_1 \prec\!\!\prec \cdots \prec\!\!\prec e_n \prec\!\!\prec e_{n+1} \prec\!\!\prec \cdots.$$

The ordering on the set of comparability classes of $K = \mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ (in fact, of any Liouville closed H-field K) is dense: if $\operatorname{Cl}(f) < \operatorname{Cl}(g)$, where $1 \prec f, g \in K$, then $\operatorname{Cl}(f) < \operatorname{Cl}(h) < \operatorname{Cl}(g)$ for some $1 \prec h \in K$.

Remark 3.7. Proposition 1.5 above remains true for extensions of H-fields, with the notion of comparability class defined above.

H-asymptotic triples. Our aim is to describe an *elimination theory* for the H-asymptotic couples of Liouville closed H-fields. Towards this goal we introduce the notions of *cut* and *H-asymptotic triple*. A **cut** of an H-asymptotic couple (Γ, ψ) is a set $P \subseteq \Gamma$ which is closed downward in Γ, contains Ψ, and is disjoint from $(\mathrm{id} + \psi)\big(\Gamma^{>0}\big)$. (So $P < (\mathrm{id} + \psi)\big(\Gamma^{>0}\big)$.) If (A1) or (A2) holds, then the set $P_1 := \Gamma \setminus (\mathrm{id} + \psi)\big(\Gamma^{>0}\big)$ is the only cut of (Γ, ψ). If (A3) holds, that is,

$$\Psi < \gamma < (\mathrm{id} + \psi)\big(\Gamma^{>0}\big),$$

for a (necessarily unique) $\gamma \in \Gamma$, then (Γ, ψ) has exactly two cuts, namely

$$P_1 := \Gamma \setminus (\mathrm{id} + \psi)\big(\Gamma^{>0}\big) = \Gamma^{\leqslant \gamma}, \qquad P_2 := P_1 \setminus \{\gamma\} = \Gamma^{<\gamma}.$$

In [3], Definition 6.2, we introduced the following notion, under the somewhat technical name "H_0-triple":

Definition 3.8. An **asymptotic triple of H-type**, or **H-asymptotic triple** for short, is a triple (Γ, ψ, P), where (Γ, ψ) is an H-asymptotic couple and P a cut of (Γ, ψ), such that

(1) Γ is divisible and
(2) there exists a positive element 1 of Γ with $\psi(1) = 1$; equivalently, $0 \in (\mathrm{id} + \psi)\big(\Gamma^{<0}\big)$.

The element $1 \in \Gamma$ in (2) is unique, since $\mathrm{id} + \psi$ is strictly increasing on Γ^*. An H-asymptotic triple (Γ, ψ, P) is said to be **closed** if

(1) $(\mathrm{id} + \psi)(\Gamma^*) = \Gamma$ and
(2) $\Psi = (\mathrm{id} + \psi)\big(\Gamma^{<0}\big)$.

(In this case, Ψ is the only cut of (Γ, ψ), so necessarily $P = \Psi$.)

Example. The derivation of an H-field K **preserves infinitesimals** if $\varepsilon' \prec 1$ for all $\varepsilon \prec 1$ in K. Suppose K is a Liouville closed H-field whose derivation preserves infinitesimals, and let (Γ, ψ) be the asymptotic couple of K. Then we associate to K the H-asymptotic triple (Γ, ψ, Ψ), with $1 = v(x^{-1})$, where $x \in K$ satisfies $x' = 1$. This asymptotic triple is closed.

The role of the element $1 \in \Gamma$ is just to provide a convenient normalization: If K is any H-field with derivation ∂, then there exists $a \in K^{>0}$ such that $a\partial$ preserves infinitesimals. (See the remarks following Definition 3.3.)

Closure of H-asymptotic triples. Let (Γ, ψ, P) and (Γ', ψ', P') be H-asymptotic triples. An **embedding**

$$\varphi \colon (\Gamma, \psi, P) \to (\Gamma', \psi', P')$$

is by definition an embedding $\varphi \colon \Gamma \to \Gamma'$ of ordered abelian groups such that $\varphi(\psi(\alpha)) = \psi'(\varphi(\alpha))$ for all $\alpha \in \Gamma^*$ and $\varphi^{-1}(P') = P$. If $\Gamma \subseteq \Gamma'$ as sets, and the natural inclusion $\Gamma \to \Gamma'$ is an embedding $(\Gamma, \psi, P) \to (\Gamma', \psi', P')$, then (Γ', ψ', P') is called an **extension** of the H-asymptotic triple (Γ, ψ, P), and we indicate this by writing $(\Gamma, \psi, P) \subseteq (\Gamma', \psi', P')$. Every H-asymptotic triple extends to a closed one ([3], Corollaries 5.3 and 6.1):

Theorem 3.9. *Every H-asymptotic triple (Γ, ψ, P) has a* **closure,** *that is, a closed H-asymptotic triple $(\Gamma^{\mathrm{c}}, \psi^{\mathrm{c}}, P^{\mathrm{c}})$ extending (Γ, ψ, P) with the property that any embedding $(\Gamma, \psi, P) \to (\Gamma', \psi', P')$ into a closed H-asymptotic triple (Γ', ψ', P') extends to an embedding $(\Gamma^{\mathrm{c}}, \psi^{\mathrm{c}}, P^{\mathrm{c}}) \to (\Gamma', \psi', P')$. Any two closures of (Γ, ψ, P) are isomorphic over (Γ, ψ, P).*

Elimination theory for closed H-asymptotic triples. The main result of [3] says that the class of closed H-asymptotic triples has an elimination theory. This kind of theorem is perhaps best understood in a model-theoretic framework, but here we state this elimination theory in terms of so-called ψ-sets and ψ-functions.

Definition 3.10. Let (Γ, ψ, P) be an H-asymptotic triple. The class of **absolute ψ-functions** (with respect to (Γ, ψ, P)) is the smallest class of functions $\Gamma_\infty^n \to \Gamma_\infty$ (for $n = 0, 1, 2, \dots$) with the following properties:

(1) the constants 0, 1, and ∞, viewed as functions $\Gamma_\infty^0 \to \Gamma_\infty$, are absolute ψ-functions;

(2) any projection map $(x_1,\dots,x_n)\mapsto x_i\colon \Gamma_\infty^n\to\Gamma_\infty$, $i\in\{1,\dots,n\}$, is an absolute ψ-function;
(3) if $F\colon\Gamma_\infty^n\to\Gamma_\infty$ and $G_1,\dots,G_n\colon\Gamma_\infty^m\to\Gamma_\infty$ are absolute ψ-functions, then

$$F(G_1,\dots,G_n)\colon\Gamma_\infty^m\to\Gamma_\infty$$

is an absolute ψ-function;
(4) addition $(\alpha,\beta)\mapsto\alpha+\beta\colon\Gamma_\infty\times\Gamma_\infty\to\Gamma_\infty$ is an absolute ψ-function;
(5) inversion $\gamma\mapsto-\gamma\colon\Gamma_\infty\to\Gamma_\infty$ is an absolute ψ-function, where $-\infty:=\infty$;
(6) $\gamma\mapsto\psi(\gamma)\colon\Gamma_\infty\to\Gamma_\infty$ is an absolute ψ-function, where $\psi(\infty)=\infty$;
(7) for any $n>0$, the map $\gamma\mapsto\frac{1}{n}\gamma\colon\Gamma_\infty\to\Gamma_\infty$ (division by n) is an absolute ψ-function, where we put $\frac{1}{n}\infty:=\infty$.

So every absolute ψ-function is given by an expression built up from variables and symbols for the constants 0, 1 and ∞, and for the functions $+$, $-$, ψ and multiplication by $\frac{1}{n}$ ($n>0$). An **absolute ψ-set** in Γ_∞^n is a boolean combination (inside Γ_∞^n) of sets of the form

$$\big\{x\in\Gamma_\infty^n: F(x)\leqslant G(x)\big\}\quad\text{and}\quad\big\{x\in\Gamma_\infty^n: F(x)\in P\big\},\tag{3.1}$$

where $F,G\colon\Gamma_\infty^n\to\Gamma_\infty$ are absolute ψ-functions. An **absolute ψ-map** is a map $\Gamma_\infty^m\to\Gamma_\infty^n$ whose graph is an absolute ψ-set in Γ_∞^{m+n}.

Theorem 3.11. *Let (Γ,ψ,Ψ) be a closed H-asymptotic triple. The image $F(S)$ of an absolute ψ-set $S\subseteq\Gamma_\infty^m$ under an absolute ψ-map $F\colon\Gamma_\infty^m\to\Gamma_\infty^n$ is an absolute ψ-set.*

Using the identity $F(S)=\Pi\big(M\cap(S\times\Gamma_\infty^n)\big)$, where $\Pi\colon\Gamma_\infty^{m+n}\to\Gamma_\infty^n$ is the obvious projection map and $M\subseteq\Gamma_\infty^{m+n}$ is the graph of F, it is easy to reduce the proof of the theorem to the case that $m=n+1$ and $F=\pi\colon\Gamma_\infty^{n+1}\to\Gamma_\infty^n$ is the projection map $(x_1,\dots,x_{n+1})\mapsto(x_1,\dots,x_n)$. In fact, this elimination can be done constructively: there is an algorithm which, given as input a "boolean" description of an absolute ψ-set $S\subseteq\Gamma_\infty^{n+1}$, outputs a similar description for $\pi(S)$. (See Corollary 6.2 in [3].)

The theorem above has the following consequence. Define a **ψ-set** in Γ_∞^n to be a set of the form $\{x\in\Gamma_\infty^n:(a,x)\in S\}$, where $a\in\Gamma_\infty^m$ and $S\subseteq\Gamma_\infty^{m+n}$ is an absolute ψ-set, and define a **ψ-map** to be a map $\Gamma_\infty^m\to\Gamma_\infty^n$ whose graph is a ψ-set.

Corollary 3.12. *Let (Γ,ψ,Ψ) be a closed H-asymptotic triple. The image of a ψ-set under a ψ-map is a ψ-set.* □

Let K be an H-field such that for every $c\in C$ and $f\in K^\times$, the equation $y^\dagger=cf^\dagger$ has a solution $y=g\in K^\times$. (If $K\supseteq\mathbb{R}$ is a Hardy field, this amounts to K being closed under powers.) Then the element $v(g)\in\Gamma=v(K^\times)$ only depends on (c,γ) where $\gamma=v(f)$ (and not on the choice of f and g), and the value group Γ has a natural structure of an *ordered vector space* over the ordered constant field C of K, with scalar multiplication given by $(c,\gamma)\mapsto c\gamma:=v(g)$. (See [5], Section 7.) The above results on H-asymptotic triples remain true, *mutatis mutandis,* for H-asymptotic triples with this extra structure; see [3] for details.

The H-field $\mathbb{R}[[x^{\mathbb{R}}]]^{\text{LE}}$ has the property above that makes its value group Γ^{LE} into an ordered vector space over $\mathbb{R}$. This vector space has *infinite* dimension. On the other hand:

Proposition 3.13. *If $K = \mathbb{R}\langle f_1,\dots,f_N\rangle$ is a differential subfield of $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ generated over $\mathbb{R}$ by finitely many elements $f_1,\dots,f_N$, then the $\mathbb{R}$-linear subspace $\mathbb{R}\Gamma$ generated by $\Gamma := v(K^\times)$ in Γ^{LE} is finite-dimensional.*

It follows that the rank r of Γ is finite (in fact $r = \dim \mathbb{R}\Gamma$); see [3], Example 3.2. Hence $\Psi_K := \big\{v(f^\dagger) : 1 \not\asymp f \in K^\times\big\}$ has at most r elements. In the proof of this proposition we use the notations in Section 2, *Construction of* $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$.

Proof. The case that $f_1,\dots,f_N \in K_0$ is clear, since K_0 has value group $\mathbb{R}v(x)$. Assume inductively that the proposition holds whenever $f_1,\dots,f_N \in K_n$. Let $f_1,\dots,f_N \in K_{n+1}$. We can assume $f_1,\dots,f_N > 0$. Then $\log f_i = g_i + c_i \log x$ with $g_i \in K_n$ and $c_i \in \mathbb{R}$, for $i = 1,\dots,N$. Below we introduce H-subfields F_i of $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$, and let Γ_i denote the value group of F_i, for $i = 0,1,2$. Put $F_0 := \mathbb{R}\langle x, g_1,\dots,g_N\rangle$; then $\dim \mathbb{R}\Gamma_0 < \infty$ by our inductive assumption. Put $F_1 := F_0\langle \log x\rangle$; then $F_1 = F_0(\log x)$, so $\dim \mathbb{R}\Gamma_1 \leqslant 1 + \dim \mathbb{R}\Gamma_0$ by Remark 3.7. The f_i's are solutions of differential equations of the form $y' = ay$ with $0 \neq a \in F_1$, so $F_2 := F_1\langle f_1,\dots,f_N\rangle$ satisfies $F_2 = F_1(f_1,\dots,f_N)$; hence

$$\dim \mathbb{R}\Gamma \leqslant \dim \mathbb{R}\Gamma_2 \leqslant N + \dim \mathbb{R}\Gamma_1 < \infty.$$

This takes care of the case that $f_1,\dots,f_N \in \mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{E}}$. For arbitrary $f_1,\dots,f_N \in \mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$, take n such that $g_i := f_i{\uparrow^n} \in \mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{E}}$ for $i = 1,\dots,N$. The automorphism ${\uparrow^n}$ of the ordered field $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ is the identity on $\mathbb{R}$ and commutes with each operation $f \mapsto f^c := \exp(c\log f)$ on $(\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}})^{>0}$, $c \in \mathbb{R}$. Hence it induces an $\mathbb{R}$-linear automorphism $v(f) \mapsto v(f{\uparrow^n})$ of Γ^{LE}. Now ${\uparrow^n}$ maps K into the ordered differential subfield $\mathbb{R}\langle g_0, g_1,\dots,g_N\rangle$ of $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{E}}$, where $g_0 := \prod_{i=1}^n x{\uparrow^i}$; thus $\mathbb{R}\Gamma$ is finite-dimensional. □

The finitely generated H-subfields of $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ are somewhat special: the H-field $\mathbb{R}\langle\varrho\rangle$, from the Example after Corollary 4.11 below, is generated as a differential field by just one element ϱ over its constant field $\mathbb{R}$, but $\Psi_{\mathbb{R}\langle\varrho\rangle}$ is infinite. Results in [3], Section 5, suggest the following question:

Question. If the H-field K is finitely generated as a differential field over C, does it follow that Ψ_K is a well-ordered set of order type $\leqslant \omega n$ for some n?

4. Algebraic Differential Equations over H-Fields

In this section we report various results motivated by the question: *Given an H-field K, which algebraic differential equations over K have solutions in H-field extensions of K?* We first discuss algebraic equations and first-order linear differential equations, for which complete answers are available. We then state some general theorems concerning higher-order equations, and finish by posing some open questions which motivate our work.

The real closure of an H-field. Let K be a pre-H-field. By basic differential algebra, the real closure K^{rc} of K carries a unique derivation extending the one on K. The constant field of K^{rc} is a real closure of the constant field of K. The dominance relation on K extends to a dominance relation on K^{rc} by setting

$$f \preccurlyeq g \quad :\Longleftrightarrow \quad |f| \leqslant h|g| \text{ for some } h \preccurlyeq 1 \text{ in } K^{>0},$$

for $f, g \in K^{\mathrm{rc}}$. The next result is proved in [4], Section 3, using valuation theory.

Theorem 4.1. *The real closure K^{rc} of a pre-H-field K is again a pre-H-field. If K is an H-field, then K^{rc} is also an H-field.*

If (Γ, ψ) is the H-couple associated to K, then the H-couple associated to K^{rc} is $(\mathbb{Q}\Gamma, \psi')$, where $\psi' \colon (\mathbb{Q}\Gamma)^* \to \Gamma$ is the unique function such that $\psi'|\Gamma^* = \psi$ and $(\mathbb{Q}\Gamma, \psi')$ is an H-asymptotic couple; see the previous section.

The Liouville closure of an H-field. In Section 1 we already mentioned that every Hardy field extends to a smallest Liouville closed Hardy field $\mathrm{Li}(K)$ containing $\mathbb{R}$ (the Liouville closure of K). A similar but more subtle statement holds for H-fields, as we explain now.

Definition 4.2. A **simple Liouville extension** of a differential field K is a differential field extension $L = K(y)$ of K such that C_L is algebraic over C and one of the following holds:

(1) y is algebraic over K,
(2) $y' \in K$,
(3) $y \neq 0$ and $y^\dagger \in K$.

A **Liouville extension** of K is a differential field extension L of K with the property that for each $a \in L$ there are differential subfields

$$K = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n$$

of L such that $a \in K_n$ and for each $i = 1, \dots, n$, K_i is a simple Liouville extension of K_{i-1}.

A **Liouville closure** of an H-field K is a Liouville closed H-field extension L of K such that L is a Liouville extension of K. (If K is a Hardy field extending $\mathbb{R}$, then $\mathrm{Li}(K)$ as defined in Section 1 is indeed a Liouville closure of K in this sense.) The following is the main result of [4]:

Theorem 4.3. *Let K be an H-field. Then one of the following occurs:*

(I) *K has exactly one Liouville closure up to isomorphism over K,*
(II) *K has exactly two Liouville closures up to isomorphism over K.*

Some remarks about this dichotomy are in order. If alternative (A2) about the asymptotic couple (Γ, ψ) of K from the last section holds, that is, if Ψ has a largest element, then Case (I) of the theorem occurs. Recall that (Γ, ψ) satisfies (A3) if and only if $\Psi < \gamma < (\mathrm{id} + \psi)(\Gamma^{>0})$ for some $\gamma \in \Gamma$. We call such an element $\gamma \in \Gamma$ a **gap** in K. Every H-field has at most one gap, and if K has a gap, then Case (II) of the theorem occurs and in one Liouville closure of K, all $s \in K$ with $v(s) = \gamma$ have the form b' with $b \succ 1$, while in another Liouville closure of K all $s \in K$ with $v(s) = \gamma$ have the form b' with $b \prec 1$. In fact, (II) is equivalent to the existence of a Liouville H-field extension of K with a gap.

Gaps in H-fields. Here are some basic facts about gaps (see [4], Section 6 and [5], Section 12):

Lemma 4.4. *Let K be an H-field.*

(1) *If Ψ has a largest element, then K has no gap.*
(2) *If every element of K has an anti-derivative in K, then K has no gap.*
(3) *Let L be an H-field extension of K such that $\Gamma^{>0}$ is coinitial in $\Gamma_L^{>0}$. Then a gap in K remains a gap in L.*

(4) *A gap in K remains a gap in the real closure of K.*
(5) *If K is a directed union of pre-H-subfields that have a smallest comparability class, then K has no gap.*

It follows that no H-subfield of $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ containing $\mathbb{R}$ and no differentially algebraic H-field extension of $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ has a gap. (See [4], Lemma 6.6 and [5], Corollary 12.2, respectively.) The first of these facts yields the following theorem in [4] about extending expansion operators (cf. Theorem 2.2):

Theorem 4.5. *Let* $\mathrm{e}\colon K \to \mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ *be an H-field embedding of the Hardy field $K \supseteq \mathbb{R}$ into the field of real LE-series such that* $\mathrm{e}(r) = r$ *for* $r \in \mathbb{R}$. *Then* e *extends to an H-field embedding of the Liouville closure* $\mathrm{Li}(K)$ *of K into* $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$.

This theorem, the remark after Proposition 1.5, and Proposition 3.13 suggest:

Question. Suppose $K \supseteq \mathbb{R}$ is a Hardy field of finite rank. Is there necessarily an H-field embedding $K \to \mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ that is the identity on $\mathbb{R}$?

Understanding how gaps appear in passing to differentially algebraic H-field extensions seems crucial for a satisfactory answer to the question posed at the beginning of this section. Many basic problems in this direction are as yet unsolved. Let us mention some things that we do know.

By [4], adjoining antiderivatives to real closed H-fields does not create gaps:

Lemma 4.6. *Let K be a real closed H-field. If $L = K(y)$ is an H-field extension of K with $y' \in K$, then K has a gap if and only if L has a gap.*

The other two types of simple Liouville extensions are less well behaved: There is an example of an H-field without a gap, whose real closure has a gap (see [5]), and there is also an example of a real closed H-field without a gap which has a Liouville extension with a gap. Here are details of the latter:

Example. Let $\mathfrak{L}$ denote the ordered subgroup of the multiplicative group $\mathfrak{M}^{\mathrm{LE}}$ of LE-monomials generated by the real powers ℓ_n^a ($a \in \mathbb{R}$) of the iterated logarithms ℓ_n of x. Thus $\mathfrak{L}$ is the set of products

$$\ell_0^{a_0}\ell_1^{a_1}\cdots\ell_n^{a_n} \quad \text{with } a_0,\dots,a_n \in \mathbb{R}.$$

The ordering on $\mathfrak{L}$ is as follows, for $a_0,\dots,a_n,b_0,\dots,b_n \in \mathbb{R}$:

$$\ell_0^{a_0}\ell_1^{a_1}\cdots\ell_n^{a_n} < \ell_0^{b_0}\ell_1^{b_1}\cdots\ell_n^{b_n} \iff (a_0,\dots,a_n) < (b_0,\dots,b_n) \text{ lexicographically.}$$

We equip the ordered field $\mathbb{R}[[\mathfrak{L}]]$ of **logarithmic transseries** with the derivation that is trivial on $\mathbb{R}$, sends ℓ_n^a to $a\ell_n^{a-1}/\ell_0\ell_1\cdots\ell_{n-1}$ (in particular x^a to ax^{a-1}), and commutes with infinite summation in $\mathbb{R}[[\mathfrak{L}]]$. This derivation makes $\mathbb{R}[[\mathfrak{L}]]$ into a real closed H-field extension of the H-subfield $E = \mathbb{R}(\ell_n : n \in \mathbb{N})$ of $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$. The H-field $\mathbb{R}[[\mathfrak{L}]]$ does not have a gap. (See [5] for proofs of these facts.) As in [25], p. 289, (7.9), we now put

$$\Lambda := \ell_1 + \ell_2 + \ell_3 + \cdots \in \mathbb{R}[[\mathfrak{L}]],$$

so

$$\lambda := \Lambda' = \frac{1}{\ell_0} + \frac{1}{\ell_0\ell_1} + \frac{1}{\ell_0\ell_1\ell_2} + \cdots \in \mathbb{R}[[\mathfrak{L}]].$$

Let $\mathfrak{M} = \exp(\mathbb{R}[[\mathfrak{L}]]^{\uparrow})$ be a multiplicative copy of the additive ordered abelian group $\mathbb{R}[[\mathfrak{L}]]^{\uparrow}$, with isomorphism $f \mapsto \exp(f)$. For $\mathfrak{l} = \ell_0^{a_0}\ell_1^{a_1}\cdots\ell_n^{a_n} \in \mathfrak{L}$ we have

$$\log \mathfrak{l} := a_0\ell_1 + \cdots + a_n\ell_{n+1} \in \mathbb{R}[[\mathfrak{L}]]^{\uparrow}.$$

This gives an ordered group embedding $\mathfrak{l} \mapsto \exp(\log \mathfrak{l}) : \mathfrak{L} \to \mathfrak{M}$, and we identify $\mathfrak{L}$ with a subgroup of $\mathfrak{M}$ via this embedding, thus making $\mathbb{R}[[\mathfrak{L}]]$ an ordered subfield of $\mathbb{R}[[\mathfrak{M}]]$. The derivation on $\mathbb{R}[[\mathfrak{L}]]$ extends uniquely to a derivation on $\mathbb{R}[[\mathfrak{M}]]$ commuting with infinite summation and satisfying $\exp(f)' = f' \exp(f)$, where $f \in \mathbb{R}[[\mathfrak{L}]]^{\uparrow}$. (See [32], Chapter 2.) With this derivation, $\mathbb{R}[[\mathfrak{M}]]$ is an H-field extension of $\mathbb{R}[[\mathfrak{L}]]$ with the same constant field $\mathbb{R}$, and $y := \exp(-\Lambda) \in \mathfrak{M}$ represents a gap in $\mathbb{R}[[\mathfrak{M}]]$: The sequence $(1/\ell_n)$ is cofinal in $\mathfrak{M}^{\prec 1}$ and

$$(1/\ell_n)' \prec \exp(-\Lambda) \prec (1/\ell_n)^{\dagger} \qquad \text{for all } n,$$

since

$$(\ell_n)^{\dagger} = (\ell_{n+1})' = \exp\big(-(\ell_1 + \ell_2 + \cdots + \ell_{n+1})\big).$$

Let now K be the real closure inside $\mathbb{R}[[\mathfrak{L}]]$ of the H-subfield $E\langle\lambda\rangle$ of $\mathbb{R}[[\mathfrak{L}]]$ generated by λ over E. The asymptotic couple of K satisfies (A1). We have $y^{\dagger} = -\lambda$, and $K(y)$ is a simple Liouville H-field extension of K with gap $v(y)$.

These examples raised the question (called the "gap problem" in [4]) whether the creation of gaps in differentially algebraic H-field extensions can be confined to Liouville extensions. More precisely, we asked the following:

Let K be a Liouville closed H-field and $L = K\langle y\rangle$ an H-field extension of K with y differentially algebraic over K. Can L have a gap?

In [6] we give an example showing that, unfortunately, the answer is "yes":

Example. We continue to use the notation introduced in the last example. Let $\varrho := 2\lambda' + \lambda^2$. A computation in $\mathbb{R}[[\mathfrak{L}]]$ shows that

$$\varrho = -\left(\frac{1}{(\ell_0)^2} + \frac{1}{(\ell_0\ell_1)^2} + \frac{1}{(\ell_0\ell_1\ell_2)^2} + \cdots + \frac{1}{(\ell_0\ell_1\cdots\ell_n)^2} + \cdots\right).$$

In [6] we consider an H-field extension $\mathbb{T}$ of $\mathbb{R}[[\mathfrak{M}]]$ in which the sequence $(1/\ell_n)$ remains cofinal in its set of positive infinitesimals, and such that $\mathbb{T}$ contains a Liouville closed H-field subfield $K \supseteq E\langle\varrho\rangle$. It follows that $K(y)$, where $y = \exp(-\Lambda)$ as above, is a differentially algebraic H-field extension of K with gap $v(y)$. The ambient H-field $\mathbb{T}$ is a certain field of transseries (called the field of "well-ordered transseries of finite exponential depth and logarithmic depth at most ω") introduced in [32]. It contains $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ as an H-subfield and comes equipped with an exponential function extending the one on $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$. Note that we have encountered the series ϱ in $\mathbb{R}[[\mathfrak{L}]]$ before: the cut in $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ determined by ϱ can be used to describe when the linear differential equation $Y'' = fY$ has a non-zero solution in an H-subfield of $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ (for $f \in \mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$) or in a Hardy field of finite rank (for f in a Hardy field of finite rank), see Theorems 1.12 and 2.4, respectively. See [49] for some observations about the role of gaps in Hardy fields, and of the transseries Λ, in the theory of o-minimal structures on the field $\mathbb{R}$.

Much remains to be done to understand how gaps appear in differentially algebraic H-field extensions, even in Liouville H-field extensions. The appearance of a gap after a *simple* Liouville extension of an H-field K is witnessed in K itself, by the realization of a certain cut in K:

Lemma 4.7. ([5], 12.4) *Let K be a real closed H-field closed under asymptotic integration. The following are equivalent for $s \in K$:*

(1) *For each $\varepsilon \in K^{\times}$ with $\varepsilon \prec 1$, we have $\varepsilon'^{\dagger} < s < \varepsilon^{\dagger\dagger}$.*

(2) *For every H-field extension $L = K(y)$ of K such that $y \neq 0$ and $y^\dagger = s$, $v(y)$ is a gap in L.*

Liouville closures and closures of H-asymptotic triples. These two closure operations are related as follows. Suppose K is a real closed H-field with an element x such that $x \succ 1$ and $x' = 1$. Let P be a cut of the asymptotic couple (Γ, ψ) of K, so (Γ, ψ, P) is an H-asymptotic triple with distinguished positive element $1 = v(x^{-1})$. If K is Liouville closed, then $P = \Psi$ and (Γ, ψ, P) is closed. In general, K has a Liouville closure L such that $\Psi_L \cap \Gamma = P$, by [4]. Fix such an L. Then $(\Gamma_L, \psi_L, \Psi_L)$ extends (Γ, ψ, P), and we have an embedding φ of the closure (Γ^c, ψ^c, P^c) of (Γ, ψ, P) into the closed H-asymptotic triple $(\Gamma_L, \psi_L, \Psi_L)$ such that φ is the identity on Γ (Theorem 3.9):

$$\begin{tikzcd}
(\Gamma^c, \psi^c, P^c) \arrow[r, "\varphi"] & (\Gamma_L, \psi_L, \Psi_L) \\
(\Gamma, \psi, P) \arrow[u, "\subseteq"] \arrow[ur, "\subseteq"'] &
\end{tikzcd}$$

There is no proper Liouville closed H-subfield of L containing K, see [4], Section 6. This is in contrast to the non-minimality of the closure of H-asymptotic triples (see [2]): if (Γ, ψ, P) is not closed, then there exists a closed H-asymptotic triple (Γ', ψ', Ψ') such that

$$(\Gamma, \psi, P) \subseteq (\Gamma', \psi', \Psi') \subseteq (\Gamma^c, \psi^c, P^c), \quad \Gamma' \neq \Gamma^c.$$

Moreover, if K has two Liouville closures that are not isomorphic over K and Ψ has no supremum in Γ, then the embedding

$$\varphi \colon (\Gamma^c, \psi^c, P^c) \to (\Gamma_L, \psi_L, \Psi_L)$$

is not surjective.

Next we give an example of a real closed H-field K with an $x \in K$ satisfying $x \succ 1$ and $x' = 1$, and with an H-cut P of its asymptotic couple (Γ, ψ), such that K has *only one* Liouville closure L, up to K-isomorphism, and such that $(\Gamma_L, \psi_L, \Psi_L)$ is *not* an H-closure of (Γ, ψ, P); in particular, the map φ above is not surjective. (In the description of the example we assume familiarity with [22].)

Example. Let $\mathrm{e} \colon K = H(\mathbb{R}_{\mathbf{an},\exp}) \to \mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ be the embedding of Theorem 2.2. Then K has only one Liouville closure, up to isomorphism over K, namely the Hardy field $L = \mathrm{Li}(K)$, and by Theorem 4.5, e can be extended to an embedding $\mathrm{Li}(K) \to \mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$. We identify $\mathrm{Li}(K)$ with its image in $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ under this embedding. It is easy to see that the H-asymptotic triple (Γ, ψ, Ψ) of K is closed. (See [39] for more on the structure of Γ.) We claim that $\Gamma_L \neq \Gamma$: Consider the function $f \colon (0, +\infty) \to \mathbb{R}$ given by

$$f(x) := \int_0^x e^{t^2}\, dt.$$

By [22], Corollaries 3.9 and 5.2, if $\sum_{n=0}^\infty a_n x^{-n} + \varepsilon \in K$ where $a_n \in \mathbb{R}$ for all n and $\varepsilon \in \mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ with $\varepsilon \prec x^{-n}$ for all n, then $\sum_n a_n X^n \in \mathbb{R}[[X]]$ converges in a neighborhood of 0. The function f has asymptotic expansion

$$f(x) \sim a := \sum_{n=0}^\infty a_{2n+1} \frac{e^{x^2}}{x^{2n+1}},$$

where the coefficients a_{2n+1} are given by

$$a_1 = 1/2, \quad a_{2n+1} = \big(1 \cdot 3 \cdots (2n-1)\big)/2^{n+1} \text{ for } n \geqslant 1.$$

(See (5.10) in [22].) Hence $f = a + \varepsilon$ with $\varepsilon \prec e^{x^2}/x^{2n+1}$ for all $n \geqslant 1$. Since the series $\sum_n a_{2n+1} X^{2n+1}$ does not converge near 0, we have $e^{-x^2} f \notin K$ and hence $f \notin K$. However, $f'(x) = e^{x^2} - 1$, so $f \in \mathrm{Li}(K)$. Let $g := \log f \in \mathrm{Li}(K)$. We claim that $v(g) \notin \Gamma$. Otherwise, we can find $h \in K$ such that $g \sim h$. Hence $\delta := f - \log h \prec 1$, and thus $\log h = a + \varepsilon - \delta \in K$ with $\varepsilon - \delta \prec e^{x^2}/x^{2n+1}$ for all $n \geqslant 1$: a contradiction.

Generalities on zeros of differential polynomials. Let now K be an H-field and $P(Y) \in K\{Y\}$ be a non-zero differential polynomial of order n. The derivation $f \mapsto f'$ on K is continuous with respect to the order topology. In particular, $P(Y)$ gives rise to a continuous function $y \mapsto P(y) \colon K \to K$. Such a differential polynomial function cannot vanish identically on any non-empty open subset of K if $C \neq K$. ([5], Lemma 3.3.)

Notation. Let L be a Liouville closed H-field. Given $f \in L$ we choose $\mathrm{E}(f) \in L^{>0}$ such that $\mathrm{E}(f)^\dagger = f'$, and given $f \in L^{>0}$ we let $\mathrm{L}(f) \in L$ such that $\mathrm{L}(f)' = f^\dagger$. (The map E behaves much like an exponential function on L, and the map L much like a logarithmic function.) If $C_L = \mathbb{R}$, then $\mathrm{E} : L \to L^{>0}$ can be chosen to be an exponential function on L extending the usual exponential function $r \mapsto e^r$ on $\mathbb{R}$. (See [5], Section 7.) We have for each n the nth iterate E_n of E, with $\mathrm{E}_0 = \mathrm{id}_L$. The function L maps $L^{>C}$ into itself, so we have also for each n the nth iterate L_n of L as a function from $L^{>C}$ into itself.

Theorem 4.8. *Let $x \in K$ be such that $x > C$ and $x' = 1$. Then there exists an element f of the subfield of K generated by x and the coefficients of P such that either $P(y) > 0$ for all $y > \mathrm{E}_n(f)$ in all Liouville closed H-field extensions of K, or $P(y) < 0$ for all $y > \mathrm{E}_n(f)$ in all Liouville closed H-field extensions of K.*

An x as in the hypothesis of the theorem exists if K is Liouville closed K and its derivation preserves infinitesimals.

Corollary 4.9. *If K has an element $x > C$ with $x' = 1$, and y in a Liouville closed H-field extension of K satisfies $P(y) = 0$, then $|y| < \mathrm{E}_n(f)$ for some $f \in K$.* □

The case $n = 0$ of this corollary is well-known. (See [8], Lemma 1.2.11.) For $n = 1$ and K the Hardy field $\mathbb{R}(x)$ the corollary is due to Borel ([10], p. 30). The proof of Theorem 4.8 in [5] generalizes the main idea of Borel's argument. See also [29], [12], [59] and [70] for related results on Hardy fields.

Another consequence of Theorem 4.8 is that if K is Liouville closed, then given $a \in K$ there exists $\varepsilon \in K^{>0}$ such that either $P(y) > 0$ for all $y \in K$ such that $a < y < a + \varepsilon$, or $P(y) < 0$ for all $y \in K$ such that $a < y < a + \varepsilon$. In particular, the zero set of P in K is discrete. (Here the condition that K is Liouville closed cannot be omitted: the conclusion fails for the Hardy field $K = \mathbb{R}(x)$ and the differential polynomial

$$P(Y) := YY''x - (Y')^2 x + YY'$$

whose zero set is $\{cx^k : c \in \mathbb{R}, k \in \mathbb{Z}\}$.) In fact, if a is a **simple zero** of P, that is, if $\frac{\partial P}{\partial Y^{(n)}}(a) \neq 0$, then there exists an interval $I = (r, s)$ around a in K such that $y \mapsto P(y) \colon I \to K$ is strictly increasing or strictly descreasing.

Corollary 4.9 can be interpreted as giving a bound on the size of *large* infinite zeros of differential polynomials. Bounding the size of *small* infinite zeros of differential polynomials in H-fields is a more delicate matter. Here is one positive result from [5]:

Theorem 4.10. *Suppose the coefficients of P lie in some pre-H-subfield of K with a smallest comparability class $\mathrm{Cl}(f)$, $f \in K^{>C}$. Then $P(y) \neq 0$ for all y in all Liouville closed H-field extensions L of K with $C_L < y < \mathrm{L}_{n+1}(f)$.*

The hypothesis on the coefficients of P is automatically satisfied if K is a directed union of pre-H-subfields each of which has a smallest comparability class. For example,

$$\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}} = \bigcup_n \mathbb{R}[[\ell_n^{\mathbb{R}}]]^{\mathrm{E}}$$

is such a representation of $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ as directed union, with ℓ_n representing the smallest comparability class of $\mathbb{R}[[\ell_n^{\mathbb{R}}]]^{\mathrm{E}}$. We conclude:

Corollary 4.11. *Let $K = \mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$. There is no element b in any differentially algebraic H-field extension L of K such that $C_L < b < a$ for all $a \in K^{>\mathbb{R}}$.* $\square$

The condition on the coefficients of P in Theorem 4.10 can be omitted if P is of order 1 or homogeneous of order 2, see [5]. The condition cannot be omitted for P of order 3:

Example. Let K, y and ϱ be as in the example preceding Lemma 4.7. If $z \succ 1$ is an antiderivative of y in a Liouville closure of the H-field $K(y)$, then z is a zero of a differential polynomial of order 3 over K, and $1 \prec z \prec \ell_n$ for all n. It follows that the H-subfield $\mathbb{R}\langle\varrho\rangle$ of K is not a directed union of pre-H-subfields each of which has a smallest comparability class. The value group of $\mathbb{R}\langle\varrho\rangle$ is $\bigoplus_n \mathbb{Z}v(\ell_n)$, and $\Psi_{\mathbb{R}\langle\varrho\rangle} = \big\{-v(\ell_0\cdots\ell_n) : n \in \mathbb{N}\big\}$ has order type ω.

Van der Hoeven [33] proves the remarkable fact that differential polynomials (in a single indeterminate) over $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ have the intermediate value property:

Theorem 4.12. *Let $K = \mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$, and let $a, b \in K$ be such that $a < b$, and $P(a)$ and $P(b)$ are non-zero of opposite sign. Then there exists an element $y \in K$ such that $a < y < b$ and $P(y) = 0$.*

The following analogue of Theorem 1.11 for the class of H-fields is shown in [5]:

Theorem 4.13. *Suppose that P has order* 1*, and $a, b \in K$ with $a < b$ are such that $P(a)$ and $P(b)$ are non-zero of opposite sign. Then there exists an element y in an H-field extension of K such that $a < y < b$ and $P(y) = 0$.*

It would be interesting to remove the "order 1" assumption in this theorem.

Linear differential equations over H-fields. Chapter 4 of J. van der Hoeven's Thèse [32] studies the solution sets in $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ of linear differential equations of any order in $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$. In [7] we extend the results of this Chapter 4 to the setting of H-fields. We mention here only one byproduct of [7], Theorem 4.14, because it can be stated with minimal prerequisites, and is new even for $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$.

Every linear differential polynomial $a_0Y + a_1Y' + \cdots + a_nY^{(n)}$ with coefficients $a_0, \ldots, a_n$ in an H-field K defines a linear differential operator $a_0 + a_1\partial + \cdots + a_n\partial^n$. These linear differential operators with coefficients in K form a ring $K[\partial]$ containing

K as a subring, with $\partial h = h\partial + h'$ for $h \in K$. This ring $K[\partial]$ is not commutative unless the derivation of K is trivial.

Every irreducible one-variable polynomial over $\mathbb{R}$ is of degree 1 or 2; this fact has an analogue for linear differential operators over $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$:

Theorem 4.14. *Let $K := \mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$. Every operator in $K[\partial] \setminus K$ is a product of order 1 factors $a\partial + b$ and order 2 factors $a\partial^2 + b\partial + c$, where $a, b, c \in K$, $a \neq 0$.*

Existentially closed H-fields. Let K be an H-field. We consider systems of equations, inequalities and asymptotic inequalities of the following form:

$$\begin{cases} A_1(Y) & \varrho_1 & B_1(Y) \\ A_2(Y) & \varrho_2 & B_2(Y) \\ \vdots & \vdots & \vdots \\ A_m(Y) & \varrho_m & B_m(Y) \end{cases} \tag{4.1}$$

where $A_i(Y), B_i(Y) \in K\{Y\}$, $Y = (Y_1, \dots, Y_n)$, and $\varrho_i \in \{=, \leqslant, <, \preccurlyeq, \prec\}$. We say that K is **existentially closed** if every system (4.1) with a solution in an H-field extension of K has a solution in K itself. (It is enough to require this for systems consisting of a *single equation*, see [5], Lemma 14.1.) Every H-field can be embedded into an existentially closed H-field; every existentially closed H-field is Liouville closed. Our work on H-fields is motivated by the following open questions:

Is the H-field $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$ existentially closed? Is the class of existentially closed H-fields an elementary class *in the sense of model theory?*

A positive answer to the second question would mean that existentially closed H-fields play a similar role in the category of H-fields as real closed fields do in the category of ordered fields, and algebraically closed fields in the the category of fields: they would truly be the *universal domains* for asymptotic differential algebra.

By Zorn's Lemma every Hardy field is contained in a maximal one; this leads naturally to the question whether every *maximal* Hardy field is an existentially closed H-field. See [5], Section 14 for more on existentially closed H-fields.

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