Definable Extension Theorems in O-minimal Structures

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Overview

1. O-minimality
   - Basic definitions and examples
   - Geometry of definable sets
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2. Definable extension theorems
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O-minimality: Basic definitions and examples

O-minimal structures were introduced 30 years in order to provide an analogue of the model-theoretic tameness notion of *strong minimality* in an ordered context ("o-minimal" = "order-minimal").

We are mainly interested in o-minimal expansions of real closed ordered fields.

Throughout this talk, we fix an expansion $\boldsymbol{R}$ of a real closed ordered field $(R; 0, 1, +, \times, <)$. If you like, think of $R = \mathbb{R}$, the usual ordered field of reals.

Unless said otherwise, "definable" means "definable in $\boldsymbol{R}$, possibly with parameters." As usual, a map $f\colon S \to R^n$, where $S \subseteq R^m$, is called definable if its graph $\Gamma(f) \subseteq R^{m+n}$ is.

O-minimality: Basic definitions and examples

Definition

One says that $\boldsymbol{R}$ is **o-minimal** if all definable subsets of R are finite unions of singletons and (open) intervals.

That is, $\boldsymbol{R}$ is o-minimal if the only one-variable sets definable in $\boldsymbol{R}$ are those that are already definable in the reduct $(R;<)$ of $\boldsymbol{R}$.

Basic examples (many more are known)

- $\mathbb{R}_{\mathrm{alg}} = (\mathbb{R}; 0, 1, +, \times, <)$ [TARSKI, 1940s]; the definable sets are the *semialgebraic* sets;
- $\mathbb{R}_{\mathrm{an}} = \mathbb{R}_{\mathrm{alg}} \cup \left\{ f \colon [-1,1]^n \to \mathbb{R} \text{ restricted analytic, } n \in \mathbb{N}^{\geqslant 1} \right\}$ [VAN DEN DRIES, 1980s]; the definable sets are the *globally* (sometimes called *finitely*) *subanalytic* sets;
- $\mathbb{R}_{\mathrm{exp}} = \mathbb{R}_{\mathrm{alg}} \cup \{\exp\}$ [WILKIE, 1990s].

O-minimality: Geometry of definable sets

In the following we assume that $\boldsymbol{R}$ is o-minimal.

Definition

$(i_1, \ldots, i_n)$**-cells** in R^n are defined inductively on n as follows:

- For $n = 0$, the set $R^0 = \{\text{pt}\}$ is an $(\ )$-cell in R^0;
- Let $C \subseteq R^n$ be an $(i_1, \ldots, i_n)$-cell ($i_k \in \{0, 1\}$).
  - An $(i_1, \ldots, i_n, 0)$**-cell** is the graph $\Gamma(f)$ of a continuous definable function $f\colon C \to R$.
  - An $(i_1, \ldots, i_n, 1)$**-cell** is a set

    $$(f, g)_C := \big\{(x, y) \in R^n \times R : x \in C,\ f(x) < y < g(x)\big\}$$

    where $f, g\colon C \to R \cup \{\pm\infty\}$ are continuous definable functions with $f < g$ on C (i.e., $f(x) < g(x)$ for all $x \in C$).

So for $n = 1$, a (0)-cell is a singleton $\{r\}$ ($r \in R$), and a (1)**-cell** is an interval (a, b) where $-\infty \leqslant a < b \leqslant +\infty$.

O-minimality: Geometry of definable sets

Cell Decomposition Theorem
(van den Dries, Pillay-Steinhorn; 1980s)

- Given definable subsets $S_1, \ldots, S_k$ of $\mathbb{R}^n$ there exists a finite partition $\mathscr{C}$ of R^n into cells such that each S_i is a union of some $C \in \mathscr{C}$.
- If $f\colon E \to R$ ($E \subseteq R^n$) is a definable function, then there is a finite partition $\mathscr{C}$ of E into cells so that $f \restriction C$ is continuous, for every $C \in \mathscr{C}$.

- As a consequence, every definable set has only finitely many *definably* connected components.
- In this theorem one can also achieve differentiability up to some fixed finite order.

Cell Decomposition yields that $\boldsymbol{R}$ has *built-in Skolem functions:*

Corollary (van den Dries)

Let $(S_a)_{a\in A}$ *be a definable family of nonempty subsets* $S_a \subseteq R^n$*, where* $A \subseteq R^N$*; that is,*

$$S = \big\{(a,x) : a \in A,\ x \in S_a\big\} \subseteq R^{N+n}$$

is definable. Then there is a definable map $f\colon A \to R^n$ *such that* $f(a) \in S_a$ *for all* $a \in A$*.*

As a consequence, one obtains *curve selection*: for each definable $E \subseteq R^n$ and $x \in \mathrm{cl}(E) \setminus E$, there is a continuous definable injective map $\gamma\colon (0,\varepsilon) \to E$, for some $\varepsilon \in R^{>0}$, such that $\lim\limits_{t\to 0^+} \gamma(t) = x$.

O-minimality: Geometry of definable sets

Many of the other classical topological finiteness theorems for semialgebraic sets and maps (triangulation, trivialization, etc.) continue to hold for definable sets in $\boldsymbol{R}$. One can develop a kind of “tame topology” (no pathologies) in $\boldsymbol{R}$.

Definition

For an $(i_1, \ldots, i_n)$-cell C, set $\quad \dim(C) := i_1 + \cdots + i_n$.

For a definable subset E of R^n, set

$$\dim(E) := \max\big\{\dim(C) : C \subseteq E,\ C \text{ is a cell}\big\},$$

where $\max(\emptyset) = -\infty$.

This notion of dimension is very well-behaved (no space-filling curves, etc.). For example, *if E, E' are definable and there is a definable bijection between E and E', then $\dim(E) = \dim(E')$.*

O-minimality: Geometry of definable sets

A deeper analysis of the *geometric* properties of definable sets usually involves gaining some control on the growth of derivatives. Let $\Omega \subseteq R^d$ be open, $d \geqslant 1$, and $\partial\Omega = \mathrm{cl}(\Omega) \setminus \Omega$.

Definition

Let $f\colon \Omega \to R^l$, $l \geqslant 1$, be definable and C^m. One says that f is Λ^m**-regular** if there exists $L > 0$ such that

$$\|D^\alpha f(x)\| \leqslant \frac{L}{d(x, \partial\Omega)^{|\alpha|-1}} \quad \text{for all } x \in \Omega,\ \alpha \in \mathbb{N}^d,\ 1 \leqslant |\alpha| \leqslant m.$$

Here $D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}$, $|\alpha| = \alpha_1 + \cdots + \alpha_d$ for $\alpha = (\alpha_1, \ldots, \alpha_d)$.

Also declare each map $R^0 \to R^l$ and the constant functions $\pm\infty$ to be Λ^m-regular.

For example, $f(x) = \frac{1}{x}$ is not Λ^1-regular on $\Omega = (0, +\infty)$.

O-minimality: Geometry of definable sets

Standard open Λ^m-regular cells in R^n are defined inductively:

1. $n = 0$: R^0 is the only standard open Λ^m-regular cell in R^0;
2. $n \geqslant 1$: a set of the form $(f, g)_D$ where $f, g \colon D \to R \cup \{\pm\infty\}$ are definable Λ^m-regular functions such that $f < g$, and D is a standard open Λ^m-regular cell in R^{n-1}.

A **standard Λ^m-regular cell in** R^n is either

1. a standard open Λ^m-regular cell in R^n; or
2. the graph of a definable Λ^m-regular map $D \to R^{n-d}$, where D is a standard open Λ^m-regular cell in R^d, and $0 \leqslant d < n$.

Thus standard Λ^m-regular cells in R^n are particular kinds of $(1, \ldots, 1, 0, \ldots, 0)$-cells in R^n.

Call $E \subseteq R^n$ a Λ^m**-regular cell in** R^n if there is an R-linear orthogonal isomorphism ϕ of R^n such that $\phi(E)$ is a standard Λ^m-regular cell in R^n.

Definition

A Λ^m**-regular stratification** of R^n is a finite partition $\mathscr{D}$ of R^n into Λ^m-regular cells such that each ∂D ($D \in \mathscr{D}$) is a union of sets from $\mathscr{D}$. Given $E_1, \ldots, E_N \subseteq R^n$, such a Λ^m-regular stratification $\mathscr{D}$ of R^n is said to be **compatible with** $E_1, \ldots, E_N$ if each E_i is a union of sets from $\mathscr{D}$.

Theorem (A. Fischer, 2007)

Let $E_1, \ldots, E_N$ be definable subsets of R^n. Then there exists a Λ^m-regular stratification $\mathscr{D}$ of R^n, compatible with $E_1, \ldots, E_N$.

Moreover, one has quite some fine control over the cells in $\mathscr{D}$; e.g., they can additionally chosen to be Lipschitz (with rational Lipschitz constant depending only on n).

At the root of this is a simple calculus lemma due to GROMOV (phrased here in $\mathbb{R}$):

Lemma

Let $h\colon I \to \mathbb{R}$ be a C^2-function on an interval I in $\mathbb{R}$ such that h, h'' are semidefinite. Let $t \in I$ and $r > 0$ with $[t-r, t+r] \subseteq I$. Then

$$\big|h'(t)\big| \leqslant \frac{1}{r} \sup\big\{\, |h(\xi)| : \xi \in [t-r, t+r]\big\}.$$

O-minimality: Why o-minimal geometry?

O-minimality

General model-theoretic properties of o-minimal structures

- valuation theory;
- definable groups;
- asymptotic expansions; . . .

Construction of examples

- elimination theory;
- resolution of singularities; . . .

Geometry of definable sets

Applications

Case in point: PILA-WILKIE (2006)

Definable extension theorems

For now, let's work in $R = \mathbb{R}$.

By an **extension problem** we will mean a situation of the following kind:

Let $\mathcal{C}$ be a class of [definable] functions $\mathbb{R}^n \to \mathbb{R}$. Find a necessary and sufficient condition for some given function $E \to \mathbb{R}$, were $E \subseteq \mathbb{R}^n$, to have an extension to a [definable] function from $\mathcal{C}$.

Earlier, A. FISCHER and I had looked at a definable version of Kirszbraun's Theorem, which concerns the extension of Lipschitz functions with a given Lipschitz constant. (Here, o-minimality turned out to be an unnecessarily strong tameness assumption on $\boldsymbol{R}$.)

Definable extension theorems: The Whitney Extension Problem

From now on, $E \subseteq \mathbb{R}^n$ is closed, and α ranges over $\mathbb{N}^n$.

Definition

A **jet of order** m **on** E is a family $F = (F^\alpha)_{|\alpha| \leqslant m}$ of continuous functions $F^\alpha \colon E \to \mathbb{R}$. For $f \in C^m(\mathbb{R}^n)$, we obtain a jet

$$J_E^m(f) := (D^\alpha f \restriction E)_{|\alpha| \leqslant m}$$

of order m on E.

Question

Let F be a jet of order m on E. *What is a necessary and sufficient condition to guarantee the existence of a* C^m*-function* $f \colon \mathbb{R}^n \to \mathbb{R}$ *such that* $J_E^m(f) = F$*?*

Definable extension theorems: The Whitney Extension Problem

Let $F = (F^\alpha)_{|\alpha| \leqslant m}$ be a jet of order m on E and $a \in E$.

$$T_a^m F(x) := \sum_{|\alpha| \leqslant m} \frac{F^\alpha(a)}{\alpha!}(x-a)^\alpha, \quad R_a^m F := F - J_E^m(T_a^m F).$$

Definition

A jet F of order m is a C^m-**Whitney field** ($F \in \mathscr{E}^m(E)$) if for $x_0 \in E$ and $|\alpha| \leqslant m$,

$$(R_x^m F)^\alpha(y) = o(|x-y|^{m-|\alpha|}) \quad \text{as } E \ni x, y \to x_0.$$

By Taylor's Formula, $J_E^m(f)$ is a C^m-Whitney field, for each $f \in C^m(\mathbb{R}^n)$.

Definable extension theorems: The Whitney Extension Problem

Whitney Extension Theorem (H. WHITNEY, 1934)

For every $F \in \mathscr{E}^m(E)$, there is an $f \in C^m(\mathbb{R}^n)$ with $J^m_E(f) = F$.

Proof outline

- Decompose $\mathbb{R}^n \setminus E$ into countably many cubes with disjoint interior satisfying some inequality regarding their diameter and distance from E. (Whitney decomposition)
- Use this to get a "special" partition of unity (ϕ_i) on $\mathbb{R}^n \setminus E$.
- Pick $a_i \in E$ such that $d(a_i, \operatorname{supp}(\phi_i)) = d(E, \operatorname{supp}(\phi_i))$.
- $f(x) = \begin{cases} F^0(x), & \text{if } x \in E; \\ \sum\limits_{i \in \mathbb{N}} \phi_i(x) T^m_{a_i} F(x), & \text{if } x \notin E. \end{cases}$

Definable extension theorems: The Whitney Extension Problem

Theorem (Kurdyka & Pawłucki, 1997)

Let $F \in \mathscr{E}^m(E)$ be subanalytic. Then there is a subanalytic C^m-function $f\colon \mathbb{R}^n \to \mathbb{R}$ such that $J^m_E(f) = F$.

Their proof used tools very specific to the subanalytic context (e.g., reduction to the case E compact; Whitney arc property).

Theorem (Pawłucki, 2008)

Let $E \subseteq \mathbb{R}^n$ be definable in $\boldsymbol{R}$. There is a linear extension operator

$$\mathscr{E}^m_{\mathrm{def}}(E) \to C^m(\mathbb{R}^n)$$

which is a finite composition of operators each of which either preserves definability or is an integration with respect to a parameter.

Definable extension theorems: The Whitney Extension Problem

Theorem (A. Thamrongthanyalak, 2012)

Let $F \in \mathscr{E}^m(E)$ be definable. Then there is a definable C^m-function $f\colon \mathbb{R}^n \to \mathbb{R}$ such that $J^m_E(f) = F$.

The construction is uniform (for definable families of Whitney fields), and works for any R, not just $R = \mathbb{R}$.

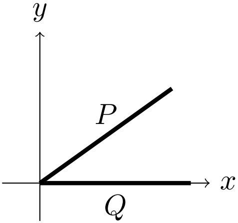
The proof follows the outline of the construction of Pawłucki, combining it with the results on Λ^m-stratification by Fischer.

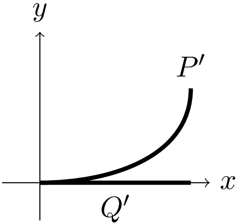
A key step is the Λ^m-regular Separation Theorem (proved by Pawłucki for $R = \mathbb{R}$), which we explain next.

Definition

Let $P, Q, Z \subseteq R^n$ be definable. Then P, Q **are Z-separated** if

$$(\exists C > 0)(\forall x \in R^n)\ d(x, P) + d(x, Q) \geqslant C \cdot d(x, Z).$$





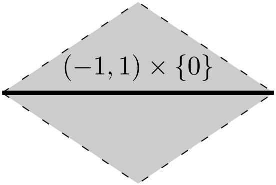
P and Q are $\{(0,0)\}$-separated; P' and Q' not $\{(0,0)\}$-separated.

Definable extension theorems: The Whitney Extension Problem

For $E' \subseteq \mathrm{cl}(E)$ and $\varepsilon > 0$, let

$$\Delta_\varepsilon(E) := \big\{x \in \mathbb{R}^n : d(x, E) < \varepsilon\, d(x, E')\big\};$$
$$\Delta_\varepsilon(E) := \Delta_\varepsilon(E, \partial E).$$

$\Delta_\varepsilon((-1,1) \times \{0\})$



Proposition (Pawłucki)

Let $E_i \supseteq E_i'$ ($i = 1, \ldots, s$) be closed definable subsets of R^n. Suppose E_i, E_j are E_i'-separated for every $i \neq j$.

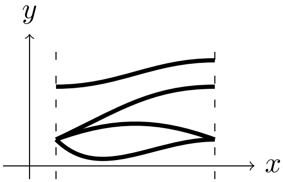
Let $F \in \mathscr{E}^m(E_1 \cup \cdots \cup E_s)$ be flat on $E_1' \cup \cdots \cup E_s'$, and $\varepsilon > 0$ be small enough. Let f_i be a definable C^m-extension of $F \upharpoonright E_i$, m-flat outside $\Delta_\varepsilon(E_i, E_i')$.

Then $\sum_i f_i$ is a C^m-extension of F.

Definable extension theorems: The Whitney Extension Problem

Definition

A Λ^m**-pancake** in R^n is a finite disjoint union of graphs of definable Lipschitz Λ^m-regular maps $\Omega \to R^{n-d}$, where $\Omega \subseteq R^d$ is an open Λ^m-regular cell.



Λ^m-regular Separation Theorem

Let $E \subseteq R^n$ be definable. Then $E = M_1 \cup \cdots \cup M_s \cup A$ where

1. each M_i is a Λ^m-pancake in a suitable coordinate system, $\dim M_i = \dim E$, and A is a definable, small, closed;
2. $\mathrm{cl}(M_i)$, $\mathrm{cl}(M_j)$ are ∂M_i-separated for $i \neq j$;
3. $\mathrm{cl}(M_i), A$ are ∂M_i-separated.

Definable extension theorems: The Whitney Extension Problem

Whitney actually asked a quite different question (and answered it for $n = 1$):

Whitney's Extension Problem

Let $f: X \to \mathbb{R}$ be a continuous function, where X is a closed subset of $\mathbb{R}^n$. How can we determine whether f is the restriction of a C^m-function on $\mathbb{R}^n$?

A complete answer was only given by C. FEFFERMAN in the early 2000s. BIERSTONE & MILMAN (2009): what about the definable case?

An answer in the case $m = 1$ was found earlier by G. GLAESER in 1958, and simplified by B. KLARTAG and N. ZOBIN (2007).

The latter can be made to work definably (A. & THAMRONGTHANYALAK, 2013).

Definition

Let $f\colon E \to R$ and $H \subseteq E \times (R \times R^n)$ be definable. We say that H is a **holding space for** f if

1. H_x is an affine subspace of $R \times R^n$ or H_x is empty, for every $x \in E$;
2. whenever $F \in C^1(R^n)$ is definable with $F \upharpoonright E = f$,

$$\left\{ \left(x, F(x), \frac{\partial F}{\partial x_1}(x), \ldots, \frac{\partial F}{\partial x_n}(x) \right) : x \in E \right\} \subseteq H.$$

Identify $R \times R^n$ with the space $\mathscr{P}_n$ of linear polynomials in n indeterminates over R. Think of a holding space for f as a collection of potential Taylor polynomials of definable C^1-extensions of f.

We always have the **trivial** holding space $H_0 := E \times \mathscr{P}_n$.

Definable extension theorems: The Whitney Extension Problem

Let $H \subseteq E \times \mathscr{P}_n$ be definable.

Definition (the (GLAESER) **refinement** $\widetilde{H}$ of H)

$(x_0, p_0) \in \widetilde{H} :\Longleftrightarrow (x_0, p_0) \in H$ and

$$(\forall \varepsilon > 0)\,(\exists \delta > 0)\,(\forall x_1, x_2 \in E \cap B_\delta(x_0))\,(\exists p_1 \in H(x_1), p_2 \in H(x_2))$$

$$|D^\alpha(p_i - p_j)(x_i)| \leqslant \varepsilon \left\| x_i - x_j \right\|^{1-|\alpha|} \text{ for } i, j = 0, 1, 2 \text{ and } |\alpha| \leqslant 1.$$

We say that H is **stable** under refinement if $\widetilde{H} = H$.

Routine to show:

- $H(x)$ affine subspace for each $x \in X \Rightarrow \widetilde{H}(x) = \emptyset$, or $\widetilde{H}(x)$ is an affine subspace of $\mathscr{P}_n$;
- f extends to a definable C^1-function $R^n \to R \Rightarrow$ every holding space for f is stable;
- $\dim \widetilde{H}(x_0) \leqslant \liminf_{X \ni x \to x_0} \dim H(x)$.

As a consequence, the sequence (H_l) where

$$H_0 = \text{trivial holding space for } f, \quad H_{l+1} := \widetilde{H_l}$$

eventually stabilizes (in fact, for $l = 2\dim \mathscr{P}_n + 1 = 2n+3$). Let H be the eventual value of this sequence.

Lemma (consequence of Definable Whitney Extension)

The function f extends to a definable C^1-function on R^n iff there is a continuous *Skolem function for the definable family (H_x).*

It is useful to think of (H_x) as a **set-valued map**

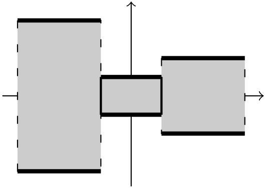
$$x \mapsto H(x) = H_x : E \rightrightarrows R \times R^n.$$

If $H(x) \neq \emptyset$ for each $x \in E$, then H is lower semi-continuous in the sense of the following definition.

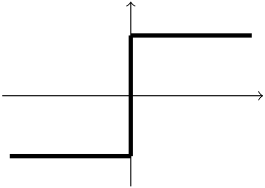
Let $E \subseteq R^m$ and $T \rightrightarrows R^n$ be a set-valued map.

Definition

One says that T is **lower semi-continuous (l.s.c.)** if, for every $x \in E$, $y \in T(x)$, and neighborhood V of y, there is a neighborhood U of x such that $T(x') \cap V \neq \emptyset$ for all $x' \in U \cap E$.



l.s.c.



not l.s.c.

Definable extension theorems: The Whitney Extension Problem

Theorem (Definable Michael's Selection Theorem)

Let E be a closed subset of R^n and $T\colon E \rightrightarrows R^m$ be a definable l.s.c. set-valued map such that $T(x)$ is nonempty, closed, and convex for every $x \in E$. Then T has a continuous definable Skolem function.

Classically, this theorem is shown by a nonconstructive iterative procedure.

It does also hold for *bounded* E in the category of semilinear sets and maps (using a different proof).

Corollary

Let $(f_a)_{a\in A}$, $A \subseteq R^N$, be a definable family of functions $f_a\colon E_a \to R$, where $E_a \subseteq R^n$ is closed. Then there is a definable $A_* \subseteq A$ such that for all $a \in A$,

$$a \in A_* \iff f_a \text{ extends to a definable } C^1\text{-function on } R^n.$$

Moreover, there is a definable family $(\widetilde{f}_a)_{a\in A_*}$ of C^1-functions on R^n such that

$$\widetilde{f}_a \restriction E_a = f_a \quad \text{for each } a \in A_*.$$