Gaps in H-Fields

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Final remarks and questions.

- Let $S(Z, Z') = 2\,Z' + Z^2$ (the "Schwarzian"). Whenever y is a non-zero solution to the linear differential equation
$$Y'' = fY,$$
then $z = 2\,y^\dagger$ satisfies $S(z, z') = f$.

- The cut in $\mathbb{R}((x^{-1}))^{\mathrm{LE}}$ determined by $\varrho = S(\lambda, \lambda') \in \mathbb{L}$ describes when $Y'' = f\,Y$ has a non-zero solution in an H-subfield of $\mathbb{R}((x^{-1}))^{\mathrm{LE}}$, for $f \in \mathbb{R}((x^{-1}))^{\mathrm{LE}}$. (Macintyre-Marker-van den Dries.)

- Let $P(Z, Z', \ldots, Z^{(n)}) \in \mathbb{R}\{Z\}$, non-constant. Up to multiplication by some monomial $\mathfrak{m} \in \mathfrak{L}$, the sum of the first ω non-zero terms of the series $P(\lambda, \lambda', \ldots, \lambda^{(n)}) \in \mathbb{L}$ is either of the form $\lambda \circ \log_r x$ or $\varrho \circ \log_r x$, for some $r \geqslant 0$. (Écalle.)

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- How can one detect in an H-field whether it has a Liouville extension with a gap?

Overview

I. Transseries

II. H-Fields

III. Gaps in H-Fields

I. Transseries

The field $\mathbb{R}((x^{-1}))$ of (formal) **Laurent series** over $\mathbb{R}$ in *descending* powers of x consists of all series

$$f(x) = \underbrace{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x}_{\text{infinite part of } f} + a_0 + \underbrace{a_{-1}x^{-1} + a_{-2}x^{-2} + \cdots}_{\text{infinitesimal part of } f}$$

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Order $\mathbb{R}((x^{-1}))$ so that $x > \mathbb{R}$, and differentiate so that $x' = 1$.

Exponentiation for *finite* elements of $\mathbb{R}((x^{-1}))$ can be defined:

$$\begin{aligned}
&\exp(a_0 + a_{-1}x^{-1} + a_{-2}x^{-2} + \cdots) \\
&= \mathrm{e}^{a_0} \sum_{n=0}^{\infty} \frac{1}{n!}(a_{-1}x^{-1} + a_{-2}x^{-2} + \cdots)^n \\
&= \mathrm{e}^{a_0}(1 + b_1 x^{-1} + b_2 x^{-2} + \cdots) \quad \text{for suitable } b_1, b_2, \ldots \in \mathbb{R}.
\end{aligned}$$

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- x^{-1} has no antiderivative $\log x$ in $\mathbb{R}((x^{-1}))$.
- $\mathbb{R}((x^{-1}))$, as a differential field, defines $\mathbb{Z}$.

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$$e^{e^x} - 3e^{x^2} + 5x^{\sqrt{2}} - (\log x)^{\pi} + 1 + x^{-1} + x^{-2} + x^{-3} + \cdots + e^{-x} + x^{-1}e^{-x}.$$

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A nonzero transseries is declared positive if its leading coefficient is positive. (Just like in $\mathbb{R}((x^{-1}))$.)

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  We obtain a derivation on the field $\mathbb{T}$, that is, a map $f \mapsto f' : \mathbb{T} \to \mathbb{T}$ with the properties

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  $$\int \frac{\mathrm{e}^x}{x}\, dx = \text{const} + \sum_{n=0}^{\infty} n!\, x^{-1-n} \mathrm{e}^x \quad \text{(diverges)}.$$

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- We have a canonical isomorphism

$$f \mapsto \exp(f) \colon (\mathbb{T}, +, 0, \leqslant) \to (\mathbb{T}^{>0}, \cdot, 1, \leqslant)$$

with inverse $g \mapsto \log(g)$, extending the exponentiation of finite Laurent series.

- The iterates of exp,

$$\mathrm{e}_0 := x,\ \mathrm{e}_1 := \exp x,\ \mathrm{e}_2 := \exp(\exp(x)),\ \ldots$$

form an increasing cofinal sequence in $\mathbb{T}$. Their formal inverses

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- The structure

$$(\mathbb{T}, +, \cdot, \leqslant, \exp)$$

is an elementary extension of

$$\mathbb{R}_{\exp} := (\mathbb{R}, +, \cdot, \leqslant, r \mapsto \mathrm{e}^r).$$

(Wilkie.)

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Origins and Applications of Transseries

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- were introduced independently by Écalle (Hilbert's 16th Problem) and by Dahn and Göring (Tarski's Problem on $\mathbb{R}_{\text{exp}}$);
- give very exact asymptotics for solutions of algebraic differential equations over $\mathbb{R}$;
- many functions occurring in analysis have an asymptotic expansion as transseries; for example, many (all?), which are definable in an exponentially bounded o-minimal expansion of $(\mathbb{R}, +, \cdot, \leqslant)$, like $\mathbb{R}_{\text{exp}}$.

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The $\mathbb{T}$-Conjecture (A., van den Dries, van der Hoeven)

$\mathbb{T}$ is model-complete.

In fact, we have a strengthened version of this conjecture, which states that $\mathbb{T}$ has quantifier elimination in a certain natural expansion of the language specified above.

Recently we have become optimistic that we are getting closer to a proof of this conjecture. (Most of the rest of this talk is joint work with van den Dries and van der Hoeven.)

II. H-Fields

H-Fields

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$$f \preccurlyeq g \quad :\Longleftrightarrow \quad \exists c \in C^{>0} : |f| \leqslant c|g| \qquad \text{“}g \text{ dominates } f\text{”}$$

$$f \prec g \quad :\Longleftrightarrow \quad \forall c \in C^{>0} : |f| \leqslant c|g| \qquad \text{“}g \text{ strictly dominates } f.\text{”}$$

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Definition

We call K an H-**field** provided that

(H1) $f > C \Rightarrow f' > 0$;

(H2) $f \preccurlyeq 1 \Rightarrow f - c \prec 1$ for some $c \in C$;

(H3) $f \prec 1 \Rightarrow f' \prec 1$.

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To prove the $\mathbb{T}$-Conjecture we need to show that the existentially closed H-fields are exactly the H-fields that share certain deeper first-order properties with $\mathbb{T}$.

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To prove the $\mathbb{T}$-Conjecture we need to show that the existentially closed H-fields are exactly the H-fields that share certain deeper first-order properties with $\mathbb{T}$.

In this talk we concentrate on one particular such property:

ω-*freeness.*

The real closure of an H-field is again an H-field. We call a real closed H-field K **Liouville closed,** if for every $a, b \in K$ there is a nonzero $y \in K$ with $y' + ay = b$.

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Theorem (A.-van den Dries, 2002)

Every H-field has exactly one or exactly two Liouville closures.

Whether there are one or two Liouville closures depends on an important trichotomy in the class of H-fields.

Let K be an H-field.

Define an equivalence relation $\asymp$ on $K^\times = K \setminus \{0\}$:

$$f \asymp g \quad :\Longleftrightarrow \quad f \preccurlyeq g \text{ and } g \preccurlyeq f.$$

Trichotomy for H-Fields

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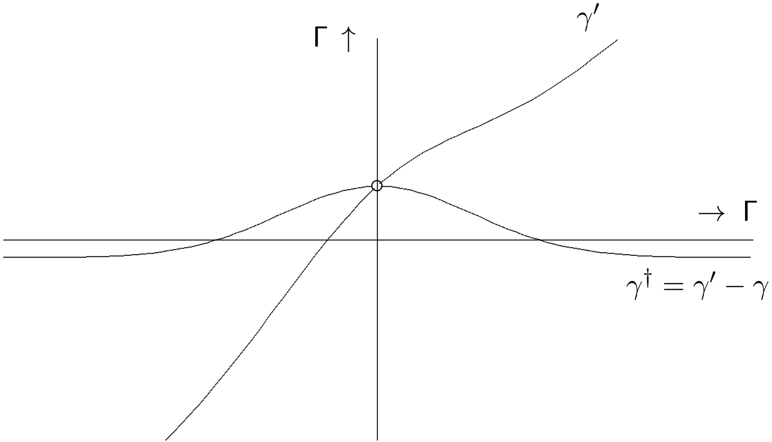
Example

For $K = \mathbb{T}$: $\quad (\Gamma, +, \leqslant) \cong (\text{group of transmonomials}, \cdot\,, \succcurlyeq)$.

The derivation ∂ induces a map

$$\gamma = vf \mapsto \gamma' = v(f'): \quad \Gamma^{\neq} := \Gamma \setminus \{0\} \to \Gamma.$$



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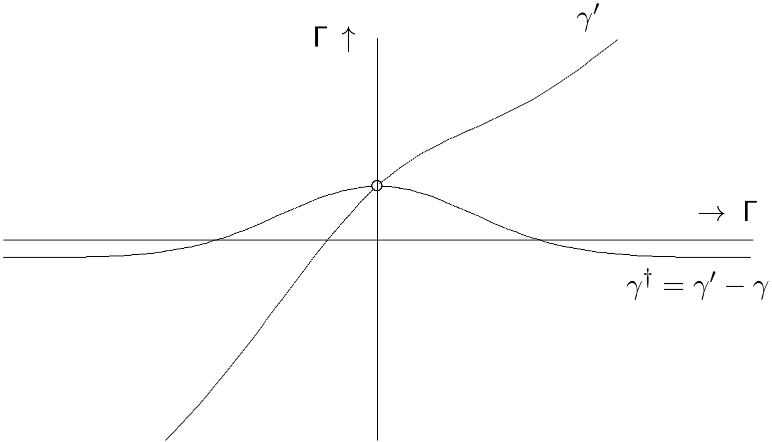
We set $\Gamma^{\dagger} := \{\gamma' - \gamma : \gamma \in \Gamma^{\neq}\}$. Then $\Gamma^{\dagger} < (\Gamma^{>0})'$.



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Examples

1. $K = C$;
2. $K = \mathbb{R}((x^{-1}))$;
3. $K = \mathbb{T}$ (or any other Liouville closed K).

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   We say that K has **asymptotic integration.**

In Case 1 we have *two* Liouville closures: if $\gamma = vg$, then we have a choice when adjoining $\int g$: make it $\succ 1$ or $\prec 1$.

In Case 2 we have *one* Liouville closure.

Obviously, Case 1 poses an obstacle for the proof of any kind of quantifier elimination. And what happens in Case 3?

III. Gaps in H-Fields

How do gaps arise under Liouville extensions?

Let K be a real closed H-field.

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  L has a gap if and only if K has a gap.

- If $L = K(z)$ with $z \neq 0$, $z^\dagger = g \in K$ ($z = \exp \int g$), then

  L may have a gap even if K does not have a gap.

  Here $z^\dagger := z'/z$ for $z \neq 0$ in K.

One can detect in K already whether some $g \in K$ **creates a gap over** K**,** i.e., $z = \exp \int g$ is a gap in $K(z)$.

Gaps under Liouville Extensions

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It is instructive to consider H-subfields $K \supseteq \mathbb{R}$ of $\mathbb{T}$: no such K can have a gap. Suppose K contains the iterated logarithms $\ell_0 = x$, $\ell_{n+1} = \log \ell_n$. Consider the “pseudo-cauchy sequence”

$$\lambda_n := -\ell_n^{\dagger\dagger} = \frac{1}{\ell_0} + \frac{1}{\ell_0 \ell_1} + \frac{1}{\ell_0 \ell_1 \ell_2} + \cdots + \frac{1}{\ell_0 \ell_1 \cdots \ell_n}$$

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in $\mathbb{T}$. Then for $\lambda \in K$,

$$\lambda \text{ is a "pseudo-limit" of } (\lambda_n) \iff -\lambda \text{ creates a gap over } K.$$

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This gap is $z = \exp(\int -\lambda)$, and then

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which is impossible by construction of $\mathbb{T}$.

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(But (λ_n) does have a pseudo-limit $\lambda = \sum_{n=0}^{\infty} \frac{1}{\ell_0\ell_1\cdots\ell_n}$ in some larger valued field.)

The property that (λ_n) does not have a pseudo-limit in $\mathbb{T}$ can be converted into a $\forall\exists$-statement about $\mathbb{T}$, and this statement has the desired property:

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Proposition

The following are equivalent, for a real closed H-field K:

1. $\forall f \exists g \big[g \succ 1 \ \&\ f - g^{\dagger\dagger} \succcurlyeq g^{\dagger} \big]$.
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We say that K is **λ-free** if it satisfies the condition in the proposition.

It is now natural to wonder whether the occurrence of gaps is concentrated in Liouville extensions:

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In the meantime we have reached a better understanding of when an H-field can have a differentially algebraic H-field extension with a gap.

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Suppose K satisfies

$$\forall f \exists g \big[g \succ 1 \ \&\ f - \omega(-g^{\dagger\dagger}) \succcurlyeq (g^{\dagger})^2 \big].$$

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Corollary

If K is ω-free, then K has exactly one Liouville closure.

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$$\omega_n := \omega(\lambda_n) = \frac{1}{\ell_0^2} + \frac{1}{(\ell_0\ell_1)^2} + \frac{1}{(\ell_0\ell_1\ell_2)^2} + \cdots + \frac{1}{(\ell_0\ell_1\cdots\ell_n)^2}$$

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The proof of the theorem has two main ingredients:

1. a proof that every pc-sequence in K has a pseudolimit in some H-field extension L of K with $\Gamma_L = \Gamma$, $C_L = C$;

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into one with a "dominant term" of the form

$$(c_0 + c_1 Y + \cdots + c_m Y^m) \cdot (Y')^n \qquad (c_0, \ldots, c_m \in \mathbb{R}).$$

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General H-fields K have no operation like $f \mapsto f{\uparrow}$.

But there is a substitute: **compositional conjugation.**

- Replacing the derivation ∂ of K by $\phi^{-1}\partial$ ($\phi \in K^{\times}$) yields a new differential field K^{ϕ}, and
- rewriting P in terms of $\phi^{-1}\partial$ yields $P^{\phi} \in K^{\phi}\{Y\}$ such that

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Theorem ($\sim$ 2009)

Let $P \in K\{Y\}$, $P \neq 0$. Then there exists $N_P \in C\{Y\}$, $N_P \neq 0$, so that for all ϕ with sufficiently large $v\phi$:

$$P^\phi = \mathfrak{d}\, N_P + R, \qquad \mathfrak{d} \in K^\times,\ R \in K^\phi\{Y\},\ R \prec \mathfrak{d}.$$

We call N_P the **Newton polynomial** *of P.*

Unfortunately (?) it is not always the case (like in $\mathbb{T}$) that $N_P \in C[Y](Y')^{\mathbb{N}}$. But we now understand exactly when it is.

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The proof of this theorem involves a deeper study of compositional conjugation.

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Such triangular automorphisms can be treated with Lie theoretic methods. Every triangular automorphism σ of $K\{Y\}$ can be represented by an upper triangular matrix $M_{\sigma} \in K^{\mathbb{N}\times\mathbb{N}}$, whose matrix logarithm $\log(M_{\sigma})$ represents a K-linear derivation of $K\{Y\}$.

A special role is played by $\phi = 1/x$ where $x' = 1$. The matrix $M_\Upsilon = (\Upsilon_{ij})$ representing

$$P(Y) \mapsto P^{1/x}(Y, xY', x^2Y'', \dots)$$

has the entries

$$\Upsilon_{ij} = (-1)^{j-i}\begin{bmatrix} j \\ i \end{bmatrix} \quad \text{(signed Stirling numbers of the first kind).}$$

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Its matrix logarithm is

$$\log(M_\Upsilon) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ & 0 & -1 & \frac{1}{2} & -\frac{1}{2} & \frac{2}{3} & -\frac{11}{12} & \cdots \\ & & 0 & -3 & 2 & -\frac{5}{2} & 4 & \cdots \\ & & & 0 & -6 & 5 & -\frac{15}{2} & \cdots \\ & & & & 0 & -10 & 10 & \cdots \\ & & & & & 0 & -15 & \cdots \\ & & & & & & \ddots & \ddots \end{pmatrix}.$$

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Fact (Jabotinsky, 1940s)

Let $K \supseteq \mathbb{Q}$ be a commutative ring. There is a group embedding

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$$[\![f]\!] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ & 1 & f_2 & f_3 & f_4 & \cdots \\ & & 1 & 3f_2 & 4f_3 + 3f_2^2 & \cdots \\ & & & 1 & 6f_2 & \cdots \\ & & & & 1 & \cdots \\ & & & & & \ddots \end{pmatrix}$$

is called the **iteration matrix** of $f = z + \sum_{n\geqslant 2} f_n \frac{z^n}{n!}$.

The matrix $\log[\![f]\!]$ has a simple form: it is the **infinitesimal iteration matrix** $\langle\!\langle h\rangle\!\rangle$ of some $h = \sum_{n=2}^{\infty} h_n \frac{z^n}{n!} \in z^2 K[[z]]$:

$$\langle\!\langle h\rangle\!\rangle = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ & 0 & h_2 & h_3 & h_4 & \cdots \\ & & 0 & 3h_2 & 4h_3 & \cdots \\ & & & 0 & 6h_2 & \cdots \\ & & & & 0 & \cdots \\ & & & & & \ddots \end{pmatrix} \quad \text{with } \langle\!\langle h\rangle\!\rangle_{ij} = \binom{j}{j-i+1} h_{j-i+1}.$$

Écalle calls $h = \operatorname{itlog}(f)$ the **iterative logarithm** of f:

$$\operatorname{itlog}(f \circ g) = \operatorname{itlog}(f) + \operatorname{itlog}(g) \quad \text{if } f \circ g = g \circ f.$$

Example

$$M_\Upsilon = (\Upsilon_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ & 1 & -1 & 2 & -6 & \cdots \\ & & 1 & -3 & 11 & \cdots \\ & & & 1 & -6 & \cdots \\ & & & & 1 & \cdots \\ & & & & & \ddots \end{pmatrix}, \quad \Upsilon_{ij} = (-1)^{j-i} \begin{bmatrix} j \\ i \end{bmatrix}.$$

Then $M_\Upsilon = [\![\log(1+z)]\!]$ and

$$\begin{aligned} \operatorname{itlog}\big(\log(1+z)\big) &= -1\frac{z^2}{2!} + \frac{1}{2}\frac{z^3}{3!} - \frac{1}{2}\frac{z^4}{4!} + \frac{2}{3}\frac{z^5}{5!} - \frac{11}{12}\frac{z^6}{6!} + \cdots \\ &= -\operatorname{itlog}\big(\mathrm{e}^z - 1\big). \end{aligned}$$

The sequence

$$0, 1, -\frac{1}{2}, \frac{1}{2}, -\frac{2}{3}, \frac{11}{12}, -\frac{3}{4}, -\frac{11}{6}, \frac{29}{4}, \frac{493}{12}, -\frac{2711}{6}, \ldots$$

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- differentially transcendental (Boshernitzan-Rubel 1986);
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A common generalization of these facts holds true:

Theorem (A.-Bergweiler)

$\operatorname{itlog}(\mathrm{e}^z - 1)$ *is differentially transcendental over* $\mathbb{C}\{z\}$.

(If $f \in z + z^2\mathbb{C}[[z]]$ is a non-linear entire function, then $\operatorname{itlog}(f)$ is differentially transcendental over the ring of entire functions.)

But this would be the topic of another talk . . .

Thank you!