DEFINABLE SMOOTHING OF CONTINUOUS FUNCTIONS

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Abstract. Let $\mathbf{R}$ be an o-minimal expansion of a real closed field. Given definable continuous functions $f\colon U \to R$ and $\epsilon\colon U \to (0, +\infty)$, where U is an open subset of R^n, we construct a definable C^m-function $g\colon U \to R$ with $|g(x) - f(x)| < \epsilon(x)$ for all $x \in U$. Moreover, we show that if f is uniformly continuous, then g can also chosen to be uniformly continuous.

1. Introduction

This paper discusses the problem of smoothing continuous functions definable in an o-minimal expansion of a real closed field. It is motivated by a series of papers by A. Fischer [5, 7] and a question posed by C. Fefferman during a meeting at the Fields Institute in Toronto in 2012, as part of its Focus Program on Whitney Problems.

Smoothing problems have been studied widely in differential topology (see [9] for classical results). Basically, the question is:

Question. Let U be an open subset of $\mathbb{R}^n$, let $f\colon U \to \mathbb{R}$ and $\epsilon\colon U \to (0, +\infty)$ be continuous functions, and $m \in \mathbb{N}$. Is there a C^m-function $g\colon U \to R$ such that $|g(x) - f(x)| < \epsilon(x)$ for all $x \in U$?

Classical methods that are used to answer this question involve convolutions and integrations (see [10]), which are non-constructive and do not generally preserve definability in the sense of first-order logic. In this paper we study smoothing of continuous functions in the category of functions $U \to \mathbb{R}$ ($U \subseteq \mathbb{R}^n$ open) which are definable in a given o-minimal expansion of the ordered field of real numbers. More generally, we fix an o-minimal expansion $\mathbf{R}$ of a real closed ordered field R (not necessarily the real field). "Definable" will mean "definable in $\mathbf{R}$, possibly with parameters." We assume that readers have some familiarity with o-minimal structures. The background required for reading this paper can be found in [2].

In [4], J. Escribano proved that in $\mathbf{R}$ it is possible, given $1 \le n \le m \in \mathbb{N}$, to find definable C^m-approximations of definable C^n-functions. The case $n = 1$ of this result can be strengthened by replacing "C^1" with the weaker condition

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"locally Lipschitz," as shown by Fischer [5, Theorem 1]. (Every definable C^1-function is locally Lipschitz.) Here, we relax the condition further to just "continuous." To formulate our main theorem, we introduce some useful terminology: Given definable subsets $X \subseteq E$ of R^n, we say that X is a **small** subset of E if $\dim(X) < \dim(E)$.

Theorem 1.1. *Let $f\colon U \to R$ be a definable continuous function, where $U \subseteq R^n$ is open. Let Z be a definable closed small subset of U such that $f \restriction (U \backslash Z)$ is C^m, where $m \geq 1$. Let $\epsilon\colon U \to R^{>0}$ be a definable continuous function. Then, for any definable open neighborhood V of Z in U, there is a definable C^m-function $g\colon U \to R$ such that*

(1) *$|g(x) - f(x)| < \epsilon(x)$ for every $x \in U$;*
(2) *$g = f$ outside V.*

Note that by the Smooth Cell Decomposition Theorem, given $m \geq 1$, such Z as in Theorem 1.1 always exists.

We say that a definable function $f\colon S \to R$ ($S \subseteq R^n$, possibly non-open), is C^m if there exists an open neighborhood U of S in R^n and an extension of f to a definable C^m-function $U \to R$. From the theorem above, Smooth Cell Decomposition, and the definable version of the Tietze Extension Theorem (see, e.g., [1]) we immediately obtain:

Corollary 1.2. *Let $f\colon S \to R$ and $\epsilon\colon S \to R^{>0}$, where $S \subseteq R^n$, be definable continuous functions. Then for each $m \geq 1$ there exists a definable C^m-function $g\colon S \to R$ such that $|g(x) - f(x)| < \epsilon(x)$ for every $x \in S$.*

We prove Theorem 1.1 in Section 3, after some preliminary lemmas in Section 2. Our proof follows the strategy to tackle smoothing problems in o-minimal structures from [4, 5, 7]. In Section 4 we discuss the smoothing of uniformly continuous maps.

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Conventions and notations. Throughout this paper, d, k, l, m, n, and N will range over the set $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ of natural numbers. Let $S \subseteq R^n$. We denote by $\mathrm{cl}(S)$ the closure of S, by $\partial(S) = \mathrm{cl}(S) \setminus S$ the frontier of S, and by $\mathrm{int}(S)$ the interior of S. We denote the Euclidean norm on R^n by $\|\cdot\|$ and the associated metric by $(x, y) \mapsto d(x, y) := \|x - y\|$. For $r \in R^{>0}$ and $x \in R^n$ we let

$$B_r(x) := \big\{ y \in R^n : d(x, y) < r \big\}$$

be the open ball of radius r around x.

2. Some Lemmas

This section contains some lemmas needed for the proof of the theorem.

2.1. **A generalization of the Łojasiewicz Inequality.** In [3], L. van den Dries and C. Miller showed that many big theorems in real analytic geometry can be modified to definable versions in o-minimal structures; in particular, a definable version of the Łojasiewicz Inequality, which is a crucial tool in the proof of Theorem 1.1, can be formulated. Given a function $f\colon E \to R$, we write $Z(f) := \{x \in E : f(x) = 0\}$ for the zero set of f.

Theorem 2.1 (Generalized Łojasiewicz Inequality [3, Theorem C.14])**.** *Let E be a non-empty, definable, closed, and bounded subset of R^n, and $f, g\colon E \to R$ definable and continuous with $Z(f) \subseteq Z(g)$. There is a definable continuous strictly increasing bijection $\phi\colon R \to R$ such that $\phi(0) = 0$ and $|\phi(g(x))| \le |f(x)|$ for all $x \in E$.*

By focusing on functions whose domains are C^m-cells, a C^m-version of the Łojasiewicz Inequality follows:

Lemma 2.2. *Let $\Omega \subseteq R^n$ be a bounded open C^m-cell and $f\colon \mathrm{cl}(\Omega) \to R^{\ge 0}$ be definable and continuous such that $f(x) > 0$ for all $x \in \Omega$. Then there is a definable continuous function $g\colon \mathrm{cl}(\Omega) \to R$ such that $g \restriction \Omega$ is C^m and $0 < g(x) < f(x)$ for all $x \in \Omega$.*

Proof. For $x = (x_1, \dots, x_n) \in R^n$ and $i = 0, \dots, n$, let $\pi_i(x) = (x_1, \dots, x_i) \in R^i$. Let $f_i, g_i\colon \pi_i(\Omega) \to R$ $(i = 0, \dots, n-1)$ be the C^m-functions defining Ω. Thus $f_i(x_1, \dots, x_i) < g_i(x_1, \dots, x_i)$ for $(x_1, \dots, x_i) \in R^i$, $i = 0, \dots, n-1$, and

$$\pi_i(\Omega) = \big\{(x_1, \dots, x_i) \in \pi_{i-1}(\Omega) \times R : \\ f_{i-1}(x_1, \dots, x_{i-1}) < x_i < g_{i-1}(x_1, \dots, x_{i-1})\big\}$$

for $i = 1, \dots, n$. Define $\rho\colon \mathrm{cl}(\Omega) \to R$ by

$$\rho(x) := \begin{cases} \prod_{i=1}^{n} \big(x_i - f_{i-1}(\pi_{i-1}(x))\big) \cdot \big(g_{i-1}(\pi_{i-1}(x)) - x_i\big), & \text{if } x \in \Omega; \\ 0, & \text{otherwise,} \end{cases}$$

where $x = (x_1, \dots, x_n) \in \mathrm{cl}(\Omega)$. Clearly, ρ is C^m on Ω. Next, we will show that ρ is continuous on $\mathrm{cl}(\Omega)$. Let $\epsilon > 0$ and $x \in \partial\Omega$. Then there is $i \in \{1, \dots, n\}$ such that $\pi_{i-1}(x) \in \pi_{i-1}(\Omega)$ and either $x_i = f_{i-1}(\pi_{i-1}(x))$ or $x_i = g_{i-1}(\pi_{i-1}(x))$. Let first $i \in \{1, \dots, n\}$ be such that $\pi_{i-1}(x) \in \pi_{i-1}(\Omega)$ and $x_i = f_{i-1}(\pi_{i-1}(x))$. Let $\|x\|_\infty := \max\{|x_1|, \dots, |x_n|\}$ for $x = (x_1, \dots, x_n) \in R^n$, and set

$$M := \max\big\{1, \sup\{\|a - b\|_\infty : a, b \in \mathrm{cl}(\Omega)\}\big\}.$$

By continuity of f_{i-1}, take $\delta_0 > 0$ so small that

$$|f_{i-1}(\pi_{i-1}(y)) - f_{i-1}(\pi_{i-1}(x))| < \frac{\epsilon}{2M^{2n}}$$

for every $y \in B_{\delta_0}(x) \cap \Omega$. Set $\delta := \min\{\delta_0, \frac{\epsilon}{2M^{2n}}\}$. Suppose $y \in B_\delta(x) \cap \mathrm{cl}(\Omega)$; we have $|\rho(y) - \rho(x)| = |\rho(y)|$ since $x \in \partial\Omega$, and we claim that $|\rho(y)| < \epsilon$. To see this, we may assume $y \in \Omega$. Then

$$\begin{aligned}|\rho(y)| \leq & \left|\big(x_i - f_{i-1}(\pi_{i-1}(x))\big)\right| \cdot M^{2n} \\ \leq & M^{2n}\big(\,|y_i - x_i| + |x_i - f_{i-1}(\pi_{i-1}(x))| \\ & \quad + |f_{i-1}(\pi_{i-1}(x)) - f_{i-1}(\pi_{i-1}(y))|\,\big) \\ \leq & M^{2n}\left(\frac{\epsilon}{2M^{2n}} + 0 + \frac{\epsilon}{2M^{2n}}\right) \\ < & \epsilon.\end{aligned}$$

Similarly one shows that if $i \in \{1, \ldots, n\}$ such that $\pi_{i-1}(x) \in \pi_{i-1}(\Omega)$ and $x_i = g_{i-1}(\pi_{i-1}(x))$, then $|\rho(y)| < \epsilon$. Thus, ρ is continuous on $\mathrm{cl}(\Omega)$. Obviously, ρ vanishes on $\partial\Omega$. By Theorem 2.1, there is a definable continuous strictly monotone bijection $\phi\colon R \to R$ such that $\phi(0) = 0$ and $0 < \phi(\rho(x)) < f(x)$ for $x \in \Omega$. By the Smooth Monotonicity Theorem, pick $0 < \delta < 1$ such that $\phi \restriction (0, \delta)$ is C^m. Define $\psi\colon R \to R$ and $g\colon \mathrm{cl}(\Omega) \to R$ by

$$\psi(t) := \phi\left(\frac{\delta t^2}{1+t^2}\right), \quad g = \psi \circ \rho.$$

Then g is C^m. We have $t \geq \frac{\delta t^2}{1+t^2}$ for $t \in (0, +\infty)$. Since ϕ is an increasing function,

$$0 < g(x) = \psi(\rho(x)) = \phi\left(\frac{\delta(\rho(x))^2}{1+(\rho(x))^2}\right) \leq \phi(\rho(x)) < f(x)$$

for $x \in \Omega$. □

2.2. **Special cases of Theorem 1.1.** The rest of this section is devoted to proving some special cases of our main theorem, before we give the proof of the general case in the next section.

Lemma 2.3. *Let $\Omega \subseteq R^n$, where $n \geq 1$, be a bounded open C^m-cell and U be a definable open set with $\Omega \times \{0\}^l \subseteq U \subseteq \Omega \times R^l$. Let $F\colon U \to R$ be definable and continuous such that $F \restriction U \setminus (\Omega \times \{0\}^l)$ and $F \restriction \Omega \times \{0\}^l$ are C^m. Let $\epsilon\colon U \to R^{>0}$ be definable and continuous, and let $O \subseteq \Omega \times R^l$ be a definable open neighborhood of $\Omega \times \{0\}^l$. Then there is a definable C^m-function $G\colon U \to R$ such that*

1. *$|G(x) - F(x)| < \epsilon(x)$ for all $x \in U$;*
2. *$G = F$ outside O;*
3. *$G = F$ on $\Omega \times \{0\}^l$.*

Proof. Since $F \restriction \Omega \times \{0\}^l$ is C^m, there are an open subset V of O and a definable C^m-function $f\colon V \to R$ such that $\Omega \times \{0\}^l \subseteq V$ and $F \restriction \Omega \times \{0\}^l =$

$f \restriction \Omega \times \{0\}^l$. Shrinking V if necessary, we can assume that $|f(x) - F(x)| < \frac{\epsilon(x)}{2}$ for all $x \in V$. Set

$$\Delta(x) = \frac{1}{2} \cdot \min\big\{d(x, \partial\Omega), d((x,0), \partial V), 1\big\} \qquad \text{for all } x \in \Omega.$$

By Lemma 2.2, there is a definable C^m-function $g\colon \Omega \to R$ such that $0 < g(x) < \Delta(x)$ for every $x \in \Omega$. Let $\sigma\colon R \to R$ be a semialgebraic increasing C^m-function such that $\sigma(x) = 0$ if $x \leq 0$ and $0 < \sigma(x) \leq 1$ if $x > 0$. Define $\psi_1, \psi_2\colon \Omega \times R^l \to R$ by

$$\psi_1(x,y) := \prod_{i=1}^{l} \big[\sigma(y_i + g(x))\sigma(g(x) - y_i)\big],$$
$$\psi_2(x,y) := \prod_{i=1}^{l} \big[\sigma\left(-y_i - \tfrac{1}{2}g(x)\right) + \sigma\left(y_i - \tfrac{1}{2}g(x)\right)\big] \quad \text{for } x \in \Omega \text{ and } y \in R^l.$$

For each $s \in (0,1]$, let

$$W_s := \big\{(x,y) : x \in \Omega,\ |y_i| < s \cdot g(x) \text{ for } i = 1, \ldots, l\big\} \subseteq V,$$

so $W_s \subseteq W_t$ for $0 < s \leq t \leq 1$. Note that $\psi_1 = 0$ outside W_1, $\psi_2 = 0$ in $W_{\frac{1}{2}}$, and $\psi_1 + \psi_2$ is positive on V. See Figure 1 for a schematic picture.

Define $G\colon U \to R$ by

$$G(z) := \begin{cases} \dfrac{\psi_1(z)f(z) + \psi_2(z)F(z)}{\psi_1(z) + \psi_2(z)}, & \text{if } z \in V; \\ F(z), & \text{otherwise.} \end{cases}$$

We will show that G satisfies the desired conditions. Since $V \subseteq O$, we have $G = F$ outside O. Moreover, $G = f = F$ on $\Omega \times \{0\}^l$, as $\psi_2 = 0$ on $\Omega \times \{0\}^l$. To prove (1), let $z \in V$. Then

$$\begin{aligned} |G(z) - F(z)| &= \left|\frac{\psi_1(z)f(z) + \psi_2(z)F(z)}{\psi_1(z) + \psi_2(z)} - F(z)\right| \\ &= \left|\frac{\psi_1(z)}{\psi_1(z) + \psi_2(z)}\right| \cdot |f(z) - F(z)| \\ &\leq \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

Therefore, it remains to prove that G is C^m. Obviously, G is C^m on $V \setminus (\Omega \times \{0\}^l)$. Since F is C^m on $U \setminus (\Omega \times \{0\}^l)$ and f is C^m on V, it is enough to show the following:

(1) $G = F$ on $U \setminus \mathrm{cl}(W_1)$;
(2) $G = f$ on $W_{\frac{1}{2}}$.

To see (1), note that by the definition,

$$G(z) = \frac{\psi_1(z)f(z) + \psi_2(z)F(z)}{\psi_1(z) + \psi_2(z)} = \frac{\psi_2(z)F(z)}{\psi_2(z)} = F(z) \quad \text{when } z \in V \setminus W_1.$$

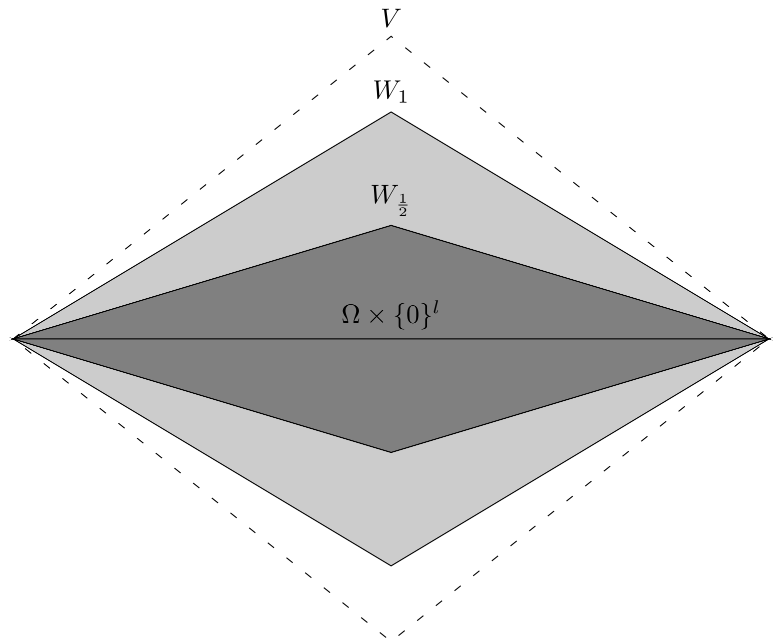

FIGURE 1. The sets $\Omega \times \{0\}^l$, $W_{\frac{1}{2}}$, W_1, and V.

Thus $G = F$ on $U \setminus W_1$. For (2), let $x \in \Omega$. Since $W_{\frac{1}{2}}$ is an open neighborhood of $\Omega \times \{0\}^l$ and $\psi_2 = 0$ on $W_{\frac{1}{2}}$, there exists an open neighborhood $V' \subseteq W_{\frac{1}{2}}$ of $(x, 0)$. On V', we have $G = f$. So, G is C^m on U. □

Lemma 2.4. *Let $U \subseteq R^n$ be open and $F\colon U \to R$ be definable continuous, and let $a \in U$ be such that $F\restriction(U \setminus \{a\})$ is C^m. Let $\epsilon\colon U \to R^{>0}$ be definable and continuous, and let O be a definable open neighborhood of a in U. Then there is a definable C^m-function $G\colon U \to R$ such that*

(1) *$|G(x) - F(x)| < \epsilon(x)$ for all $x \in U$; and*
(2) *$G = F$ outside O.*

Proof. We may assume that $a = 0 \in U$. Let $\epsilon_0 := \min\{\epsilon(x) : \|x\| \leq 1\}$. Let $V \subseteq O \cap B_1(0)$ be a definable open neighborhood of 0 such that $|F(x) - F(0)| < \frac{\epsilon_0}{2}$ for every $x \in V$. Take a positive $r \in R$ such that $(-2r, 2r)^n \subseteq V$. We may assume $r = 1$. Let $\sigma\colon R \to R$ be a semialgebraic increasing C^m-function such that $\sigma(x) = 0$ if $x \leq 0$ and $0 < \sigma(x) \leq 1$ if

$x > 0$. Define $\psi_1, \psi_2 \colon R^n \to R$ by

$$\psi_1(x) := \prod_{i=1}^{n} \big[\sigma(x_i - 1)\sigma(1 - x_i)\big],$$
$$\psi_2(x) := \prod_{i=1}^{n} \big[\sigma(-x_i - \tfrac{1}{2}) + \sigma(x_i - \tfrac{1}{2})\big]$$

For each $s \in (0, 1]$, let

$$W_s := \big\{(x \in R^n : |x_i| < s \text{ for } i = 1, \dots, l\big\} \subseteq V.$$

Similar to the proof of Lemma 2.3, we know that $\psi_1 = 0$ outside W_1, $\psi_2 = 0$ in $W_{\frac{1}{2}}$, and $\psi_1 + \psi_2$ is positive on V. Define $G \colon U \to R$ by

$$G(z) := \begin{cases} \dfrac{\psi_1(z)F(0) + \psi_2(z)F(z)}{\psi_1(z) + \psi_2(z)}, & \text{if } z \in V; \\ F(z), & \text{otherwise.} \end{cases}$$

Since $V \subseteq O$, $G = F$ outside O. To prove (1), let $z \in V$. Then

$$\begin{aligned} |G(z) - F(z)| &= \left| \frac{\psi_1(z)F(0) + \psi_2(z)F(z)}{\psi_1(z) + \psi_2(z)} - F(z) \right| \\ &= \left| \frac{\psi_1(z)}{\psi_1(z) + \psi_2(z)} \right| \cdot |F(0) - F(z)| \\ &\le \frac{\epsilon_0}{2} < \epsilon(z). \end{aligned}$$

Thus, it remains to prove that G is C^m. Fortunately, by the same argument as in Lemma 2.3, we can also show that G is C^m on $U \setminus \{0\}$ and on an open neighborhood of 0 contained in $W_{\frac{1}{2}}$. Hence, G is C^m on U. ☐

The previous lemma now allows us to show Theorem 1.1 for a finite set Z:

Corollary 2.5. *Let $U \subseteq R^n$ be open and $F \colon U \to R$ be definable continuous, and let Z be a finite subset of U such that $F \restriction (U \setminus Z)$ is C^m. Let $\epsilon \colon U \to R^{>0}$ be definable and continuous, and let O be a definable open neighborhood of Z in U. Then there is a definable C^m-function $G \colon U \to R$ such that*

1. *$|G(x) - F(x)| < \epsilon(x)$ for all $x \in U$; and*
2. *$G = F$ outside O.*

Proof. Let $Z = \{z_1, \dots, z_k\}$, where $k = |Z|$, and let i, j range over $\{1, \dots, k\}$. For each i let $U_i := U \setminus (Z \setminus \{z_i\})$, an open subset of R^n. For each i choose a definable open neighborhood O_i of z_i in O with $O_i \cap O_j = \emptyset$ for $i \neq j$, and further a definable open neighborhood O_i' of z_i in O_i with $\mathrm{cl}(O_i') \subseteq O_i$. By the previous lemma applied to $F \restriction U_i$, z_i, $\epsilon \restriction U_i$ and O_i' in place of F, a, ϵ and O, respectively, pick a definable C^m-function $G_i \colon U_i \to R$ such that $|G_i(x) - F_i(x)| < \epsilon(x)$ for all $x \in U_i$, and $G_i = F \restriction U_i$ outside O_i'. Now define

$G\colon U \to R$ by $G(x) := F(x)$ if $x \in U \setminus \bigcup_i O_i$ and $G(x) := G_i(x)$ if $x \in O_i$. One easily verifies that then G is C^m and satisfies (1) and (2). □

3. Proof of Theorem 1.1

Our main tool in the proof is the main theorem of [6], which we state next. We need some definitions:

Definition 3.1. Let $f = (f_1, \dots, f_n)\colon \Omega \to R^n$ be a C^m-map, where Ω is an open subset of R^d, $d \geq 1$. We say that f is **Λ^m-regular** if there is some $L \in R^{>0}$ such that

$$\|D^\alpha f(x)\| \leq \frac{L}{d(x, \partial\Omega)^{|\alpha|-1}} \qquad \text{for all } x \in \Omega \text{ and } \alpha \in \mathbb{N}^d \text{ with } 1 \leq |\alpha| \leq m.$$

Here, for $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ we set

$$D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}, \qquad |\alpha| := \alpha_1 + \cdots + \alpha_d,$$

and we let $D^\alpha f := (D^\alpha f_1, \dots, D^\alpha f_n)$ if $|\alpha| \leq m$ (so $D^0 f = f$).

We also define every map $R^0 \to R^n$ to be Λ^m-regular.

Notation. Let $\Omega \subseteq R^d$ be definable and open. Set

$$\Lambda^m(\Omega) := \{f\colon \Omega \to R : f \text{ is definable and } \Lambda^m\text{-regular}\},$$
$$\Lambda^m_\infty(\Omega) := \Lambda^m(\Omega, R) \cup \{-\infty, +\infty\}.$$

where $-\infty$ and $+\infty$ are considered as constant functions on Ω. For $f, g \in \Lambda^m_\infty(\Omega)$ we write $f < g$ if $f(x) < g(x)$ for all $x \in \Omega$.

Definition 3.2. Standard open Λ^m-regular cells in R^n are defined inductively on n as follows:

(1) $n = 0$: R^0 is a standard open Λ^m-regular cell in R^0;
(2) $n \geq 1$: a set of the form

$$(f, g) := \big\{(x, y) : x \in D,\ f(x) < y < g(x)\big\},$$

where $f, g \in \Lambda^m_\infty(D)$ such that $f < g$, and D is a standard open Λ^m-regular cell in R^{n-1}.

We say that a subset of R^n is a **standard Λ^m-regular cell in R^n** if it is either a standard open Λ^m-regular cell in R^n or one of the following:

(1) a singleton; or
(2) the graph of a definable Λ^m-regular map $D \to R^{n-d}$, where D is a standard open Λ^m-regular cell in R^d, and $1 \leq d < n$.

A subset $E \subseteq R^n$ is called a **Λ^m-regular cell in R^n** if there is a linear orthogonal transformation $\phi\colon R^n \to R^n$ such that $\phi(E)$ is a standard Λ^m-regular cell in R^n.

Definition 3.3. Let $R^{n\times n}$ be the space of all $n\times n$ matrices with entries from R, where $n\geq 1$, equipped with the operator norm $\|\cdot\|$. For each $d\leq n$, let

$$\mathbb{H}_{n,d}=\{A\in R^{n\times n}: A^{\mathrm{t}}=A,\ A^2=A,\ \operatorname{tr}(A)=d\}$$

be the subset of $R^{n\times n}$ consisting of the matrices (with respect to the standard basis of R^n) of orthogonal projections of R^n onto a subspace of R^n, having trace d. Note that $\mathbb{H}_{n,d}$ is an algebraic subset of $R^{n\times n}$ (where $R^{n\times n}$ is identified with R^{n^2} as usual) and hence definable in $\mathbf{R}$. Consider

$$\delta\colon \mathbb{H}_{n,d}\times\mathbb{H}_{n,d}\to R,\quad \delta(A,B)=\left\|B^{\perp}A\right\| \qquad \text{where } B^{\perp}=\mathrm{id}-B.$$

In [6] it is shown that δ is a metric on $\mathbb{H}_{n,d}$. For $A\in\mathbb{H}_{n,d}$ and $\epsilon>0$ let

$$B_\epsilon(A)=\left\{B\in\mathbb{H}_{n,d}:\delta(B,A)<\epsilon\right\}$$

be the open ball of radius ϵ centered at A in $\mathbb{H}_{n,d}$.

Definition 3.4. Let M a d-dimensional (embedded) C^1-submanifold of R^n. We view the tangent bundle $T(M)$ of M as a subbundle of $T(R^n)\cong R^n\times R^n$ in the natural way. Define $\tau_M\colon M\to\mathbb{H}_{n,d}$ by letting $\tau_M(x)$ be the matrix (w.r.t. the standard basis of R^n) of the orthogonal projection $R^n\to T_x(M)$. Let $A\in\mathbb{H}_{n,d}$ and $\epsilon>0$. We say that M is **ϵ-flat with respect to** A if $\tau_M(M)\subseteq B_\epsilon(A)$.

Note that every Λ^m-regular cell C of dimension d is a d-dimensional C^1-submanifold of R^n, hence the previous definition applies to C. A standard Λ^m-regular cell of dimension d is called **ϵ-flat** if it is ϵ-flat with respect to the projection of R^n onto the first d coordinates. In addition, we call a Λ^m-regular cell **ϵ-flat** if there is a linear orthogonal transformation $\phi\colon R^n\to R^n$ such that the image of this set under ϕ is an ϵ-flat standard Λ^m-regular cell.

In [11], the author proved a simpler version of a result in [6]:

Lemma 3.5. *Let $0<\epsilon<\frac{1}{32d^{\frac{3}{2}}}$ be rational and suppose Ω is an ϵ^d-flat standard Λ^1-regular cell in R^d. Let $f\colon\Omega\to R^n$ be a definable C^1-map. Suppose all derivatives of f are bounded by a rational $L\in R^{>0}$. Then f is Lipschitz.*

Definition 3.6. A **Λ^m-regular stratification of** R^n is a finite partition $\mathscr{D}$ of R^n into Λ^m-regular cells such that each ∂D ($D\in\mathscr{D}$) is a union of sets from $\mathscr{D}$. Given $\epsilon>0$ and definable $E_1,\dots,E_N\subseteq R^n$, such a Λ^m-regular stratification $\mathscr{D}$ of R^n is said to be **ϵ-flat** if each $D\in\mathscr{D}$ is an ϵ-flat Λ^m-regular cell, and **compatible with** $E_1,\dots,E_N$ if each E_i is a union of sets from $\mathscr{D}$.

Theorem 3.7 (Fischer, [6, Theorem 1.4])**.** *Let $E_1,\dots,E_N$ be definable subsets of R^n and $\epsilon>0$ be rational. There exists an ϵ-flat Λ^m-regular stratification of R^n which is compatible with $E_1,\dots,E_N$.*

We now use Lemma 3.5 and Theorem 3.7 to show Theorem 1.1. First, another lemma based on results from Section 2.

Lemma 3.8. *Suppose* $0 < \epsilon_0 < \frac{1}{32d^{\frac{3}{2}}}$ *be rational. Let* U *be a definable bounded open subset of* R^n *and* $Z_1, \dots, Z_N \subseteq U$, $N \geq 1$, *be disjoint* ${\epsilon_0}^d$*-flat* Λ^m*-regular cells. Suppose* $U_0 := U \setminus \bigcup_{i=1}^{N} Z_i$ *is open and*

$$\dim(Z_1) \leq \cdots \leq \dim(Z_N) = d < n, \quad d \geq 1.$$

Let $f\colon U \to R$ *be a definable continuous function such that* $f \restriction U_0$ *and* $f \restriction Z_i$ $(i = 1, \dots, N)$ *are* C^m. *Let* $\epsilon\colon U \to (0, +\infty)$ *be a definable continuous function, and let* V *be a definable open neighborhood of* $\bigcup_{i=1}^{N} Z_i$. *Then there is a definable continuous function* $g\colon U \to R$ *such that*

(1) $U_0 \cup Z_N$ *is open in* R^n*;*
(2) $g \restriction (U_0 \cup Z_N)$ *is* C^m*;*
(3) $|g(x) - f(x)| < \epsilon(x)$ *for every* $x \in U$*;*
(4) $g = f$ *outside* V.

Proof. Since Z_N is a Λ^m-regular cell of dimension d, after applying a suitable orthogonal transformation, we may assume that $Z_N = \Gamma(h)$ where Ω is an open ${\epsilon_0}^d$-flat Λ^m-regular cell and $h\colon \Omega \to R^{n-d}$ is a definable Lipschitz C^m-map. (Lemma 3.5.) Let

$$\Delta(x) := \min\big\{d((x,h(x)), Z_i) : i = 1, \dots, N-1\big\} \text{ for } x \in X;$$
$$U' := \big\{(x,y) \in \Omega \times R^{n-d} : \|y\| < \Delta(x)\big\}; \text{ and,}$$
$$O' := \big\{(x,y) \in V \cap (\Omega \times R^{n-d}) : \|y\| < \tfrac{1}{2} \cdot \min\{d(x, \partial\Omega), \Delta(x)\}\big\}.$$

Note that U' is a definable open neighborhood of Z_N with $U' \cap Z_i = \emptyset$ for $i = 1, \dots, N-1$. Hence, $U_0 \cup Z_N = U_0 \cup U'$ is open in R^n. For $E \subseteq \Omega \times R^{n-d}$, let

$$\widetilde{E} := \big\{(x,y) \in \Omega \times R^{n-d} : \big(x, y + h(x)\big) \in E\big\},$$

and for a function $\phi\colon E \to R$ define $\widetilde{\phi}\colon \widetilde{E} \to R$ by

$$\widetilde{\phi}(x,y) = \phi\big(x, y + h(x)\big) \qquad \text{for } (x,y) \in \widetilde{E}.$$

Clearly, $\widetilde{Z_N} = \Omega \times \{0\}^{n-d}$. By Lemma 2.3, there is a definable C^m function $G\colon \widetilde{U'} \to R$ such that

(1) $\Big|G(x) - \widetilde{f}(x)\Big| < \widetilde{\epsilon}(x)$ for all $x \in U'$;
(2) $G = \widetilde{f}$ outside $\widetilde{O'}$;
(3) $G = \widetilde{f}$ on $\widetilde{Z_N}$.

Thus, define $g\colon U \to R$ by

$$g(x,y) := \begin{cases} G(x, y - h(x)), & \text{if } (x,y) \in U'; \\ f(x,y), & \text{otherwise.} \end{cases}$$

By the choice of O, for every $z \in U \cap \partial U'$, there is a neighborhood V' of z such that $V' \cap O' = \emptyset$. Therefore, g satisfies the desired properties. □

Corollary 3.9. *Suppose $0 < \epsilon_0 < \frac{1}{32d^{\frac{3}{2}}}$ be rational. Let U be a definable bounded open subset of R^n and $Z_1, \dots, Z_N \subseteq U$, $N \geq 1$, be disjoint ${\epsilon_0}^d$-flat Λ^m-regular cells. Suppose $U_0 := U \setminus \bigcup_{i=1}^{N} Z_i$ is open and*

$$\dim(Z_1) \leq \dots \leq \dim(Z_N) = d < n.$$

Let $f\colon U \to R$ be a definable continuous function such that $f \restriction U_0$ and $f \restriction Z_i$ $(i = 1, \dots, N)$ are C^m. Let $\epsilon\colon U \to R^{>0}$ be a definable continuous function, and let V be a definable open neighborhood of $\bigcup_{i=1}^{N} Z_i$. Then there is a definable C^m-function $g\colon U \to R$ such that

1. *$|g(x) - f(x)| < \epsilon(x)$ for every $x \in U$;*
2. *$g = f$ outside V.*

Proof. We prove this by induction on N. For $N = 1$, Z_1 is either a singleton or an ${\epsilon_0}^d$-flat Λ^m-regular cell of positive dimension; since $U_0 \cup Z_1 = U$, this case immediately follows from Corollary 2.5 and Lemma 3.8. Assume $N > 1$. By Corollary 2.5 again, we may assume that $d \geq 1$. By Lemma 3.8, take $g_0\colon U \to R$ with

1. g_0 is C^m on $U_0 \cup Z_N$ and on $\bigcup_{i=1}^{N-1} Z_i$;
2. $|g_0(x) - f(x)| < \frac{\epsilon(x)}{2}$ for all $x \in U$;
3. $g_0 = f$ outside V.

By induction hypothesis, there is $g\colon U \to R$ such that

1. g is C^m on U;
2. $|g_0(x) - g(x)| < \frac{\epsilon(x)}{2}$ for $x \in U$.
3. $g = g_0$ outside V.

Therefore, for all $x \in U$ we have

$$|g(x) - f(x)| \leq |g(x) - g_0(x)| + |g_0(x) - f(x)| < \frac{\epsilon(x)}{2} + \frac{\epsilon(x)}{2} = \epsilon(x),$$

and $g = g_0 = f$ outside V; so we're done. □

Proof of Theorem 1.1. Let $U \subseteq R^n$ definable and open, let Z be a definable closed small subset of U, and let $f\colon U \to R$ be definable and continuous such that $f \restriction (U \setminus Z)$ is C^m, where $m \geq 1$. We need to show that for each definable continuous $\epsilon\colon U \to R^{>0}$ and each definable open neighborhood V of Z in U, there exists a definable C^m-function $g\colon U \to R$ such that

1. $|g(x) - f(x)| < \epsilon(x)$ for every $x \in U$;
2. $g = f$ outside V.

Let $\tau\colon R^n \to (-1,1)^n$ be given by

$$\tau(x) = \left(\frac{x_1}{\sqrt{1+{x_1}^2}}, \dots, \frac{x_n}{\sqrt{1+{x_n}^2}}\right) \qquad \text{for } x = (x_1, \dots, x_n) \in R^n.$$

After replacing U, Z, f by $\tau(U)$, $\tau(Z)$, $f \circ \tau^{-1}$, respectively, we may assume that U is bounded. By the Λ^m-Regular Stratification Theorem, we can partition Z into ${\epsilon_0}^d$-flat Λ^m-regular cells $Z_1, \dots, Z_N$, for some small rational ϵ_0, such that $f \restriction Z_i$ is C^m for $i = 1, \dots, N$ and $\dim(Z_i) \le \dim(Z_{i+1})$ for $i = 1, \dots, N-1$. By Corollary 3.9, this completes the proof. □

From Theorem 1.1, which deals with smoothing of definable *functions,* we immediately obtain a version for definable *maps*:

Corollary 3.10. *Let $U \subseteq R^n$ be open and $f\colon U \to R^k$ be a definable continuous map, and let Z be a definable closed small subset of U such that $f \restriction (U \backslash Z)$ is C^m. Let $\epsilon\colon U \to R^{>0}$ be a definable continuous function. Then, for any open neighborhood V of Z, there exists a definable continuous C^m-map $g\colon U \to R^k$ with*

1. *$\|g(x) - f(x)\| < \epsilon(x)$ for every $x \in U$;*
2. *$g = f$ outside V.*

Naturally, once the above theorem is known, one may ask whether it is possible to simultaneously approximate a definable family of continuous functions. An answer to this question can be obtained by redoing the above proof "uniformly in parameters," or more elegantly, by simply appealing to the Compactness Theorem of first-order logic, as in the proof of the following corollary:

Corollary 3.11. *Let $(f_a)_{a\in A}$, where $A \subseteq R^l$, be a definable family of continuous maps $f_a\colon U_a \to R^k$, and let $(\epsilon_a)_{a\in A}$ be a definable family of continuous functions $\epsilon_a\colon U_a \to R^{>0}$. There is a definable family $(g_a)_{a\in A}$ of C^m-maps $g_a\colon U_a \to R^k$ with*

$$\|g_a(x) - f_a(x)\| < \epsilon_a(x) \qquad \text{for every } x \in U_a.$$

Proof. Let $\mathscr{L}$ be the language of $\mathbf{R}$; we assume that $\mathscr{L}$ includes a name for each element of R, so that each set and map definable in $\mathbf{R}$ is definable by an $\mathscr{L}$-formula. Let x, y, z be tuples of pairwise distinct variables of length n, k and l, respectively, and let t be a variable distinct from each of the variables in x, y, z. Let $\phi_f(x,y,z)$ and $\phi_\varepsilon(x,t,z)$ be $\mathscr{L}$-formulas such that for each $a \in A$, $\phi_f(x,y,a)$ defines the graph of f_a and $\phi_\epsilon(x,t,a)$ defines the graph of ϵ_a. Let also $\alpha(z)$ be an $\mathscr{L}$-formula which defines A in $\mathbf{R}$.

For each $\mathscr{L}$-formula $\psi(x,y,z)$, let $\chi_\psi(z)$ be a formula such that, for each $a \in R^l$, $\chi_\psi(a)$ holds precisely when $a \in A$ and $\psi(x,y,a)$ defines a the graph of a C^m-map $g_a\colon U_a \to R^k$ such that $\|g_a - f_a\| < \epsilon$. Next, add l fresh constants

$c_1, \dots, c_l$ to $\mathscr{L}$ and call the resulting language $\mathscr{L}'$. For notational convenience, we write $c = (c_1, \dots, c_l)$. By Corollary 3.10,

$$\mathrm{Th}(\mathbf{R}) \cup \{\neg\chi_\psi(c) : \psi = \psi(x, y, z) \text{ is an } \mathscr{L}\text{-formula}\}$$

is inconsistent. Therefore, by the Compactness Theorem, there are formulas

$$\psi_1(x, y, z), \dots, \psi_N(x, y, z)$$

such that, for each $a \in A$, one of $\psi_i(x, y, a)$ defines the graph of a C^m-approximation of f_a. From the ψ_i one easily constructs a single formula $\psi(x, y, z)$ which works for every $a \in A$. □

4. Smoothing of Uniformly Continuous Maps

In [5], the constructions of approximation maps preserve the local Lipschitz property and the Lipschitz property, respectively. Therefore, it is natural to ask:

Is there an approximation method that preserves uniform continuity?

Below, we will give such a construction.

Corollary 4.1. *Let $U \subseteq R^n$ be open and $f \colon U \to R^k$ be a definable uniformly continuous map. Let Z be a definable closed small subset of U such that $f \restriction (U \backslash Z)$ is C^m. Let $\epsilon \colon U \to R^{>0}$ be a definable continuous function and V be an open neighborhood of Z. Then there exists a definable uniformly continuous C^m-map $g \colon U \to R^k$ with*

(1) $\|g(x) - f(x)\| < \epsilon(x)$ *for every* $x \in U$*;*
(2) $g = f$ *outside* V*.*

In the proof, for $x, y \in R^n$, we write

$$[x, y] := \{ty + (1 - t)x \in R^n : t \in [0, 1]\}$$

for the line segment connecting x and y.

Proof. First, define $\epsilon_0 \colon U \to R$ by

$$\epsilon_0(x) := \min\left\{1, d(x, \partial U), \tfrac{1}{\|x\|}, \epsilon(x)\right\}$$

with the conventions that $d(x, \partial U) = +\infty$ if $\partial U = \emptyset$, and $\frac{1}{\|x\|} = +\infty$ if $x = 0$. By Corollary 3.10, we can find a definable C^m-map $g \colon U \to R^k$ such that

(1) $\|g(x) - f(x)\| < \epsilon_0(x)$ for all $x \in U$;
(2) $g = f$ outside V.

To prove that g is uniformly continuous, let $\bar{\epsilon} > 0$ be given. Set

$$K_{\bar{\epsilon}} := \left\{x \in U : d(x, \partial U) \geq \tfrac{\bar{\epsilon}}{6},\ \|x\| \leq \tfrac{6}{\bar{\epsilon}}\right\}.$$

Clearly, $\epsilon_0(x) \leq \frac{\bar{\epsilon}}{6}$ for each $x \in \mathrm{cl}(U \setminus K_{\bar{\epsilon}})$ and $K_{\bar{\epsilon}}$ is definable, closed, and bounded. Thus $g \restriction K_{\bar{\epsilon}}$ is uniformly continuous. Pick $\delta_1 > 0$ such that, for every $x, y \in K_{\bar{\epsilon}}$, if $\|x - y\| < \delta_1$, then $\|g(x) - g(y)\| < \frac{\bar{\epsilon}}{2}$. Since f is uniformly continuous on U, there exists $\delta_2 > 0$ such that, for $x, y \in U$, if $\|x - y\| < \delta_2$, then $\|f(x) - f(y)\| < \frac{\bar{\epsilon}}{6}$. Therefore, for every $x, y \in \mathrm{cl}(U \setminus K_{\bar{\epsilon}})$,

$$\begin{aligned}\|g(x) - g(y)\| &\leq \|g(x) - f(x)\| + \|f(x) - f(y)\| + \|f(y) - g(y)\| \\ &< \epsilon_0(x) + \frac{\bar{\epsilon}}{6} + \epsilon_0(x) \leq \frac{\bar{\epsilon}}{6} + \frac{\bar{\epsilon}}{6} + \frac{\bar{\epsilon}}{6} = \frac{\bar{\epsilon}}{2} < \bar{\epsilon}.\end{aligned}$$

Let $\delta := \min\{\delta_1, \delta_2, \frac{\bar{\epsilon}}{6}\}$. From the above discussion, it is sufficient to show that for $x \in \mathrm{cl}(U \setminus K_{\bar{\epsilon}})$ and $y \in K_{\bar{\epsilon}}$ with $\|x - y\| < \delta$, we have $\|g(x) - g(y)\| < \bar{\epsilon}$. Let such x, y be given. Since $\delta \leq \frac{\bar{\epsilon}}{6}$ and $d(y, \partial U) \geq \frac{\bar{\epsilon}}{6}$, we have $B_\delta(y) \subseteq U$. Hence, $[x, y] \subseteq U$. Therefore, there exists $z \in [x, y]$ such that $z \in \mathrm{cl}(U \setminus K_{\bar{\epsilon}}) \cap K_{\bar{\epsilon}}$. Thus,

$$\|g(x) - g(y)\| \leq \|g(x) - g(z)\| + \|g(z) - g(y)\| < \frac{\bar{\epsilon}}{2} + \frac{\bar{\epsilon}}{2} = \bar{\epsilon}.$$

So, g is uniformly continuous. □

Next, we use the same trick as in Corollary 3.11 to prove the following.

Corollary 4.2. *Let $(f_a)_{a \in A}$, where $A \subseteq R^l$, be a definable family of uniformly continuous maps $f_a \colon U_a \to R^k$, where $U_a \subseteq R^n$ is open, and let $(\epsilon_a)_{a \in A}$ be a definable family of continuous functions $\epsilon_a \colon U_a \to R^{>0}$. Then there is a definable family $(g_a)_{a \in A}$ of uniformly continuous C^m-maps $g_a \colon U_a \to R^k$ such that*

$$\|g_a(x) - f_a(x)\| < \epsilon_a(x) \qquad \textit{for every } x \in U_a.$$

The following theorem is shown in [1]:

Theorem 4.3. *Every definable bounded uniformly continuous function $E \to R$, where $E \subseteq R^n$, extends to a definable uniformly continuous function $R^n \to R$.*

Therefore, by a combination of Corollary 4.1 and Theorem 4.3 we obtain:

Corollary 4.4. *Let $U \subseteq R^n$ be open and $f \colon U \to R^k$ be a definable bounded uniformly continuous map. Let Z be a definable closed small subset of U such that $f \restriction (U \setminus Z)$ is C^m. Let $\epsilon \colon U \to R^{>0}$ be a definable continuous function, and let V be an open neighborhood of $Z \cup \partial U$. There exists a definable uniformly continuous C^m-map $g \colon R^n \to R^k$ with*

(1) *$\|g(x) - f(x)\| < \epsilon(x)$ for every $x \in U$;*
(2) *$g = f$ outside V.*

References

[1] M. Aschenbrenner, A. Fischer, *Definable versions of theorems by Kirszbraun and Helly*, Proc. Lond. Math. Soc. (3) **102** (2011), no. 3, 468–502.
[2] L. van den Dries, *Tame Topology and O-Minimal Structures*, London Mathematical Society Lecture Note Series, vol. 248, Cambridge University Press, Cambridge, 1998.
[3] L. van den Dries, C. Miller, *Geometric categories and o-minimal structuress*, Duke Math. J. **84** (1996), no. 2, 497–540.

[4] J. Escribano, *Approximation theorems in o-minimal structures*, Illinois J. Math. **46** (2002), no. 1, 111–128.
[5] A. Fischer, *Definable smoothing of Lipschitz continuous functions*, Illinois J. Math. **52** (2008), no. 2, 583–590.
[6] ———, *o-minimal Λ^m-regular stratification*, Ann. Pure Appl. Logic **147** (2007), no. 1-2, 101–112.
[7] ———, *Smooth approximation of definable continuous functions*, Proc. Amer. Math. Soc. **136** (2008), no. 7, 2583–2587.
[8] J. Kurzweil, *On approximation in real Banach spaces*, Studia Math. **14** (1954), 214–231.
[9] J. Margalef Roig, E. Outerelo Domínguez, *Differential Topology*, North-Holland Mathematics Studies, vol. 173, North-Holland Publishing Co., Amsterdam, 1992.
[10] H. Shapiro, *Smoothing and Approximation of Functions*, Van Nostrand Reinhold Mathematical Studies, vol. 24, Van Nostrand Reinhold Co., New York-Toronto, Ont.-London, 1969.
[11] A. Thamrongthanyalak. *Whitney's Extension Theorem in o-minimal structures*, in preparation.

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