Abstract. We show that the natural embedding of the differential field of
transseries into Conway’s field of surreal numbers with the Berarducci-Mantova
derivation is an elementary embedding. We also prove that any Hardy field
embeds into the field of surreals with the Berarducci-Mantova derivation.

Introduction

Berarducci and Mantova [3, Theorem B] have recently constructed a derivation
(denoted by $\partial_{BM}$ below) on Conway’s ordered field $\mathbb{N}_0$ of surreal numbers that
makes the latter a Liouville closed $H$-field with constant field $\mathbb{R}$. The standard
example of such an object is the ordered differential field $T$ of transseries, and the
question arises whether $\mathbb{N}_0$ with $\partial_{BM}$ is elementarily equivalent to $T$. Below we
give a positive answer in a stronger form: Theorem 1. Throughout this paper we
consider $\mathbb{N}_0$ as a differential field with derivation $\partial_{BM}$.

Both $\mathbb{N}_0$ and $T$ are also exponential fields; the exponential function $\exp$ on $\mathbb{N}_0$
defined in Gonshor [9]. We refer to [2, Appendix A] for the precise construction
of $T$, but the “generating element” $x$ of $T$ there will be denoted by $x_T$ here, since
we prefer to have $x$ range here over arbitrary surreal numbers. It is folklore (but
see Section 5 for a proof) that there is a unique embedding $\iota: T \to \mathbb{N}_0$ of ordered
exponential fields with $\iota(x_T) = \omega$ that is the identity on $\mathbb{R}$ and respects infinite
sums. It follows easily from Wilkie’s theorem [13] and other known facts that $\iota$
is an elementary embedding of ordered exponential fields; see Section 5 for details.
The analogue for the derivation instead of the exponentiation requires more effort:

Theorem 1. The mapping $\iota: T \to \mathbb{N}_0$ is an elementary embedding of ordered
differential fields.

This answers a question posed in [3]. The main tools for proving this result come
from [2, Theorems 15.0.1 and 16.0.1]. These tools enable us to reduce the proof
of Theorem 1 to exhibiting $\mathbb{N}_0$ as a directed union of subfields $\mathbb{R}[\![\omega^\Gamma]\!]$ that are
closed under $\partial_{BM}$ and where $\Gamma$ is an ordered additive subgroup of $\mathbb{N}_0$ having a
smallest nontrivial archimedean class; exhibiting $\mathbb{N}_0$ as such a directed union makes
up an important part of our paper. (As a byproduct we get a new proof that
$\partial_{BM}(\mathbb{N}_0) = \mathbb{N}_0$.) We use the same kind of reduction to obtain:

Theorem 2. The surreals of countable length form a subfield of $\mathbb{N}_0$ closed un-
der $\partial_{BM}$. As a differential subfield of $\mathbb{N}_0$ it is an elementary submodel of $\mathbb{N}_0$.

This also uses a result of Esterle [8] and its consequence that for any countable
ordinal $\alpha$, any well-ordered set of surreals of length $< \alpha$ is countable: Lemma 4.3.

Finally, we establish an embedding result for $H$-fields:
Theorem 3. Every $H$-field with small derivation and constant field $\mathbb{R}$ can be embedded over $\mathbb{R}$ as an ordered differential field into $\mathbb{No}$.

Thus every Hardy field extending $\mathbb{R}$ embeds over $\mathbb{R}$ as an ordered differential field into $\mathbb{No}$. Despite these excellent properties of $\partial_{BM}$, Schmeling’s thesis [12] gives us reason to believe that $\partial_{BM}$ is not yet the “best” derivation on $\mathbb{No}$. We expect to address this issue in later papers.

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1. Preliminaries

Here we fix notation and terminology and summarize the results from [2, 3, 9] that we need as background material and as tools in our proofs.

Notations and terminology. Below, $m, n$ range over $\mathbb{N} = \{0, 1, 2, \ldots\}$, and $\alpha, \beta$ and $\mu, \nu$ range over ordinals. (The letter $\lambda$ will serve another purpose, as in [3].)

As in [9], a surreal number is by definition a function $a : \mu \to \{-, +\}$ on an ordinal $\mu = \{\alpha : \alpha < \mu\}$. For such $a$ we let $l(a) := \mu$ be the length of $a$. From now on we let $a, b, x, y$ be surreal numbers. The class $\mathbb{No}$ of surreal numbers carries a canonical linear ordering $< : a < b$ iff $a$ is lexicographically less than $b$, where by convention we set $a(\mu) := 0$ for $\mu \geq l(a)$ and linearly order $\{-, 0, +\}$ by $- < 0 < +$. We also have the canonical partial ordering $<_{s}$ on $\mathbb{No}$ given by: $a <_{s} b$ (“$a$ is simpler than $b$”) iff $a$ is a proper initial segment of $b$, that is, $l(a) < l(b)$, and $a|_{l(b)} = b|_{l(b)}$ for $\mu := l(a)$. For sets $A, B \subseteq \mathbb{No}$ with $A < B$ (that is, $a < b$ for all $a \in A$ and $b \in B$) we let $x = A|B$ mean that $x$ is the simplest surreal with $A < x < B$, as in [9] and [3]. We also use the terms “canonical representation” and “monomial representation” (of a surreal number) as in [3].

The ordinal $\alpha$ is identified with the surreal $a : \alpha \to \{-, +\}$ with $a(\beta) = +$ for all $\beta < \alpha$. A useful fact is the equivalence $\alpha < x \iff \alpha + 1 <_{s} x$, where $\alpha + 1$ is the successor ordinal to $\alpha$. The subclass of $\mathbb{No}$ consisting of the ordinals is denoted by $\mathbb{On}$. A set $S \subseteq \mathbb{No}$ is said to be initial if $x \in S$ whenever $x <_{s} y \in S$. As in [5] we set $\mathbb{No}(\alpha) = \{x : l(x) < \alpha\}$, an initial subset of $\mathbb{No}$.

We refer to [9] or [3] for the inductive definitions of the binary operations of addition and multiplication on $\mathbb{No}$ that make $\mathbb{No}$ into a real closed field, with the ordinal 0 as its zero element and the ordinal 1 as its multiplicative identity. The field ordering of this real closed field is the above lexicographic linear ordering $<$. This field $\mathbb{No}$ contains $\mathbb{R}$ as an initial subfield in the way specified in [9]. The field sum $\alpha + n$ equals the ordinal sum $\alpha + n$. Each initial set $\mathbb{No}(\omega^{n})$ underlies an additive subgroup of $\mathbb{No}$; see [5].

Let $\Gamma$ be an (additively written) ordered abelian group. Then we set

$$\Gamma^{>} := \{\gamma \in \Gamma : \gamma > 0\}.$$ 

We use this notation also for the underlying additive groups of $\mathbb{No}$ and $\mathbb{R}$, so $\mathbb{No}^{>} = \{a : a > 0\}$, and $\mathbb{R}^{>} := \{r \in \mathbb{R} : r > 0\}$. For $\gamma \in \Gamma$ we define

$$[\gamma] := \{\delta \in \Gamma : |\delta| \leq n|\gamma| \text{ and } |\gamma| \leq n|\delta| \text{ for some } n \geq 1\},$$
the archimedean class of $\gamma$ (in $\Gamma$). The archimedean classes of elements of $\Gamma$ partition the set $\Gamma$, and we totally order this set of archimedean classes by

$$|\gamma_1| < |\gamma_2| \iff n|\gamma_1| < |\gamma_2|$$

for all $n \geq 1$.

Thus the least archimedean class is $[0] = \{0\}$, the trivial archimedean class.

The convex hull of $\mathbb{R}$ in $\mathbb{No}$ is a valuation ring $V$ of the field $\mathbb{No}$. We consider $\mathbb{No}$ accordingly as a valued field whose (Krull) valuation $v$ has $V$ as its valuation ring. For any (Krull) valued field $K$ with valuation $v$ and elements $f, g \in K$ we let $f \preceq g$, $f \prec g$, $f \asymp g$ abbreviate $v(f) \geq v(g)$, $v(f) > v(g)$, $v(f) = v(g)$, and $v(f - g) > v(f)$. (See [2, Section 3.1].) We shall use these notations in particular for the valued field $\mathbb{No}$.

The omega map, the Conway normal form, and summability. We assume familiarity with Conway’s omega map $x \mapsto \omega^x : \mathbb{No} \to \mathbb{No}^\omega$. Recall that $\omega^x$ is the simplest positive element in its archimedean class; so $\omega^x \preceq \omega^y$ whenever $x < y$. See [9] for details, including the proof that each $\omega$ has a unique representation

$$a = \sum x a_x \omega^x \quad \text{(the Conway normal form of } a)$$

with real coefficients $a_x$ such that $E(a) := \{x : a_x \neq 0\}$ is a subset of $\mathbb{No}$ (not just a subclass) and is reverse well-ordered. This will be the meaning of $E(a)$ and $a_x$ throughout. The leading monomial of $a$ is $\omega^x$ with $x = \max E(a)$, for $a \neq 0$. The terms of $a$ are the $a_x \omega^x$ with $a_x \neq 0$. The omega map extends the usual ordinal exponentiation $\alpha \mapsto \omega^\alpha$. Given any set $S \subset \mathbb{No}$ we let $\mathbb{R}[[\omega^S]]$ denote the additive subgroup of $\mathbb{No}$ consisting of the surreals $a$ with $E(a) \subseteq S$.

Let $(a_i)_{i \in I}$ be a family of surreals; this includes $I$ being a set. We say that $(a_i)$ is summable (or that $\sum_i a_i$, exists) if $\bigcup_i E(a_i)$ is reverse well-ordered, and for each $x$ there are only finitely many $i \in I$ with $x \in E(a_i)$; in that case we set $\sum_i a_i := \sum_x (\sum_i a_{i,x}) \omega^x$. If $S$ is a subset of $\mathbb{No}$, then for any summable family $(a_i)$ in $\mathbb{R}[[\omega^S]]$ we have $\sum_i a_i \in \mathbb{R}[[\omega^S]]$.

As in [3], we let $\mathfrak{M}$ denote the class of monomials $\omega^x$; so $\mathfrak{M}$ is a multiplicative subgroup of $\mathbb{No}^\omega$. The Conway normal form allows us to consider any surreal number $a$ as a generalized series

$$a = \sum m \in \mathfrak{M} a_m m$$

with coefficients $a_m \in \mathbb{R}$, monomials $m \in \mathfrak{M}$, and reverse well-ordered support

$$\text{supp } a := \{m \in \mathfrak{M} : a_m \neq 0\} = \omega^{E(a)}.$$ 

This makes the above notion of summability for surreal numbers coincide with the corresponding notion for generalized series from [12, Section 1.5].

Next, $\mathbb{J} := \{a : E(a) \subseteq \mathbb{No}^\omega\}$ is the class of purely infinite surreals, an additive subgroup of $\mathbb{No}$ that is moreover closed under multiplication. Thus $\mathfrak{M} \cap \mathbb{J} = \mathfrak{M}^\omega$, and $\mathbb{No} = \mathbb{J} \oplus \mathbb{R} \oplus \mathbb{No}^{-1}$.

Exponentiation, and the functions $g$ and $h$. Gonshor [9] gave an inductive definition of the exponential function $\exp : \mathbb{No} \to \mathbb{No}^\omega$, and established its basic properties. These include $\exp$ being an order-preserving isomorphism from the additive group of $\mathbb{No}$ onto its multiplicative group of positive elements. The inverse of $\exp$ is of course denoted by $\log : \mathbb{No}^\omega \to \mathbb{No}$. The $n$th iterate of the map $\exp : \mathbb{No} \to \mathbb{No}$ is denoted by $\exp^n$, so $\exp^0$ is the identity map on $\mathbb{No}$, and
\[ \exp_1(x) = \exp(x). \] Also \( e^x := \exp(x) \). The logarithmic map \( \log : \mathbb{N}^\times \to \mathbb{N}^\times \) maps \( \mathbb{N}^\times \) into itself; the \( n \)th iterate of the restriction of \( \log \) to a map \( \mathbb{N}^\times \to \mathbb{N}^\times \) is denoted by \( \log_n \), so \( \log_0 \) is the identity map on \( \mathbb{N}^\times \) and \( \log_1(x) = \log(x) \) for \( x \in \mathbb{N}^\times \).

The exponential map \( \exp \) and the omega-map \( x \mapsto \omega^x \) are related by the order preserving bijection \( g : \mathbb{N}^\times \to \mathbb{N} \), which satisfies
\[ \exp(\omega^x) = \omega^{g(x)} \quad \text{for all } x > 0. \]

We have \( g(n) = n \) for all \( n \). More generally, Theorem 10.14 in [9] says that \( g(\alpha) = \alpha \) unless \( \varepsilon \leq \alpha < \varepsilon + \omega \) for some \( \varepsilon \)-number, in which case \( g(\alpha) = \alpha + 1 \). (An \( \varepsilon \)-number is an ordinal \( \varepsilon \) such that \( \omega^\varepsilon = \varepsilon \).) We shall need \( g(x) \) mainly in the other extreme case where \( x \) has the form \( \omega^{-\alpha} \). Here Theorem 10.15 in [9] gives \( g(\omega^{-\alpha}) = -\alpha + 1 \).

We also use the inverse \( h : \mathbb{N} \to \mathbb{N}^\times \) of \( g \). Note that
\[ \omega^{\omega^y} = \exp(\omega^y) \quad \text{for all } y. \]

The result above for \( g(\omega^{-\alpha}) \) yields \( h(-\alpha + 1) = \omega^{-\alpha} \), from which we get
\[ \log(\omega^{-\alpha + 1}) = \omega^{-\alpha}. \]

Applying this to the ordinal \( \alpha + 1 \) instead of \( \alpha \) we get
\[ \log(\omega^{-\alpha}) = \omega^{-\alpha - 1}. \]

From [9] we have \( \exp(J) = \mathfrak{M} \). Thus besides the Conway normal form and the series representation, any surreal number \( a \) also has a unique representation
\[ a = \sum_{j \in J} a_j e^j \quad \text{(exponential normal form of } a) \]
with real coefficients \( a_j \) and reverse well-ordered \( \{ j \in J : a_j \neq 0 \} \); this is also called the Ressayre form of \( a \). For nonzero \( a \) with leading monomial \( e^b \), \( b \in J \), we set \( \ell(a) := b \). Then \( -\ell : \mathbb{N}^\times \to \mathbb{J} \) is a (Krull) valuation on the field \( \mathbb{N} \), and
\[ \{ a : -\ell(a) \geq 0 \} = \{ a : |a| \leq r \text{ for some } r \in \mathbb{R}^{>0} \} = V, \]
so we may consider \( -\ell \) as the valuation of our valued field \( \mathbb{N} \). Important in [3] is also the class \( \mathcal{A} \) of log-atomic surreals, consisting of the \( a > \mathbb{N} \) all whose iterated logarithms \( \log_n a \) lie in \( \mathfrak{M} \). We have \( \mathcal{A} \subseteq \mathfrak{M}^{-1} \) and \( \exp(\mathcal{A}) = log(\mathcal{A}) = \mathcal{A} \). It follows from \( \mathcal{A} \subseteq \mathfrak{M} \) that if \( x, y \in \mathcal{A} \) and \( x < y \), then \( x < y \). (In [3] the class of log-atomic surreals is denoted by \( \mathbb{L} \), but this notation conflicts with ours in other papers.)

**Surreal derivations.** We summarize here some results from [3] as needed, and add a few remarks. A *surreal derivation* is a derivation \( \partial \) on the field \( \mathbb{N} \) such that
\begin{align*}
(SD1) \quad \{ a : \partial(a) = 0 \} & = \mathbb{R}; \\
(SD2) \quad \partial(a) > 0 \quad \text{for all } a > \mathbb{R}; \\
(SD3) \quad \partial(\exp(a)) = \partial(a) \exp(a) \quad \text{for all } a; \\
(SD4) \quad \text{for any summable family } (a_i) \text{ of surreals, the family } (\partial(a_i)) \text{ is also summable, and } \partial \left( \sum_i a_i \right) = \sum_i \partial(a_i). \\
\end{align*}

The ordered field \( \mathbb{N} \) equipped with any surreal derivation is an \( H \)-field; this doesn’t need \( (SD3) \) or \( (SD4) \). The particular derivation \( \partial_{BM} \) is surreal, maps \( \mathcal{A} \) into \( \mathfrak{M} \), and is obtained in [3] as a special case of a rather general construction. Before we get to that, we mention Proposition 6.5 and Theorem 6.32 from that paper:
(BM1) If \( \partial \) is a surreal derivation, then for all \( x, y > N \) with \( x - y > N \) we have
\[
\log \partial(x) - \log \partial(y) < x - y.
\]

(BM2) Any map \( D: \mathbb{A} \to \mathbb{R}^>\mathbb{M} \) such that for all \( x, y \in \mathbb{A} \),
\[
D(\exp x) = D(x) \exp x, \quad \log D(x) - \log D(y) < \max(x, y),
\]
extends to a surreal derivation.

Thus (BM2) is a partial converse to (BM1), although the condition in (BM2) that \( D \) takes only values in \( \mathbb{R}^>\mathbb{M} \) seems a rather severe restriction. We define a pre-derivation to be a map \( D: \mathbb{A} \to \mathbb{R}^>\mathbb{M} \) as in (BM2). Note that if \( D \) is a pre-derivation, then
\[
D(a) = \left( \prod_{m<n} \log_m a \right) \cdot D(\log_n a) \quad \text{for all } a \in \mathbb{A} \text{ and all } n. \quad (\ast)
\]

A pre-derivation \( D \) actually extends canonically to a surreal derivation \( \partial_D \). To define \( \partial_D \) in terms of \( D \) we rely on the notion of path derivatives, introduced in [10], further developed in [12], and adapted to the surreal setting in [3]. A path is a function \( P: \mathbb{N} \to \mathbb{R}^x\mathbb{M} \) such that \( P(n+1) \) is a term of \( \ell(P(n)) \), for all \( n \). Given \( x \), the paths \( P \) such that \( P(0) \) is a term of \( x \) are the elements of a set \( \mathcal{P}(x) \). For \( x \in \mathbb{A} \) there is a unique path \( P \in \mathcal{P}(x) \); it is given by \( P(n) = \log_n x \). Thus if \( P \) is a path and \( P(m) \in \mathbb{A} \), then \( P(n) = \log_{n-m} P(m) \) for all \( n \geq m \), so \( P(n) \in \mathbb{A} \) for all \( n \geq m \).

Let \( D \) be a pre-derivation. The path derivative \( \partial_D(P) \in \mathbb{R}^\mathbb{M} \) for a path \( P \) is defined as follows, with \( (\ast) \) guaranteeing independence of \( n \) in (1):
1. if \( P(n) \in \mathbb{A} \), then \( \partial_D(P) := \left( \prod_{m<n} P(m) \right) \cdot D(P(n)) \);
2. if \( P(n) \notin \mathbb{A} \) for all \( n \), then \( \partial_D(P) := 0 \).

The rationale behind path derivatives is the following proposition:

(BM3) For each \( a \) the family \( (\partial_D(P))_{P \in \mathcal{P}(a)} \) is summable.

This result is stated in [3, Proposition 6.20] only for one particular pre-derivation, but, as the authors mention, the proof extends to any pre-derivation. In view of (BM3) we can now define \( \partial_D: \mathbb{N}^\mathbb{O} \to \mathbb{N}^\mathbb{O} \) by
\[
\partial_D(a) := \sum_{P \in \mathcal{P}(a)} \partial_D(P).
\]

It follows from \( (\ast) \) that \( \partial_D \) extends \( D \), and the arguments in [3, Section 6] show that \( \partial_D \) is a surreal derivation.

Results from [2]. To state the relevant facts, we recall from [1] or [2] that an \( H \)-field is by definition an ordered differential field \( K \) with derivation \( \partial \) and constant field \( C = \{ f \in K : \partial(f) = 0 \} \) such that:

(H1) \( \partial(f) > 0 \) for all \( f \in K \) with \( f > C \);
(H2) \( \mathcal{O} = C + \sigma \), where \( \mathcal{O} \) is the convex hull of \( C \) in \( K \), and \( \sigma \) is the maximal ideal of the valuation ring \( \mathcal{O} \).

Let \( K \) be an \( H \)-field, and let \( \mathcal{O} \) and \( \sigma \) be as in (H2). Thus \( K \) is a valued field with valuation ring \( \mathcal{O} \). We consider \( K \) in the natural way as an \( \mathcal{L} \)-structure, where
\[
\mathcal{L} := \{ 0, 1, +, - , \times, \partial, \leq, \prec \}.
\]
Proof. A smallest nontrivial archimedean class.

Lemma 1.1. Assume \( K \) has constant field \( C = \mathbb{R} \). Then \( K \) is grounded iff \( \Gamma \) has a smallest nontrivial archimedean class.

Proof. Let \( f, g \in K \), \( f, g > C \). Suppose the archimedean class \( [v(f)] = [v(1/f)] \) of \( v(f) \) is greater than \([v(g)]\). This means: \( v(f) < nv(g) = v(g^n) < 0 \) for all \( n \geq 1 \). Hence \( f^\dagger > (g^n)^\dagger = ng^\dagger > 0 \) for all \( n \geq 1 \), by [1, Lemma 1.4], so \( v(f^\dagger) < v(g^\dagger) \). A similar argument (which doesn’t need \( C = \mathbb{R} \)) shows that if \([v(f)] = [v(g)]\), then \( v(f^\dagger) = v(g^\dagger) \). Thus we have an order-reversing bijection \([v(f)] \mapsto v(f^\dagger) \) (\( f \in K \), \( f > C \)) from the set of nontrivial archimedean classes of \( \Gamma \) onto \( \Psi \).

An \( H \)-subfield of \( K \) is by definition an ordered differential subfield of \( K \) that is an \( H \)-field. In [2] we axiomatized the elementary (= first-order) theory of the \( H \)-field \( \mathcal{T} \) of transseries. This (complete) theory is called \( T_{\text{small}}^{\text{ul}} \) there and its models are exactly the \( H \)-fields \( K \) satisfying the following (first-order) conditions:

1. The derivation of \( K \) is small, that is, \( \partial \sigma \subseteq \sigma \);
2. \( K \) is Liouville closed;
3. \( K \) is \( \omega \)-free;
4. \( K \) is newtonian.

(An \( H \)-field \( K \) is said to be Liouville closed if it is real closed and for all \( f \in K \) there exists \( g \in K \) with \( g' = f \) and an \( h \in K^\times \) such that \( h^\dagger = f \); for the definition of “\( \omega \)-free” and “newtonian” we refer to the Introduction of [2].) Dropping the smallness axiom (1), we get the incomplete but model complete theory \( T_{\text{ul}}^{\text{ul}} \); see [2, Chapter 16]. The \( H \)-field \( \mathcal{T} \) satisfies (3) and (4) by [2, Corollary 11.7.15 and Theorem 15.0.1], which for an arbitrary \( H \)-field \( K \) amount to the following:

If \( \partial K = K \) and \( K \) is a directed union of spherically complete grounded \( H \)-subfields, then \( K \) is \( \omega \)-free and newtonian.

The condition \( \partial K = K \) is automatically satisfied if \( K \) is a directed union of spherically complete grounded \( H \)-subfields \( E \) such that for some \( \phi \in E \) we have \( v(\phi) = \max \Psi_E \) and \( \phi \in \partial K \), by [2, Corollary 15.2.4].

2. Infinite Products and Log-atomic Surreals

The pre-derivation \( D \) in [3] with \( \partial D = \partial_{\text{BM}} \) is defined by a certain identity. Towards the end of this section we give this identity a more suggestive form, which we found useful. But we begin with some remarks on \( \varepsilon \)-numbers, which play an important role in the next sections.
Remarks on ε-numbers. Throughout this paper ε will denote an ε-number, that is, ε is an ordinal such that ω^ε = ε.

Lemma 2.1. For any α there is a least ε-number ε(α) ⩾ α. Moreover, if α is infinite, then \( \text{card}(ε(α)) = \text{card}(α) \).

Proof. The recursion defining \( ω^α \) as a function of α easily yields that this function is strictly increasing, with \( ω^α ⩾ α, \text{card}(ω^α) = \max\{N_0, \text{card}(α)\} \), and thus \( \text{card}(ω^α) = \text{card}(α) \) if α is infinite. Now define α_n as a function of n by the recursion α_0 = α and α_{n+1} = ω^{α_n}. Then sup_n α_n is clearly the least ε-number ⩾ α, and it has the same cardinality as α if the latter is infinite. □

If κ is an uncountable cardinal, then by the remarks in the proof above we have ω^α < κ for all α < κ. Thus uncountable cardinals are ε-numbers. The least ε-number is denoted by ε_0, as usual, so ε_0 = sup_n ω_n where the ω_n are defined by the recursion ω_0 = ω and ω_{n+1} = ω^{ω_n}.

Infinite products of monomials. Recall that \( M \) is the multiplicative group of monomials \( ω^α \). For a family \( (m_i) \) in \( M \) we say that \( \prod_i m_i \) exists if \( \sum_i a_i \) exists, with \( m_i = ω^{a_i} \) for all i, and in that case, we set

\[
\prod_i m_i := ω^{\sum_i a_i} \in M.
\]

The rules for manipulating these infinite products are easy consequences of those for infinite sums, and we shall freely use them below. Note in particular that if \( (m_i) \) is a family in \( M \) and \( \prod_i m_i \) exists, then \( \prod_i m_i^{-1} \) exists and equals \( (\prod_i m_i)^{-1} \).

In our definition of infinite products we could have represented monomials as exponentials of elements in \( J \) instead of as powers of ω. Indeed, the equivalence between these options follows from the next two lemmas:

Lemma 2.2. Let \( (a_i) \) be a summable family in \( J \). Then \( \prod_i \exp(a_i) \) exists, and

\[
\exp\left(\sum_i a_i\right) = \prod_i \exp(a_i).
\]

Proof. We have \( a_i = \sum_{x > 0} a_i x^x \), so by [9, Theorem 10.13],

\[
\exp(a_i) = ω^{b_i}, \quad b_i := \sum_{x > 0} a_i x^x g(x),
\]

so \( E(b_i) = g(E(a_i)) \). Since \( \sum_i a_i \) exists, so does \( \sum_i b_i \), and hence \( \prod_i \exp(a_i) = \prod_i ω^{b_i} \) exists, and \( \prod_i \exp(a_i) = ω^{\sum_i b_i} \). Moreover, with \( \sum_i a_i = \sum_{x > 0} a_x ω^x \), we have \( \prod_i b_i = \sum_{x > 0} a_x ω^x g(x) \). Hence again by [9, Theorem 10.13],

\[
\prod_i \exp(a_i) = ω^{\sum_{x > 0} a_x ω^x} = \exp\left(\sum_{x > 0} a_x^x \right) = \exp\left(\sum_i a_i\right),
\]

as claimed. □

Lemma 2.3. Let \( (m_i) \) be a family in \( M \) such that \( \prod_i m_i \) exists. Then \( \sum_i \log m_i \) exists, and \( \log \prod_i m_i = \sum_i \log m_i \).
Proof. We have $m_i = \exp(a_i)$ with $a_i \in J$, so $a_i = \sum_{x > 0} a_{i,x} \omega^x$, hence

$$m_i = \omega^{b_i}, \quad b_i := \sum_{x > 0} a_{i,x} \omega^g(x)$$

by [9, Theorem 10.13]. Since the product $\prod_i m_i$ exists, so does $\sum_i b_i$, and therefore $\sum_i a_i = \sum_i \log m_i$ exists. Moreover, and again by [9, Theorem 10.13],

$$\prod_i m_i = \omega^{\sum_i b_i} = \omega^{\sum_{x > 0} a_{i,x} \omega^x} = \exp\left(\sum_{x > 0} a_x \omega^x\right), \quad a_x := \sum_i a_{i,x},$$

and so $\log \prod_i m_i = \sum_{x > 0} a_x \omega^x = \sum_i a_i$. □

Log-atomic surreals. Recall that $\mathfrak{A} \subseteq \mathcal{M}^{-1}$ is the class of log-atomic surreals. See [3, Sections 1, 5] for the order-preserving bijection $x \mapsto \lambda_x: \mathfrak{N} \to \mathfrak{A}$ and for the fact that $\lambda_x \preceq_s \lambda_y$ iff $x \leq_s y$. It follows from $\exp(\omega^x) = \omega^{\omega^g(x)}$ that $\mathfrak{A} \subseteq \omega^{\mathfrak{N}}$. Thus for any well-ordered index set $I$ and strictly decreasing map $i \mapsto \lambda_i: I \to \mathfrak{A}$ the product $\prod_i \lambda_i$ exists. We shall use Proposition 2.6 and Corollary 2.9 below to define the pre-derivation $\partial_{BM}\mathfrak{A}$.

Lemma 2.4. Let $m = A|B$ be a monomial representation with $m > 1$. Then

$$\exp(m) = (m^\mathfrak{N} \cup \exp(A)) \exp(B).$$

Proof. For $m' < m$ with $m' \leq a$ for some $a \in A$ (since $A \subseteq m < B$ gives $m \leq a'$, for $m < m' < m < B$ gives $m \leq a'$, for $m' < m < m'' < a$, we have $b \leq m$ for some $b \in B$. It follows that for $m'$ as above and $k \in \mathbb{N}^\geq 1$ we have $\exp(m')^k \leq \exp(a)$ for some $a \in A$, and that for $m''$ as above and $k \in \mathbb{N}^\geq 1$ we have $\exp(b) \leq \exp(m'')^{1/k}$ for some $b \in B$. This yields the desired result in view of [3, Theorem 3.8 (1)]. □

The monomial representation $\omega = \mathbb{N}[\emptyset]$ shows that in the conclusion of Lemma 2.4 we cannot drop $\mathfrak{N}$. Below we use the binary relations $\succ^L$ and $\succ^L$ from [3]. Let $x = \{x'\} \{x''\}$ be the canonical representation of $x$, and let $j, k$ range over $\mathbb{N}^\geq 1$. Then by [3, Definition 5.12], the defining representation of $\lambda_x$ is given by

$$\lambda_x = \{k, \exp_j\left(k \log_j(\lambda_{x'})\right)\} \| \{\exp_j\left(\frac{1}{k} \log_j(\lambda_{x''})\right)\}.$$

Proposition 2.5. We have $\lambda_{x+1} = \exp(\lambda_x)$, and thus $\lambda_{x-1} = \log(\lambda_x)$.

Proof. Let $x = \{x'\} \{x''\}$ be the canonical representation of $x$. Then $1 = 0|\emptyset$ gives $x + 1 = \{x, x' + 1\} \{x'' + 1\}$. Assume inductively that $\lambda_{x'+1} = \exp(\lambda_{x'})$ and $\lambda_{x''+1} = \exp(\lambda_{x''})$ for all $x'$ and $x''$. With $j, k$ ranging over $\mathbb{N}^\geq 1$, [3, 5.15] gives

$$\lambda_{x+1} = \{k, \exp_j\left(k \log_j(\lambda_x)\right), \exp_j\left(k \log_j(\lambda_{x'+1})\right)\} \| \{\exp_j\left(\frac{1}{k} \log_j(\lambda_{x''+1})\right)\}$$

$$= \{k, \exp_j\left(k \log_j(\lambda_x)\right), \exp_j\left(k \log_j(\lambda_{x'+1})\right)\} \| \{\exp_j\left(\frac{1}{k} \log_j(\lambda_{x''})\right)\}. $$

The defining representation $\lambda_x = A|B$ is monomial, and the above gives $\lambda_{x+1} = \mathbb{N} \cup S \cup \exp(A) \exp(B)$ where $S$ includes $A^\mathfrak{N}$ and all elements of $S$ are $\succ^L$ $\lambda_x$. Since $\lambda_x \succ^L \exp(\lambda_x)$, it follows easily from Lemma 2.4 that $\lambda_{x+1} = \exp(\lambda_x)$. □

Lemma 2.6. We have $\lambda_{-\alpha} = \omega^{\omega^{\alpha}}$.

Proof. By induction on $\alpha$. The case $\alpha = 0$ holds since $\lambda_0 = \omega$. Assuming it holds for a certain $\alpha$, we have

$$\lambda_{-(\alpha+1)} = \log \lambda_{-\alpha} = \log \omega^{\omega^{-\alpha}} = \omega^{\omega^{-\alpha+1}}.$$
Next, let $\mu$ be an infinite limit ordinal. Then $-\mu = 0\{\alpha : \alpha < \mu\}$, and so by [3, 5.15] and with $j, k$ ranging over $\mathbb{N}\geq 1$ we have
\[ \lambda_{-\mu} = \mathbb{N}\left\{\exp\left(\frac{1}{k}\log_j \lambda_{-\alpha}\right)\right\}. \]

Now $\exp\left(\frac{1}{k}\log_j \lambda_{-\alpha}\right) \succeq^L \lambda_{-\alpha} \succeq^L \lambda_{-\beta}$ when $\alpha < \beta$, so by cofinality and the inductive assumption we have
\[ \lambda_{-\mu} = \mathbb{N}\left\{\omega^{\omega^{-\alpha}} : \alpha < \mu\right\}. \]

From $\mathbb{N} < \omega^{\omega^{-\mu}} < \omega^{\omega^{-\alpha}}$ for all $\alpha < \mu$, we get $\lambda_{-\mu} \leq_s \omega^{\omega^{-\mu}}$. Take $\alpha$ such that $\lambda_{-\mu} = \omega^{\omega^{-\alpha}}$. Then $\lambda_{-\mu} < \omega^{\omega^{-\alpha}}$ for $\alpha < \mu$ gives $\omega^{-\alpha} < \omega^{-\alpha}$ for all $\alpha < \mu$, and thus $\alpha > \alpha$ for all $\alpha < \mu$. This yields $\mu \leq_s \alpha$, and thus $\omega^{\omega^{-\mu}} \leq_s \lambda_{-\mu}$, hence $a = \mu$. \qed

**Lemma 2.7.** For $\lambda \in \mathcal{L}$ we have: $\lambda < \lambda_{-\alpha} \iff \lambda_{-(\alpha+1)} \leq_s \lambda$.

**Proof.** For $\lambda = \lambda_{\omega}$ we have the equivalences
\[
\lambda_{\omega} < \lambda_{-\alpha} \iff x < -\alpha \iff \alpha < -x \iff \alpha + 1 \leq_s -x \\
\iff -(\alpha + 1) \leq_s x \iff \lambda_{-(\alpha+1)} \leq_s \lambda_{\omega}.
\]

**Transfinite iteratively the logarithm function.** In view of $\lambda_{-\alpha} = \log_n \omega$ and the proof of Lemma 2.6 it is suggestive to think of $\lambda_{-\alpha}$ as the $\alpha$ times iterated function $\log$ evaluated at $\omega$. Accordingly we set $\log_{\alpha} \omega := \lambda_{-\alpha}$. We note that for $\beta < \alpha$ we have $-\beta < -\alpha$, so $\omega^{-\beta} < s \omega^{-\alpha}$, and thus $\log_{\alpha} \omega < s \log_{\alpha} \omega$.

**Lemma 2.8.** Suppose $\alpha$ is an infinite limit ordinal. Then $\log_{\alpha} \omega$ is the simplest surreal $x > \mathbb{N}$ such that $x < \log\beta \omega$ for all $\beta < \alpha$.

**Proof.** First, $\mathbb{N} < \log_{\alpha} \omega < \log_{\beta} \omega$ for all $\beta < \alpha$. Let $x$ be the simplest surreal $> \mathbb{N}$ such that $x < \log_{\beta} \omega$ for all $\beta < \alpha$. Then $x$ is the simplest positive element in its archimedean class, so $x = \omega^y$ with $y > 0$. Then $x = \omega^y < \omega^{\omega^{-\beta}}$ for $\beta < \alpha$ gives $\omega^{-\beta} < s \omega$ for all $\beta < \alpha$. Then $y$ is the simplest positive element in its archimedean class: if $0 < y_0 \leq_s y$ and $y_0 < n y_0$, then $\omega^{y_0} \leq_s \omega^y = x$ and $\mathbb{N} < \omega^{y_0} \leq s \omega^{-\beta}$ for all $\beta < \alpha$, so $\omega^y = \omega^z$, and thus $y_0 = y$. Hence $y = \omega^z$ with $z < -\beta$ for all $\beta < \alpha$, and thus $z \leq -\alpha \leq_s z$. Therefore, $\omega^{-\alpha} \leq_s \omega^z = y$, so
\[
\log_{\alpha} \omega = \omega^{\omega^{-\alpha}} \leq_s \omega^y = x,
\]
and thus $\log_{\alpha} \omega = x$. \qed

The surreal numbers $\log_{\alpha} \omega$ occur in the definition of $\partial_{\text{BM}}$ later in this section.

**The $\kappa$-numbers.** The definition of $\partial_{\text{BM}}$ in [3] also involves the surreals $\kappa_x \in \mathcal{L}$ defined by Kuhlmann and Matusinski [11]. This is only needed for $x = -\alpha$, and it follows from the results in [11] that $\kappa_{-\alpha} = \omega^{\omega^{-\alpha}}$, where $\omega^\alpha$ is the usual ordinal product. Thus in view of Lemma 2.6:

**Corollary 2.9.** We have $\kappa_{-\alpha} = \lambda_{-\omega^\alpha} = \omega^{\omega^{-\alpha}} = \log_{\omega^\alpha} \omega$.

We also use the binary relations $\prec^K, \succ^K, \asymp^K$ on $\mathbb{N}^{>\mathbb{N}}$ defined by
\[
x \prec^K y \iff x \leq \exp_n(y) \text{ for some } n, \\
x \succ^K y \iff x > \exp_n(y) \text{ for all } n, \\
x \asymp^K y \iff x \asymp^K y \text{ and } y \asymp^K x.
\]
We refer to [3, 5.3] for proofs of some basic facts about these relations and the $\kappa_x$ such as: $\asymp^K$ is an equivalence relation on $\text{No}^{>\mathbb{N}}$ with convex equivalence classes, every $\asymp^K$-equivalence class has a unique element $\kappa_x$ in it, and this element is the simplest element of this equivalence class. Also, $\kappa_x \leq_s \kappa_y$ iff $x \leq_s y$.

**Defining the pre-derivation for $\partial_{BM}$**. The pre-derivation $D$ with $\partial_D = \partial_{BM}$ is denoted by $\partial_L$ in [3, Definition 6.7], and by $\partial_\mathfrak{A}$ in this paper. It is given by

$$\partial_\mathfrak{A}(\lambda) := \prod_n \log_n \lambda / \prod_\alpha \log_\alpha \omega$$

with $\alpha$ in the denominator ranging over the ordinals such that $\log_\alpha \omega \geq \log_n \lambda$ for some $n$; to facilitate comparison with [3] we note that this condition on $\alpha$ is equivalent to $\lambda \asymp^K \log_\alpha \omega$. (The products on the right exist, since $\log_\alpha \lambda$ and $\log_\alpha \omega$ are strictly decreasing as functions of $n$ and $\alpha$, respectively.) The above defining identity for $\partial_\mathfrak{A}$ simplifies the expression in [3] by our use of infinite products (instead of exponentials of infinite sums), and of Lemma 2.6 and Corollary 2.9 (to get rid of $\kappa$-numbers). As [3, Section 9] shows, $\partial_\mathfrak{A}$ is in a certain technical sense the simplest pre-derivation.

If $\lambda > \exp_n \omega$ for all $n$, then $\partial_\mathfrak{A}(\lambda) = \prod_n \log_n \lambda$. Another special case is $\partial_\mathfrak{A}(\log_\omega \omega) = 1/\prod_{\beta<\alpha} \log_\beta \omega$, in particular, $\partial_\mathfrak{A}(\omega) = 1$. For $\varepsilon$-numbers we get the following (not needed later, but included as an example):

**Lemma 2.10**. We have $\log_n \varepsilon = \omega^{\varepsilon-n}$. Hence $\varepsilon \in \mathfrak{A}$ and

$$\partial_\mathfrak{A}(\varepsilon) = \omega^{\varepsilon+\varepsilon^(-1)+\varepsilon^(-2)+\cdots} = \omega^{\varepsilon/(1-\varepsilon^{-1})}.$$

**Proof**. From [9, pp. 179, 180] we get that if $b$, as a sequence of pluses and minuses, equals $\varepsilon$ followed by $\varepsilon \omega n$ minuses, with $n \geq 1$ and $\varepsilon \omega n$ being the ordinal product, then $b = \omega^{\varepsilon-n}$, and $g(b) = \varepsilon - (n - 1)$. In other words,

$$g(\omega^{\varepsilon-n}) = \varepsilon - (n - 1) \quad (n \geq 1).$$

Using this we prove the lemma by induction on $n$. The case $n = 0$ is clear. Assume inductively that $\log_n \varepsilon = \omega^{\varepsilon-n}$. Since $g(\omega^{\varepsilon-(n+1)}) = \varepsilon - n$, this gives

$$\exp(\omega^{\varepsilon-(n+1)}) = \omega^{\varepsilon-n},$$

from which we get $\log_{n+1} \varepsilon = \omega^{\varepsilon-(n+1)}$, as desired. \qed

### 3. Exhibiting $\text{No}$ as a Suitable Directed Union

At the end of Section 1 we explained how proving $T \equiv \text{No}$ (as differential fields) reduces to representing $\text{No}$ as a directed union of spherically complete grounded $H$-subfields. In this section we obtain such a representation. The reader should beware of considering $\text{No}$ itself as spherically complete, even though the Conway normal form is sometimes summarized as “$\text{No} = \mathbb{R}(\omega^{\omega^\omega})$”. This is misleading, however, since it suggests that a series like $\sum_\alpha \omega^{-\alpha}$, where the sum is over all ordinals $\alpha$, is a surreal number. It might perhaps be viewed as a surreal number in a strictly larger set-theoretic universe, but not in the one we are (tacitly) working in. A better way of understanding $\text{No}$ as a valued field is as the directed union $\bigcup_{\Gamma} \mathbb{R}[\omega^\Gamma]$ with $\Gamma$ ranging over the subsets of $\text{No}$ that underly an additive subgroup of $\text{No}$; for example, any $\alpha$ gives $\text{No}(\omega^{\alpha})$ as such a $\Gamma$. For any such $\Gamma$ the corresponding $\mathbb{R}[\omega^\Gamma]$
is indeed a spherically complete valued subfield of $\mathbf{No}$, but in general $\mathbb{R}[[\omega^R]]$ is not closed under $\partial_{BM}$, and even if it is, it might not be grounded.

In this section we show that for $S = \mathbf{No}(\varepsilon) \cup \{-\varepsilon\}$, with $\varepsilon$ any $\varepsilon$-number, the Hahn subgroup $\Gamma = \mathbb{R}[[\omega^S]]$ of $\mathbf{No}$ gives rise to a spherically complete valued subfield $\mathbb{R}[[\omega^H]]$ that is closed under $\partial_{BM}$ and grounded as an $H$-subfield of $\mathbf{No}$.

**A length bound for $h$.** This very useful bound is as follows:

**Lemma 3.1.** $l(h(y)) \leq \omega^{l(y)+1}$.

**Proof.** By [9, p. 172] the canonical representation $y = \{y'\} \setminus \{y''\}$ yields

$$h(y) = \{(0,h(y')) \mid \{h(y''),\omega^y/2^n\}\}.$$ 

We can assume inductively that the lemma holds for the $y'$ and $y''$ instead of $y$, and thus $l(h(y')) \leq \omega^{l(y')+1} < \omega^{l(y)+1}$ for all $y'$, and likewise with $y''$ instead of $y'$. Also, $l(\omega^y/2^n) \leq l(\omega^y)(1/2^n) < \omega^{l(y)}\omega = \omega^{l(y)+1}$, using [5, Lemmas 3.6 and 4.1]. Now appeal to [9, Theorem 2.3]. \qed

Recall from Section 1 that $h(-\alpha) = \omega^{-(\alpha+1)}$, and so $h(0) = \omega^{-1}$ shows that for $y = 0$ the upper bound in Lemma 3.1 is attained.

**Some spherically complete initial subfields of $\mathbf{No}$.** In this subsection we fix an initial subset $I$ of $\mathbf{No}$. Then $\Gamma := \mathbb{R}[[\omega^I]]$ is an initial additive subgroup of $\mathbf{No}$ by the proof of Theorem 18 in [7]. (That theorem considers Hahn fields rather than the Hahn group $\Gamma$, but the same ideas work; we stress that it is the proof of that theorem rather than its statement that matters here.) Moreover, as Philip Ehrlich mentioned to one of us:

**Lemma 3.2.** Suppose $I$ has a least element $a$. Then $a = -\alpha$ for some $\alpha$, and $\Gamma$ has a least nontrivial archimedean class represented by $\omega^\alpha$.

**Proof.** Taking the longest initial segment of $a$ consisting of minus signs we get the largest ordinal $\alpha$ with $-\alpha \leq a$. Then $-\alpha \in I$ and $-\alpha \leq a$, so $-\alpha = a$. \qed

Since $\Gamma$ is initial and an ordered additive group it leads to the initial subfield $K := \mathbb{R}[[\omega^K]]$ of $\mathbf{No}$. Note that $K$ is spherically complete, and if $(a_i)$ is a family in $K$ for which $\sum_i a_i$ exists, then $\sum_i a_i \in K$. Now $\Gamma := \mathbb{R}[[\omega^I]]$ is also closed under infinite sums, so if $(m_i)$ is a family in $\mathcal{M} \cap K$ such that $\prod_i m_i \in K$, then $\prod_i m_i \in K$. Thus $K$ is closed under infinite sums, and also under infinite products of monomials. This is very useful in showing that for suitable choices of $I$ the field $K$ is closed under certain surreal derivations. Note however, that if $I$ has a least element, then $K^{>N}$ is not closed under log: if $-\alpha$ is the least element of $I$, then $\log_{-\alpha} \omega = \omega^{-\alpha} \in K$, but $\log_{-\alpha+1} \omega \notin K$, as $-(\alpha+1) \notin I$.

In order to discuss examples we set $a^r := \exp(r \log a)$ for $a > 0$ and $r \in \mathbb{R}$, and note agreement with the previously defined $\omega^r$ when $a = \omega$. Moreover,

$$(\log_{\omega} \omega)^r = \omega^{r\omega^{-\alpha}} \quad (r \in \mathbb{R}),$$

by the definition of $a^r$, using also $g(\omega^{-(\alpha+1)}) = -\alpha$ and [9, Theorem 10.13].

**Examples.** For $I = \{0\}$ we get $\Gamma = \mathbb{R}$ and $K = \mathbb{R}[[\omega^R]]$; note that $K$ is closed under $\partial_{BM}$, but $\omega \in K$ and $\log \omega = \omega^{1/\omega} \notin K$.

For $I = \{0, -1\}$ we have $\Gamma = \mathbb{R} + \mathbb{R} \omega^{-1}$, so $\omega^\Gamma = \omega^\mathbb{R}(\log \omega)^\mathbb{R}$, and thus $K = \mathbb{R}[[\omega^{\mathbb{R}(\log \omega)^\mathbb{R}}]]$, which is again closed under $\partial_{BM}$. 


Let \( I = \{ \alpha : \alpha \leq \varepsilon \} \). Then \( \varepsilon = \omega^\omega \in K \), but Lemma 2.10 gives \( \log \varepsilon \notin K \), since \( \varepsilon - 1 \notin I \) and so \( \omega^{\varepsilon - 1} \notin \Gamma \). Likewise we get \( \partial_{BM}(\varepsilon) \notin K \).

**Lemma 3.3.** If \( I = \{ a : l(a) < \alpha \} \) or \( I = \{ a : l(a) \leq \alpha \} \), then \( I \subseteq \Gamma \subseteq K \).

**Proof.** Suppose \( I = \{ a : l(a) < \alpha \} \). (The case \( I = \{ a : l(a) \leq \alpha \} \) is handled in the same way.) Let \( a \in I \). Then \( a = \sum_x a_x \omega^x \), and if \( x \in E(a) \), then \( l(x) \leq l(\omega^x) \leq l(a) < \alpha \) by \([5, \text{Lemmas 3.4, 4.1, and 4.2}]\), so \( x \in I \). Thus \( a \in \Gamma \). This proves \( I \subseteq \Gamma \).

Next, if \( b \in \Gamma \), then \( b = \sum_{x \in I} b_x \omega^x \), and so \( b \in K \) in view of \( I \subseteq \Gamma \). \( \square \)

The next lemma will also be crucial:

**Lemma 3.4.** Suppose \( h(I) \subseteq \Gamma \). Then \( \log K^> \subseteq K \) and for each \( a \in K \) and term \( t \) of \( a \) we have: \( t \) and all terms of \( \ell(t) \) lie in \( K \).

**Proof.** Let \( a \in K^> \) have leading monomial \( m = \omega^b \) with \( b = \sum_{y \in I} b_y \omega^y \); to get \( \log a \in K \), it is enough that \( \log m \in K \); the latter holds because \( \log m = \sum_y b_y \omega^{h(y)} \).

This proves \( K^> \subseteq K \).

Next, let \( a \in K \) and let \( t \) be a term of \( a \); we have to show that \( t \) and all terms of \( \ell(t) \) lie in \( K \). As \( K \supseteq \mathbb{R} \) is initial, it does contain the term \( t \) of its element \( a \). We have \( t = r \omega^b \) with \( r \in \mathbb{R}^\omega \) and \( b \in \Gamma \), so \( b = \sum_{y \in I} b_y \omega^y \), and thus \( \omega^b = \exp \left( \sum_{y \in I} b_y \omega^{h(y)} \right) \). Hence \( \ell(t) = \ell(r \omega^b) = \sum_{y \in I} b_y \omega^{h(y)} \) and each of its terms \( b_y \omega^{h(y)} \) lies obviously in \( K \). \( \square \)

**Corollary 3.5.** If \( h(I) \subseteq \Gamma \) and \( D \) is a pre-derivation with \( D(K \cap \mathfrak{A}) \subseteq K \), then \( \partial_D(K) \subseteq K \).

**Proof.** Use the definition of \( \partial_D \) from Section 1, the fact that \( K \) is closed under infinite sums, and Lemma 3.4. \( \square \)

**Corollary 3.6.** Suppose \( h(I) \subseteq \Gamma \). Then \( \partial_{BM}(K) \subseteq K \).

**Proof.** Let \( \lambda \in K \cap \mathfrak{A} \); by Corollary 3.5 we just need to get \( \partial_\mathfrak{A}(\lambda) \in K \). Since \( K \) is closed under infinite products, it is enough for this to get \( \log_n \lambda \in K \) for all \( n \) (which is the case by Lemma 3.4), and \( \lambda_{\alpha} \in K \) for all \( \alpha \) such that \( \lambda \prec^K \lambda_{\alpha} \).

Given such \( \alpha \), take \( n \) with \( \log_n \lambda < \lambda_{\alpha} \). Then \( \lambda_{\alpha} \leq \lambda_{\alpha+1} \leq s \log_n \lambda \in K \) by Lemma 2.7, and so \( \lambda_{\alpha} \in K \) because \( K \) is initial. \( \square \)

It can happen that \( h(I) \not\subseteq \Gamma \) and that \( K \) is nevertheless closed under \( \partial_{BM} \). The next lemma gives a useful criterion for that. To see why that lemma holds, consider a surreal derivation \( \partial \), and note that from \( \omega^\omega = \exp(\omega^{h(y)}) \) we get

\[
\partial(\omega^y) = \omega^y \cdot \partial(\omega^{h(y)}),
\]

so for any monomial \( m = \omega^b \in K \) we have \( b = \sum_{y \in I} b_y \omega^y \), and thus

\[
m = \exp \left( \sum_{y \in I} b_y \omega^{h(y)} \right), \quad \partial(m) = m \cdot \sum_{y \in I} b_y \partial(\omega^{h(y)}).
\]

This leads to:

**Lemma 3.7.** Given a surreal derivation \( \partial \), the following are equivalent:

1. \( K \) is closed under \( \partial \);
2. \( \partial(\omega^y) \in K \) for all \( y \in I \);
3. \( \partial(\omega^{h(y)}) \in K \) for all \( y \in I \).
The surreal fields $K_\varepsilon$. Given the $\varepsilon$-number $\varepsilon$, we have the initial set $I := \mathbf{No}(\varepsilon)$, with the corresponding $\Gamma := \mathbb{R}[[\omega^I]]$ and $K := \mathbb{R}[[\omega^I]]$. In view of Lemmas 3.1 and 3.3 we have $h(I) \subseteq I \subseteq \Gamma$, so $\partial BM(K) \subseteq K$ by Corollary 3.6. Thus $K$ is a spherically complete initial $H$-subfield of $\mathbf{No}$. However, $I$ has no least element, so $K$ is not grounded. We repair this by just augmenting $I$ by $-\varepsilon$: set $I_\varepsilon := I \cup \{-\varepsilon\}$. Then $I_\varepsilon$ is still initial, with least element $-\varepsilon$, and so we have the corresponding $\Gamma_\varepsilon := \mathbb{R}[[\omega^{I_\varepsilon}]]$ and $K_\varepsilon := \mathbb{R}[[\omega^{I_\varepsilon}]]$. To get $\partial BM(K_\varepsilon) \subseteq K_\varepsilon$ we note that $K \subseteq K_\varepsilon$, and so it suffices by Lemma 3.7 that $\partial BM(\omega^{\omega^{-\varepsilon}}) \in K_\varepsilon$. But $\omega^{\omega^{-\varepsilon}} = \log_\varepsilon \omega$, and

$$\partial BM(\log_\varepsilon \omega) = 1 / \prod_{\alpha < \varepsilon} \log_\alpha \omega,$$

which lies in $K$, and hence in $K_\varepsilon$. Thus $K_\varepsilon$ is a grounded $H$-subfield of $\mathbf{No}$, and

$$\mathbf{No} = \bigcup_\varepsilon K_\varepsilon.$$

Note that Corollary 3.6 does not apply to $I_\varepsilon$, since $h(-\varepsilon) = \omega^{-(\varepsilon+1)} \notin \Gamma$; this is why we did the less direct construction via $I = \mathbf{No}(\varepsilon)$.

Since $\omega^{-\varepsilon}$ represents the smallest archimedean class of $\Gamma_\varepsilon$, we have

$$\max \Psi_{K_\varepsilon} = \nu((\omega^{\omega^{-\varepsilon}})^\dagger) = \nu((\log_\varepsilon \omega)^\dagger)$$

by the proof of Lemma 1.1. In view of $(\log_\varepsilon \omega)^\dagger = (\log_{\varepsilon+1} \omega)^\dagger$ and the remarks at the end of Section 1, the representation of $\mathbf{No}$ as an increasing union $\bigcup K_\varepsilon$ of spherically complete grounded $H$-subfields now gives $\partial BM(\mathbf{No}) = \mathbf{No}$. (The proof of $\partial BM(\mathbf{No}) = \mathbf{No}$ in [3, Section 7] is different.) Thus by the results stated at the end of Section 1 we conclude that $\mathbf{No} \equiv \mathbb{T}$, as differential fields.

4. The Case of Restricted Length

A set $S \subseteq \mathbf{No}$ is said to be of countable type if $l(a)$ is countable for all $a \in S$, and all well-ordered subsets of $S$ as well as all reverse well-ordered subsets of $S$ are countable. (Note that $l(a)$ is countable for every $a \in \mathbf{No}(\omega_1)$, but that $\mathbf{No}(\omega_1)$ is not of countable type, since it has the set of countable ordinals as an uncountable well-ordered subset.)

**Proposition 4.1.** Suppose the subset $S$ of $\mathbf{No}$ is of countable type. Then the additive subgroup $\mathbb{R}[[\omega^S]]$ of $\mathbf{No}$ is also of countable type.

**Proof.** The case $\alpha = 1$ of Esterle [8, Lemme 2.2] and the remarks following it yield that every well-ordered subset of $\mathbb{R}[[\omega^S]]$ is countable. Hence every reverse well-ordered subset of $\mathbb{R}[[\omega^S]]$ is countable as well. Let $a \in \mathbb{R}[[\omega^S]]$. Then $a = \sum_{s \in E(a)} a_s \omega^s$. Now $E(a) \subseteq S$ is countable, so the well-ordered set $-E(a)$ has order type $\mu < \omega_1$. Since $\omega_1$ is regular, we have a countable ordinal $\nu$ such that $l(s) \leq \nu$ for all $s \in E(a)$. Then $l(\omega^s) \leq \omega^\nu$ for all $s \in E(a)$ by [5, Lemma 4.1], hence $l(a_\omega \omega^s) \leq \omega^{\nu+1}$ for all $s \in E(a)$ by [5, Proposition 3.6]. Thus

$$l(a) \leq \mu \cdot \omega^{\nu+1} < \omega_1,$$

by [9, Theorem 5.12], or [5, Lemma 4.2, (3)].

As a consequence, consider $S := \mathbf{No}(\omega)$, the set of of dyadic numbers. Then $S$ is of countable type, and so $\mathbb{R}[[\omega^S]]$ is of countable type. Nevertheless, $l(\mathbb{R}[[\omega^S]])$ is
cofinal in $\omega_1$: given any countable ordinal $\mu$, take an order reversing injective map $\alpha \mapsto s_\alpha: \mu \to S$; then $a := \sum_\alpha \omega^{s_\alpha} \in \mathbb{R}[\omega^S]$ has $l(a) \geq \mu$, by [9, p. 63].

Let $\kappa$ be any infinite cardinal. Esterle [8, Lemme 2.2] actually tells us for any set $S \subseteq \mathbb{N}$: if all well-ordered subsets of $S$ have size $\leq \kappa$, then this remains true for the set $\mathbb{R}[\omega^S] \subseteq \mathbb{N}$. The next cardinal $\kappa^+$ is regular, so the arguments in the proof of Proposition 4.1 go through to give the following, where we call $S \subseteq \mathbb{N}$ of type $\kappa$ if $l(a) \leq \kappa$ for all $a \in S$ and all well-ordered subsets of $S$ and all reverse well-ordered subsets of $S$ have size $\leq \kappa$.

**Corollary 4.2.** If $S \subseteq \mathbb{N}$ is of type $\kappa$, then so is $\mathbb{N}[\omega^S]$.\hspace{1cm}□

Next we show that for countable $\mu$ the set $\mathbb{N}(\mu)$ is of countable type. Every element of $\mathbb{N}(\mu)$ has clearly countable length, for countable $\mu$, and $\mathbb{N}(\mu)$ is closed under $x \mapsto -x$, so the assertion above reduces to:

**Lemma 4.3.** Suppose the ordinal $\mu$ is countable. Then every well-ordered subset of $\mathbb{N}(\mu)$ is countable.

This may remind the reader of the well-known property of the ordered set $\mathbb{R}$ that every well-ordered subset of $\mathbb{R}$ is countable. Here is a quick proof using that $\mathbb{R}$ has a countable dense subset $\mathbb{Q}$: given any embedding $\alpha \mapsto r_\alpha$ of an infinite cardinal $\kappa$ into $\mathbb{R}$, pick for each $\alpha < \kappa$ a rational $q_\alpha$ such that $r_\alpha < q_\alpha < r_{\alpha+1}$; it follows that $\kappa = \aleph_0$. However, such a countable density argument cannot be used for ordered sets $\mathbb{N}(\mu)$ when $\mu$ is a countable limit ordinal $> \omega$.

**Lemma 4.4.** Let $\mu$ be an infinite limit ordinal. Then the ordered set $\mathbb{N}(\mu)$ is dense without endpoints. If $\mu > \omega$, then there exists a collection of $2^\aleph_0$ pairwise disjoint open intervals in $\mathbb{N}(\mu)$, which has therefore no countable dense subset.

**Proof.** The ordinals $\alpha < \mu$ are cofinal in this ordered set, and there is no largest such $\alpha$. For $a < b$ in this ordered set, take $\alpha \leq l(a), l(b)$ such that $a|_\alpha = b|_\alpha$ and $a(\alpha) < b(\alpha)$. If $l(b) > \alpha$, then $b(\alpha) = +$, so $a < b_- < b$. If $l(a) > \alpha$, then $a(\alpha) = -$, so $a < a+ < b$. Note that $b_-, a+ \in \mathbb{N}(\mu)$, as $\mu$ is a limit ordinal.

Next, assume $\mu > \omega$. For each nondyadic $r \in \mathbb{R} \subseteq \mathbb{N}$, we have the surreals $r_-$ and $r_+$ of length $\omega+1$, and so we obtain the pairwise disjoint open intervals $(r_-, r_+)$ in $\mathbb{N}(\mu)$.

**Proof of Lemma 4.3.** For $a \in \mathbb{N}(\mu)$ we define $\hat{a}: \mu \to \mathbb{R}$ by

\[
\hat{a}(\alpha) = \begin{cases} 
-1 & \text{if } a(\alpha) = -, \\
0 & \text{if } a(\alpha) = 0, \\
1 & \text{if } a(\alpha) = +,
\end{cases}
\]

For $S = \{\alpha: \alpha < \mu\}$ this yields an order-preserving injective map $a \mapsto \sum_{\alpha < \mu} \hat{a}(\alpha)\omega^{-\alpha}: \mathbb{N}(\mu) \to \mathbb{R}[\omega^S]$.\hspace{1cm}□

It remains to appeal to Proposition 4.1.

Essentially the same argument yields the following generalization:

**Corollary 4.5.** If $\kappa$ is an infinite cardinal and $\mu$ is an ordinal of cardinality $\leq \kappa$, then each well-ordered subset of $\mathbb{N}(\mu)$ has cardinality $\leq \kappa$.\hspace{1cm}□
Note that for a countable $\varepsilon$-number $\varepsilon$ the initial set $I_\varepsilon = \text{No}(\varepsilon) \cup \{-\varepsilon\}$ is of countable type by Lemma 4.3, and hence $\Gamma_\varepsilon$ and $K_\varepsilon$ are as well by Proposition 4.1. Taking the union over all such countable $\varepsilon$ we obtain the set $\text{No}(\omega_1)$ of all surreals of countable length as an increasing union of spherically complete grounded $H$-subfields $K_\varepsilon$ of $\text{No}$. As in Section 3 and using also the model completeness of $T_\text{small} = \text{Th}(\mathbb{T})$ this yields Theorem 2. The results above lead moreover to the following generalization:

**Corollary 4.6.** Let $\kappa$ be any uncountable cardinal. Then the subfield $\text{No}(\kappa)$ of $\text{No}$ is closed under $\partial_{\text{BM}}$, and $\text{No}(\kappa) \prec \text{No}$, as ordered differential fields.

**Proof.** If $\kappa$ is regular we can argue as for $\omega_1$, using Corollaries 4.2 and 4.5 instead of Proposition 4.1 and Lemma 4.3. If $\kappa$ is singular, use that it is the supremum of the uncountable regular cardinals below it. $\square$

5. Constructing Embeddings

So far we have just worked inside $\text{No}$ and established Theorem 2. In this section we turn to $\mathbb{T}$ and prove the embedding results: Theorems 1 and 3.

**Embedding $\mathbb{T}$ into $\text{No}$.** Given a Hahn field $\mathbb{R}[[G]]$ over $\mathbb{R}$ we define a map $F: \mathbb{R}[[G]] \to \text{No}$ to be strongly additive if for every summable family $(f_i)_{i \in I}$ in $\mathbb{R}[[G]]$ the family $(F(f_i))$ is summable in $\text{No}$ and $F(\sum_i f_i) = \sum_i F(f_i)$. We refer to [2, Appendix A] for the construction of $\mathbb{T}$ as an exponential ordered field. In this construction $\mathbb{T}$ is a subfield of a Hahn field $\mathbb{R}[[G^\text{LE}]]$: in fact, $G^\text{LE}$ is a certain directed union of ordered subgroups $G_m \downarrow n$, and $\mathbb{T}$ is the corresponding directed union of the Hahn fields $\mathbb{R}[[G_m \downarrow n]]$. A map $F: \mathbb{T} \to \text{No}$ is said to be strongly additive if its restriction to each $\mathbb{R}[[G_m \downarrow n]]$ is strongly additive.

**Proposition 5.1.** There is a unique strongly additive embedding $\iota: \mathbb{T} \to \text{No}$ of exponential ordered fields that is the identity on $\mathbb{R}$ and such that $\iota(x_T) = \omega$.

**Proof.** We use the notations from [2, Appendix A] except that the $x$ there is $x_T$ here. The construction of $\mathbb{T}$ there begins with the Hahn field $E_0 = \mathbb{R}[[x_T^E]]$, and there is clearly (unique) strongly additive ordered field embedding $i_0: E_0 \to \text{No}$ such that $i_0(r) = r$ and $i_0(x_T^E) = \omega^r$ for all $r \in \mathbb{R}$. Moreover, $i_0(e^b) = \exp(i_0(b))$ for all $b \in B_0$, and $\exp(i_0(a)) > i_0(E_0)$ for all $a \in A_0$. Assume inductively that we have an extension of $i_0$ to a strongly additive ordered field embedding $i_m: E_m = \mathbb{R}[[G_m]] \to \text{No}$ such that $i_m(e^b) = \exp(i_m(b))$ for all $b \in B_m$, and $\exp(i_m(a)) > i_m(E_m)$ for all $a \in A_m$. Then one checks easily that $i_m$ extends (uniquely) to a strongly additive ordered field embedding $i_{m+1}: E_{m+1} \to \text{No}$ such that $i_{m+1}(e^b) = \exp(i_{m+1}(b))$ for all $b \in B_{m+1}$, and $\exp(i_{m+1}(a)) > i_{m+1}(E_{m+1})$ for all $a \in A_{m+1}$. Taking a union over all $m$ we obtain an embedding

$$\iota_0 := \bigcup_m i_m : \mathbb{R}[[x_T^E]]^E = \bigcup_m \mathbb{R}[[G_m]] \to \text{No}$$

of ordered exponential fields. Replacing in the above $\ell_0 = x_T$, $G_m$, $\omega$, by $\ell_n = \log_n x_T$, $G_m \downarrow n$, $\log_n \omega$, respectively, we obtain likewise an embedding

$$\iota_n : \mathbb{R}[[\ell_n^E]]^E = \bigcup_m \mathbb{R}[[G_m \downarrow n]] \to \text{No}$$

of ordered exponential fields with $\iota_n(\ell_n) = \log_n \omega$. Each $\iota_{n+1}$ extends $\iota_n$, so we can take the union over all $n$ to get an embedding $\iota: \mathbb{T} \to \text{No}$ as claimed. The
uniqueness holds because the smallest subfield of \( T \) that contains \( \mathbb{R}(x_T) \) and is closed under exponentiation, taking logarithms of positive elements, and summation of summable families is \( T \) itself.

Next we apply the model completeness of the theory of the exponential ordered field of real numbers (Wilkie [13]). By [6] and [5], respectively, the ordered exponential fields \( T \) and \( \mathbb{N} \) are models of this theory, and so \( \iota : T \to \mathbb{N} \) is an elementary embedding of ordered differential fields.

It is easy to check that \( \iota : T \to \mathbb{N} \) is also an embedding of ordered differential fields. In view of \( T \equiv \mathbb{N} \) (as differential fields), and the model completeness of \( T_{\text{small}} \) mentioned at the end of Section 1 we conclude that \( \iota \) is an elementary embedding of ordered differential fields: Theorem 1.

Is \( \iota \) an elementary embedding of ordered differential exponential fields? We don’t know; this is related to the open problem from [2] to extend the model-theoretic results there about \( T \) as a differential field to \( T \) as a differential exponential field.

It follows easily from the construction of \( T \) and \( \iota \) that all surreal derivations \( \partial \) with \( \partial(\omega) = 1 \) agree on \( \iota(T) \).

**Proposition 5.2.** Here are some further properties of the map \( \iota \):

1. \( \iota(G^{\text{LE}}) = \mathfrak{M} \cap \iota(T) \);
2. \( \iota(T) \) is truncation closed;
3. \( \iota(T) \) is of countable type; in particular, \( \iota(T) \subseteq \mathbb{N} \omega_1 \).

**Proof.** Induction on \( m \) gives \( \iota(G_m) \subseteq \mathfrak{M} \), where we use at the inductive step that \( G_{m+1} = \exp(A_m)G_m \) and \( \iota(A_m) \subseteq J \), the latter being a consequence of \( \iota(G_m) \subseteq \mathfrak{M} \). Likewise, \( \iota(G_{m\downarrow n}) \subseteq \mathfrak{M} \) for all \( m, n \), and thus \( \iota(G^{\text{LE}}) \subseteq \mathfrak{M} \). Since \( \iota \) respects infinite sums of monomials, this yields (1), and (2) is then an immediate consequence using also that \( T \) is truncation closed in \( \mathbb{R}[[G^{\text{LE}}]] \). As to (3), using the results in Section 4 one shows by induction on \( m \) that \( \iota(G_m) \), and likewise each \( \iota(G_{m\downarrow n}) \), has countable type. Hence \( \iota(G^{\text{LE}}) \) has countable type, and so does \( \iota(T) \). \( \square \)

**Question** (Elliot Kaplan): can (2) be improved to \( \iota(T) \) being initial?

**Embedding \( H \)-fields into \( \mathbb{N} \).** Let \( \varepsilon \) be an \( \varepsilon \)-number; for example, \( \varepsilon \) could be any uncountable cardinal. We recall from [5] that \( \mathbb{N}(\varepsilon) \) is a real closed subfield of \( \mathbb{N} \) containing \( \mathbb{R} \). We consider \( \mathbb{N}(\varepsilon) \) as a valued subfield of \( \mathbb{N} \) with (divisible) ordered value group \( \mathbb{V}(\mathbb{N}(\varepsilon)^{\times}) \). We shall need an easy auxiliary result:

**Lemma 5.3.** Let \( \kappa \) be a regular uncountable cardinal. Then the underlying ordered sets of \( \mathbb{N}(\varepsilon)^{\times} \) and \( \mathbb{N}(\varepsilon)^{\times} \) are \( \kappa \)-saturated.

**Proof.** Let \( A, B \subseteq \mathbb{N}(\varepsilon) \) have cardinality \( \kappa \), with \( A < B \). The regularity of \( \kappa \) yields an ordinal \( \alpha < \kappa \) such that \( l(A \cup B) < \alpha \). By [9, Theorem 2.3] this gives a surreal \( a \) with \( l(a) \leq \alpha \) such that \( A < a < B \), and then \( a \in \mathbb{N}(\varepsilon) \). Thus \( \mathbb{N}(\varepsilon) \) is \( \kappa \)-saturated as an ordered set. Next, let \( P, Q \subseteq \mathbb{N}(\varepsilon)^{\times} \) have cardinality \( \kappa \), with \( v(P) > v(Q) \). Set \( A := \{ np : n \geq 1, p \in P \} \) and \( B := \{ q/n : n \geq 1, q \in Q \} \). Then \( A < B \), and so the above gives \( a \in \mathbb{N}(\varepsilon) \) with \( A < a < B \). Then \( v(P) > v(a) > v(Q) \), showing that \( v(\mathbb{N}(\varepsilon)^{\times}) \) is \( \kappa \)-saturated as an ordered set. \( \square \)

For Theorem 3 we need a sharpening of the model completeness of the theory \( T_{\text{nil}}^{\text{nil}} \) of \( \omega \)-free newtonian Liouville closed \( H \)-fields, namely, the quantifier elimination (QE) explained in [2, Introduction to Chapter 16]. The relevant first-order language for
QE has in addition to $L$ extra unary predicate symbols $I, \Lambda, \Omega,$ to be interpreted in a model $L$ of $T^\text{ul}$ as sets $I(L), \Lambda(L), \Omega(L) \subseteq L$ according to their defining axioms:

$I(a) \iff a = y'$ for some $y < 1$ in $L,$

$\Lambda(a) \iff a = -y^{\dagger\dagger}$ for some $y > 1$ in $L,$

$\Omega(a) \iff 4y'' + ay = 0$ for some $y \in L^\times.$

The sets $I(L), \Lambda(L), \Omega(L) \subseteq L$ are convex; their role with respect to QE is like that of the set of squares in a real closed field. For more on this, see [2, Introduction]. A $\Lambda\Omega$-field is a substructure $K = (K, I, \Lambda, \Omega)$ of such an expanded model $(L, \ldots)$ of $T^\text{ul}$ for which $K$ is an $H$-subfield of $L.$ This notion of a $\Lambda\Omega$-field is studied in detail in [2, Section 16.3], from which we take in particular the fact that any $\omega$-free $H$-field $K$ has a unique expansion to a $\Lambda\Omega$-field $K = (K, I, \Lambda, \Omega).$ The proof below assumes familiarity with several other results from [2, Section 16.3].

**Proof of Theorem 3.** Let $\mathbf{No}_{\Lambda\Omega}$ be the expansion of $\mathbf{No}$ to a $\Lambda\Omega$-field, and let $K$ be any $H$-field with small derivation and constant field $\mathbb{R}.$ In order to embed $K$ over $\mathbb{R}$ into $\mathbf{No},$ we first expand $K$ to a $\Lambda\Omega$-field $K = (K, I, \Lambda, \Omega)$ with $1 \not\in I;$ this can be done in at least one way, and at most two ways, and $1 \not\in I$ guarantees that all $\Lambda\Omega$-field extensions of $K$ have small derivation. We claim that $K$ can be embedded into $\mathbf{No}_{\Lambda\Omega}.$ The ordered field $\mathbb{R}$ with the trivial derivation is an $H$-field and expands to the $\Lambda\Omega$-field $R := (\mathbb{R}, \{0\}, (-\infty, 0], (-\infty, 0]).$ The inclusion of $\mathbb{R}$ into $K$ and into $\mathbf{No}$ are embeddings of $R$ into $K$ and $\mathbf{No}_{\Lambda\Omega},$ respectively. By taking $E := R,$ our claim reduces therefore to proving the following more general statement:

**Claim.** Let $E \subseteq K$ be an extension of $\Lambda\Omega$-fields with $\mathbb{R}$ as their common constant field, and let $i: E \to \mathbf{No}_{\Lambda\Omega}$ be an embedding of $\Lambda\Omega$-fields that is the identity on $\mathbb{R}.$ Then $i$ extends to an embedding $K \to \mathbf{No}_{\Lambda\Omega}$ of $\Lambda\Omega$-fields.

To prove this we first extend $K$ to make it $\omega$-free, newtonian, and Liouville closed; by [2, 16.4.1 and 14.5.10] this can be done without changing its constant field. Next we apply [2, 16.4.1] again, but this time to $E,$ to arrange that $E$ is $\omega$-free. Take a regular uncountable cardinal $\kappa > \text{card}(K)$ such that $i(E) \subseteq \mathbf{No}(\kappa),$ where $E$ is the underlying set of $E.$ By Corollary 4.6 we have $\mathbf{No}(\kappa) \prec \mathbf{No}.$ In view of Lemma 5.3 and [2, 16.2.3] we can then extend $i$ to an embedding $K \to \mathbf{No}(\kappa).$ □

**Final remarks.** Suppose the $H$-field $K$ has small derivation and constant field $\mathbb{R}.$ Then Theorem 3 yields an embedding $i: K \to \mathbf{No}$ over $\mathbb{R}.$ Under some reasonable further conditions, like $K$ being $\omega$-free and newtonian, can we take $i$ such that $i(K)$ is truncation closed, or even initial? The interest of such a result would depend on how canonical the derivation $\partial_{BM}$ is deemed to be. As already mentioned at the end of the introduction, we doubt that $\partial_{BM}$ is optimal: the condition on pre-derivations to take values in $\mathbb{R} \supseteq \mathbb{R}$ seems too narrow. But even with this restriction one can construct pre-derivations $D \neq \partial_\omega$ such that Theorems 1 and 3 go through for $\mathbf{No}$ equipped with $\partial_D$ instead of with $\partial_{BM},$ with only minor changes in the proofs.

**References**


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