Problem Set 1
Due Friday, September 15.

Foundations of Number Theory

Math 435, Fall 2006

1. By induction, show that
$$3|(n^3+2n), \quad 24|(5^{2n}-1) \qquad \text{for all } n \in \mathbb{N},\ n \geq 1.$$
2. Prove that $(1+x)^n \geq 1+nx$ if $x \in \mathbb{R}$ with $x > -1$, for all natural numbers $n \geq 1$. (Bernoulli's Inequality.)
3. Show that the square of an odd natural number has remainder 1 upon division by 8. Use this fact to prove that $(m+n)(m-n)$ is always divisible by 8, for odd $m, n \in \mathbb{Z}$.
4. Here is a proposed proof for the statement "all cats have the same color". We proceed as follows: By induction on n we show that all cats in a set consisting of n cats have the same color. *Base step:* A set consisting of one cat only clearly satisfies the claim. *Inductive step:* Suppose the theorem has been proven for all sets of n cats. Consider a set of $n+1$ cats. Take one cat out of the set; call it "first cat". By the inductive hypothesis, the rest of the cats (consisting of n cats), are the same color, say color x. Take another cat away; call it "second cat". Remember that the rest consists of $n-1$ cats of color x. Now return the first cat to the set. The set has now n cats, $n-1$ of which are color x; the first cat has some unknown color. Since by inductive hypothesis a set of cats with n elements has just one color, the first cat must also have color x. Now bring the second cat back in, and we get back our set of $n+1$ cats, all of which have color x. To summarize, we assumed any n cats were of color x, and we have proven that any $n+1$ cats are of color x also. *Conclusion:* By virtue of mathematical induction, all cats are of the same color. Since this conclusion is obviously false, only one of the following three options is possible: the principle of mathematical induction is false or inapplicable to cats, logic is false or inapplicable to cats, or there is an error in the proof. Which one? Explain why!
5. Prove that $1^2+2^2+\cdots+n^2 = \frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n = \frac{1}{6}n(n+1)(2n+1)$, for all natural numbers $n \geq 1$.
6. Into how many regions do n straight lines divide the Euclidean plane, provided that no two lines are parallel and no more than two lines intersect in the same point? (Hint: use mathematical induction; how many regions are split into two by drawing one more line?)
7. (Extra credit.) Show that for every $n \geq 1$, among the first n^2 Fibonacci numbers $F_1, F_2, \ldots, F_n$ there is one which is divisible by n.