Problem Set 1
Due Friday, Sept. 17.

Formal Logic

Math 430, Fall 2004

1. Specify a symbol set appropriate for vector spaces over the rational numbers $\mathbb{Q}$.
2. Let S be a symbol set. Show that every S-term has length > 0.
3. Compute the complexity $\mathrm{Cp}(t)$ of the following S_F-term t (using the definition of Cp given in class):

$$\cdot \cdot + x1 + x + 11 + \cdot + 11x1$$

4. Let $S = \{E\}$ be the symbol set appropriate for equivalence relations introduced in class. Prove or disprove: the word

$$\forall v_0 \exists v_0 (\neg E v_0 v_1 \wedge v_2 = v_2)$$

   is an S-formula.
5. Let S be a symbol set and $\mathbf{A}'_S := \mathbf{A}_S \setminus \{(,)\}$. Define **$S$-formulas in Polish notation** (S-P-formulas) to be the smallest subset F of ${\mathbf{A}'_S}^*$ which satisfies (1), (2), (3), (5) in Lemma 1.5.7 (in the lecture notes), and instead of (4) the following rule:

   (4') If $\varphi, \psi \in F$, then $\wedge\varphi\psi, \vee\varphi\psi, \rightarrow \varphi\psi, \leftrightarrow \varphi\psi \in F$.

   Formulate a Unique Readability Theorem for S-P-formulas. (You are not required to prove the theorem. Note that this notation for formulas does away with parentheses!)
6. Write down the $S_{V(\mathbb{R})}$-sentences that express the axioms for vector spaces over $\mathbb{R}$.
7. Let S be a symbol set consisting only of the 2-place relation symbol R. Write down an S-formula which expresses that R is the graph of a 1-place function. (The *graph* of a 1-place function $f\colon A \to B$, where A and B are sets, is the set $\{(a, b) \in A \times B : b = f(a)\}$.)
8. Show that every subset of a countable set is countable.
9. Show that if M and N are countable sets, then $M \times N$ is countable. Use this to show, by induction on k, that if M is countable, then so is M^k for every k.

(please turn)

10. Let $M_0, M_1, \ldots$ be an infinite sequence of countable sets. Show that the union $\bigcup_{n \in \mathbb{N}} M_n$ is also countable. Use this together with the previous problem to give a proof of the fact that if $\mathbf{A}$ is a countable alphabet, then $\mathbf{A}^*$ is countable.

11. (Extra credit.) Consider the alphabet $\mathbf{A} = \{M, U, I\}$. Let P be the smallest subset of $\mathbf{A}^*$ which satisfies the following rules. Here x, y are words, and concatenated words are denoted by writing them one after the other.

    (P1) $MI \in P$;

    (P2) if $xI \in P$, then $xIU \in P$;

    (P3) if $Mx \in P$, then $Mxx \in P$;

    (P4) if $xIIIy \in P$, then $xUy \in P$;

    (P5) if $xUUy \in P$, then $xy \in P$.

    Prove the following claims:

    (a) $MUUIU \in P$.

    (b) $MU \notin P$.