FINAL EXAM

Math 31B, Fall Quarter 2008

Integration and Infinite Series

December 6, 2008

ANSWERS

Problem 1.

Determine the slope of the tangent line to the graph of the function

$$y = e^{x^x}$$

at $x = 1$. (5 points.)

Answer:

$$\frac{d}{dx}\, e^{x^x} = e^{x^x} \frac{d}{dx}\, x^x = e^{x^x} \frac{d}{dx}\, e^{x \ln x} = e^{x^x} x^x \frac{d}{dx}\, (x \ln x) = e^{x^x} x^x (\ln x + 1)$$

Substituting $x = 1$ yields the answer: the slope is e.

Problem 2. Find

$$\lim_{x\to 0} \frac{1-\cos 2x}{2x^2}$$

(5 points.)

Answer:
We use l'Hôpital's Rule twice:

$$\lim_{x\to 0} \frac{1-\cos 2x}{2x^2} = \lim_{x\to 0} \frac{\sin 2x}{2x} = \lim_{x\to 0} \frac{2\cos 2x}{2} = 1.$$

Problem 3.

1. Determine constants A and B such that
$$\frac{1}{x^2+x-6} = \frac{A}{x-2} + \frac{B}{x+3}.$$
2. Compute
$$\int \frac{x}{x^2+x-6}\,dx$$

(5+5 points.)

Answer:

1. We have $x^2+x-6 = (x-2)(x+3)$. Let $f(x) = \frac{1}{(x-2)(x+3)}$. We substitute $x = 0$ and $x = 1$ to get $f(0) = -\frac{1}{6}$ and $f(1) = -\frac{1}{4}$. This yields two equations for A, B:
$$-\frac{1}{6} = -\frac{1}{2}A + \frac{1}{3}B, \quad -\frac{1}{4} = -A + \frac{1}{4}B$$
From the second equation we obtain $A = \frac{1}{4}(B+1)$. Substituting this into the first equation we get $-\frac{1}{6} = -\frac{1}{8}(B+1)+\frac{1}{3}B$ and hence $B = -\frac{1}{5}$. Therefore $A = \frac{1}{5}$.
2. By part (1):
$$\frac{x}{x^2+x-6} = \frac{1}{5}\left(\frac{x}{x-2} - \frac{x}{x+3}\right)$$
and hence
$$\int \frac{x}{x^2+x-6}\,dx = \frac{1}{5}\left(\int \frac{x}{x-2}\,dx - \int \frac{x}{x+3}\,dx\right).$$
By long division $\frac{x}{x-2} = 1 + 2\frac{1}{x-2}$ and $\frac{x}{x+3} = 1 - 3\frac{1}{x+3}$, hence
$$\int \frac{x}{x-2}\,dx = x + 2\ln|x-2| + C_1, \quad \int \frac{x}{x+3}\,dx = x - 3\ln|x+3| + C_2$$
and thus
$$\int \frac{x}{x^2+x-6}\,dx = \frac{1}{5}\ln\left|(x-2)^2(x+3)^3\right| + C.$$
(Alternatively, you could have applied partial fraction decomposition to $\frac{x}{x^2+x-6}$.)

Problem 4.
Give the definition of the improper integral

$$\int_1^\infty \frac{1}{x^{3/2}}\,dx.$$

Then use your answer to evaluate the integral if it converges, or else show it diverges.

(5 points.)

Answer:
The given integral is improper because of the upper limit being infinite. We define the improper integral as a limit of definite integrals:

$$\int_1^\infty \frac{1}{x^{3/2}}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^{3/2}}\,dx = \lim_{b\to\infty}\left.\frac{-2}{x^{1/2}}\right|_1^b = -2\lim_{b\to\infty}\frac{1}{b^{1/2}} + 2 = 2.$$

Therefore the improper integral converges and has the value 2.

Problem 5.
A hard-boiled egg at 98°C is put in a sink filled with 18°C cold water. After 5 minutes, the temperature of the egg is 38°C. Assume that the water has not warmed appreciably and that the temperature of the egg changes at a rate proportional to the difference between its temperature and that of the water. (Newton's Law of Cooling.)

1. Write the differential equation and initial condition which model the temperature $E(t)$ of the egg t minutes after being placed in the water.
2. Find an explicit formula for $E(t)$.
3. How long does it take for the temperature $E(t)$ of the egg to reach 20°C?

(5+3+2 points.)

Answer:

1. $$\frac{dE}{dt} = -k(E-18), \qquad E(0) = 98,$$
   where k is a positive constant of proportionality.
2. The function $y = E(t) - 18$ satisfies the differential equation $y' = -ky$, hence $y = Ce^{-kt}$ for some constant C. This means that
   $$E(t) = Ce^{-kt} + 18.$$
   Since $E(0) = 98$ we must have $C = 80$. Also, $E(5) = 38$, hence $38 = 18 + 80e^{-5k}$ or $k = \frac{\ln 4}{5}$. Therefore
   $$E(t) = 18 + 80e^{-(\ln 4)t/5}.$$
3. Set $E = 20$ and solve for t. The result is $t = \frac{5 \ln 40}{\ln 4} \approx 13.3048$ minutes.

Problem 6.
Suppose $f(x) = \sin(x^2)$, and M_N denotes the value of the approximation of the integral

$$I = \int_0^1 f(x)\,dx$$

using the midpoint rule with N subintervals of $[a, b] = [0, 1]$. Recall the error bound

$$|M_N - I| \le \frac{K_2(b-a)^3}{24N^2}$$

where $K_2 \ge |f''(x)|$ for all $x \in [a, b]$.

1. Show that $|M_N - I| \le \frac{1}{4N^2}$. (Use the triangle inequality $|c + d| \le |c| + |d|$.)

2. Find an N such that $|M_N - I| \le \frac{1}{100}$.

(5+5 points.)

Answer:

1. We have

$$f'(x) = 2x\cos(x^2), \qquad f''(x) = 2(\cos(x^2) + x(-\sin(x^2))2x).$$

Hence, using the triangle inequality:

$$|f''(x)| \le 2(1 + 2) = 6.$$

This yields

$$|M_N - I| \le \frac{6}{24N^2} = \frac{1}{4N^2}.$$

2. It is enough to have $\frac{1}{4N^2} \le \frac{1}{100}$, or equivalently, $N^2 \ge \frac{100}{4} = 25$, that is, $N \ge 5$.

Problem 7. Use a trigonometric substitution to evaluate the integral

$$\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\, dx.$$

(10 points.)

Answer:
We use the substitution $x = \sin\theta$, where θ is in $[0, \pi/2]$. Then $\frac{dx}{d\theta} = \cos\theta$, and

$$\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta,$$

and therefore

$$\int \frac{x^2}{\sqrt{1-x^2}}\, dx = \int \sin^2\theta\, d\theta.$$

The latter integral can be found, e.g., using the reduction formula for sine:

$$\int \sin^2\theta\, d\theta = \frac{\theta}{2} - \frac{1}{2}\sin\theta\cos\theta + C$$

Note that $\theta = \sin^{-1} x$. Moreover $\cos\theta = \sqrt{1-x^2}$. Hence

$$\int \frac{x^2}{\sqrt{1-x^2}}\, dx = \frac{\sin^{-1} x}{2} - \frac{x\sqrt{1-x^2}}{2} + C$$

and therefore

$$\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\, dx = \frac{1}{2}\left(\sin^{-1} x - x\sqrt{1-x^2}\right)\Big|_0^1 = \frac{\pi}{4}.$$

Problem 8.

1. Find the sum of the infinite series

$$2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots + \left(\frac{2}{3}\right)^n + \cdots$$

2. Does the infinite series

$$\sum_{n=1}^{\infty} \frac{n^3}{n^5+1}$$

converge? Explain why or why not.

(3+2 points.)

Answer:

1.

$$\begin{aligned} 2 + \left(\frac{2}{3}\right)^2 + \cdots + \left(\frac{2}{3}\right)^n + \cdots &= 2 + \left(\frac{2}{3}\right)^2 \left[1 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots + \left(\frac{2}{3}\right)^n + \cdots\right] \\ &= 2 + \left(\frac{2}{3}\right)^2 \left[\frac{1}{1-\frac{2}{3}}\right] = 2 + \frac{4}{3} = \frac{10}{3}. \end{aligned}$$

2. The infinite series $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges and

$$0 < \frac{n^3}{n^5+1} \leq \frac{n^3}{n^5} = \frac{1}{n^2}$$

for all $n \geq 1$. Hence the given infinite series converges by the Comparison Test.

Problem 9.

1. Find the radius of convergence R of the power series
$$\sum_{n=0}^{\infty} \frac{3^n}{n!} x^n.$$

2. Use the Integral Test to show that
$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$
converges.

3. The power series
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n^3}(x-1)^n$$
has radius of convergence $R = 2$. Determine whether this power series converges for the endpoints of its interval of convergence.

(5+5+5 points.)

Answer:

1. We have
$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}}\right| = \frac{3^{n+1}}{(n+1)!}\frac{n!}{3^n} = \frac{3}{n+1}$$
and thus
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{3}{n+1} = 0.$$
Hence $R = \infty$. That is, this series converges for all x.

2. We have $\int \frac{1}{x^3}\,dx = -\frac{1}{2x^2} + C$, hence
$$\int_1^{\infty} \frac{1}{x^3}\,dx = \lim_{b\to\infty} -\frac{1}{2x^2} + \frac{1}{2} = \frac{1}{2}$$
converges. So by the Integral Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.

3. The endpoints of the interval of convergence of this power series are $x = -1$ and $x = 3$. We test for convergence:

$$x = 3: \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n^3}(3-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3},$$

which converges by the Leibniz alternating series test or by the fact that this series converges absolutely, namely $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by part (2).

$$x = -1: \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n^3}(-1-1)^n = \sum_{n=1}^{\infty} \frac{1}{n^3},$$

which converges by (2).

Problem 10. Evaluate

$$\int x^3 \sin(x^2)\,dx.$$

(10 points.)

Answer:
We use integration by parts:

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx.$$

We take

$$u(x) = x^2, \quad v'(x) = x\sin(x^2),$$

so

$$u'(x) = 2x, \quad v(x) = -\frac{1}{2}\cos(x^2),$$

thus

$$\int x^3 \sin(x^2)\,dx = -\frac{1}{2}x^2\cos(x^2) + \int x\cos(x^2)\,dx.$$

Now

$$\int x\cos(x^2)\,dx = \frac{1}{2}\sin(x^2)$$

and therefore

$$\int x^3 \sin(x^2)\,dx = \frac{1}{2}(\sin(x^2) - x^2\cos(x^2)) + C.$$

Problem 11. Consider the function $f(x) = 30\,x^2 \cos(\sqrt{x})$.

1. Find the Maclaurin series $T(x)$ for $f(x)$ in the form $T(x) = \sum\limits_{n=0}^{\infty} a_n x^n$.
2. Use $T(x)$ to find $f^{(5)}(0)$.
3. Estimate
$$|f(0.1) - 0.3|.$$
(Hint: look at the alternating series $T(0.1)$.)

(5+3+2 points.)

Answer:

1. The Maclaurin series for $\cos(x)$ is given by $\sum\limits_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, with radius of convergence ∞. Hence $\cos(\sqrt{x})$ has Maclaurin series
$$\sum_{n=0}^{\infty} (-1)^n \frac{(\sqrt{x})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n)!}.$$
Thus $f(x) = 30x^2 \cos(\sqrt{x})$ has Maclaurin series
$$T(x) = 30 \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+2}}{(2n)!} = \sum_{n=2}^{\infty} \frac{30\,(-1)^n}{(2n-4)!} x^n.$$
2. By part (1),
$$\frac{f^{(5)}(0)}{5!} = \frac{30\,(-1)^5}{(2 \cdot 5 - 4)!} = -\frac{1}{24}, \qquad \text{so } f^{(5)}(0) = -5.$$
3. We note that $0.3 = 30 \cdot 0.1^2$ is the first term $(-1)^2 b_2$ of the alternating series
$$T(0.1) = \sum_{n=2}^{\infty} (-1)^n \underbrace{\frac{30 \cdot 0.1^n}{(2n-4)!}}_{b_n} = \sum_{n=2}^{\infty} (-1)^n b_n,$$
which satisfies the hypotheses of the alternating series test. Thus, by the error bound in the alternating series test discussed in class,
$$|f(0.1) - 0.3| \leq b_3 = \frac{30 \cdot 0.1^3}{(2 \cdot 3 - 4)!} = \frac{15}{1000} = 0.015.$$

Problem 12. Consider the sequence $\{c_n\}$ given by

$$c_n = \frac{1}{n^2+1} + \frac{1}{n^2+2} + \cdots + \frac{1}{n^2+n}.$$

1. Show that
$$c_n \leq \frac{n}{n^2+1}.$$

2. Determine whether $\{c_n\}$ converges, and if so, compute its limit.

(3+2 points.)

Answer:

1. We have
$$n^2 + i \geq n^2 + 1 \qquad \text{for } i = 1, \ldots, n,$$
so
$$\frac{1}{n^2+i} \leq \frac{1}{n^2+1} \qquad \text{for } i = 1, \ldots, n$$
and hence
$$c_n = \frac{1}{n^2+1} + \frac{1}{n^2+2} + \cdots + \frac{1}{n^2+n} \leq \frac{n}{n^2+1}.$$

2. Both sequences $a_n = 0$ and $b_n = \frac{n}{n^2+1}$ converge to 0, since
$$\lim_{x\to\infty} \frac{x}{x^2+1} = \lim_{x\to\infty} \frac{1}{2x} = 0$$
by l'Hôpital, and
$$a_n \leq c_n \leq b_n$$
by part (1). Hence c_n also converges to 0, by the Squeeze Theorem.