MIDTERM EXAM 1

Math 31B, Spring Quarter 2011

Integration and Infinite Series

April 15, 2011

ANSWERS

Problem 1. Consider the function

$$f(x) = x^3 + 2x + 4.$$

1. Why is f invertible? Justify your answer.
2. Denoting the inverse of f by g, calculate $g'(4)$.

(5+5 points.)

Answer:

1. We have $f'(x) = 3x^2 + 2 > 0$ for every x, so f is strictly increasing and hence one-to-one, i.e., invertible.
2. We have $g'(x) = \frac{1}{f'(g(x))}$ for every x. To find $g(4)$ we solve $f(x) = 4$ for x and obtain $x = 0$, so $g(4) = 0$ and hence

$$g'(4) = \frac{1}{3 \cdot 0^2 + 2} = \frac{1}{2}.$$

Problem 2. Find the y-coordinate of the intersection of the line $4x = 5$ with the tangent line to the curve $y = 2^{x^2}$ at the point $(1, 2)$. (10 points.)

Answer:
Observe that
$$\frac{d}{dx}(2^{x^2}) = (\ln 2)2^{x^2} \cdot 2x.$$
Therefore, the slope of the tangent line at $x = 1$ is $4(\ln 2)$. Now the tangent line satisfies
$$y - 2 = 4(\ln 2)(x - 1).$$
To find the y-coordinate of the intersection of this line with the line $4x = 5$ amounts to setting $x = 5/4$ and solving for y:
$$y = 2 + 4(\ln 2)(5/4 - 1) = 2 + \ln 2.$$

Problem 3.
The atmospheric pressure $P(h)$ (in pounds per square inch) at a height h (in miles) above sea level on earth satisfies the differential equation

$$P' = -kP \qquad \text{for some positive constant } k.$$

Measurements with a barometer show that

$$P(0) = 14.7 \qquad \text{and} \qquad P(10) = 2.$$

Find k. (20 points.)

Answer:
We have

$$P(h) = P_0 e^{-kh}$$

where $P_0 = P(0) = 14.7$. Since $P(10) = 2$ we have

$$2 = 14.7 \cdot e^{-10k}$$

and hence, taking ln on both sides of the equation and solving for k:

$$k = -\frac{1}{10} \ln\left(\frac{2}{14.7}\right).$$

Problem 4. Compute the following limits:

1.
$$\lim_{x\to\infty} \frac{(1+\ln x)^{1/3}}{1+x}.$$

2.
$$\lim_{x\to 0} (1+2x)^{2/x}.$$

(10+10 points.)

Answer:

1. We have
$$\lim_{x\to\infty}(1+\ln x) = \infty$$
and hence
$$\lim_{x\to\infty}(1+\ln x)^{1/3} = \infty.$$
Since also
$$\lim_{x\to\infty}(1+x) = \infty,$$
l'Hôpital's Rule applies, and we have
$$\lim_{x\to\infty} \frac{(1+\ln x)^{1/3}}{1+x} = \lim_{x\to\infty} \frac{(1/3)(1+\ln x)^{1/3-1}\cdot(1/x)}{1} =$$
$$\frac{1}{3}\lim_{x\to\infty} \frac{1}{(1+\ln x)^{2/3}\cdot x} = 0.$$

2. We have
$$\ln(1+2x)^{2/x} = \frac{2}{x}\ln(1+2x) = \frac{2\ln(1+2x)}{x}.$$
Now as $x \to 0$ both numerator and denominator of this fraction approach 0, so l'Hôpital's Rule applies and we obtain
$$\lim_{x\to 0} \frac{2\ln(1+2x)}{x} = \lim_{x\to 0} \frac{2/(1+2x)\cdot 2}{1} = 4.$$
Hence
$$\lim_{x\to 0}(1+2x)^{2/x} = \lim_{x\to 0} e^{\ln((1+2x)^{2/x})} = e^4.$$

Problem 5.
Evaluate the following indefinite integrals, using substitution if necessary.

1. $\displaystyle\int \sin x \; 4^{\cos x} dx;$
2. $\displaystyle\int \frac{4x-2}{9-4x+4x^2} dx.$

(10+10 points.)

Answer:

1. Let $u = \cos x$. Then $du = -\sin x dx$, so:
$$\int \sin x \; 4^{\cos x} dx = -\int 4^u du = -\frac{1}{\ln 4}\int (\ln 4)4^u du = -\frac{4^u}{\ln 4} + C = -\frac{4^{\cos x}}{\ln 4} + C.$$
2. Let $u = 9-4x+4x^2$. Then $du = (-4+8x)dx$, and $-\frac{1}{2}du = (4x-2)dx$, so:
$$\int \frac{4x-2}{9-4x-4x^2} dx = -\frac{1}{2}\int \frac{1}{u} du = -\frac{1}{2}\ln|u| + C = -\frac{1}{2}\ln|9-4x+4x^2| + C.$$

Problem 6.
How large should N be to guarantee that the error in the Trapezoidal Rule approximation T_N to

$$\int_0^2 x^3\,dx$$

is accurate within 0.1? (10 points.)

Answer:
Let $f(x) = x^3$. Then

$$f'(x) = 3x^2, \qquad f''(x) = 6x.$$

Note that f'' is increasing and $f''(x) \geq 0$ on the interval $0 \leq x \leq 2$. So

$$|f''(x)| \leq |f''(2)| = 12 \qquad \text{for } 0 \leq x \leq 2,$$

hence we can choose $K_2 = 12$. We now use the error bound for T_N:

$$\text{Error}(T_N) = \left| \int_0^2 x^3\,dx - T_N \right| \leq \frac{K_2(b-a)^3}{12N^2} = \frac{8}{N^2}.$$

We'd like to have $\text{Error}(T_N) \leq 0.1$, and this is certainly the case if

$$\frac{8}{N^2} \leq 0.1,$$

or equivalently

$$N^2 \geq 80.$$

So $N = 9$ is enough.

Problem 7. Consider the differential equation

$$y'(t) = 5(y(t) + 13).$$

1. Without justification, write down the general solution of this differential equation.
2. Find a formula for the solution $y(t)$ satisfying $y(0) = a$, where a is a given constant. (Your answer should be a formula for $y(t)$ that depends on the parameter a.)

(5+5 points.)

Answer:

1. The solution is $y(t) = -13 + Ce^{5t}$, where C is a constant.
2. Evaluate $y(t) = -13 + Ce^{5t}$ at $t = 0$:

$$a = y(0) = -13 + Ce^{5\cdot 0} = -13 + C,$$

so $C = a + 13$. Therefore $y(t) = -13 + (a + 13)e^{5t}$.