
FINAL EXAM

Math 31B, Spring Quarter 2011

Integration and Infinite Series

June 8, 2011

ANSWERS

Problem 1. Find derivative of

$$\ln\left(\frac{\sin(x)+1}{x^2+2}\right).$$

(5 points.)

Answer:

$$\frac{d}{dx}\ln\left(\frac{\sin(x)+1}{x^2+2}\right) = \frac{x^2+2}{\sin(x)+1} \times \frac{\cos(x)(x^2+2)-(\sin(x)+1)(2x)}{(x^2+2)^2}$$

Problem 2.

The population of a city grows exponentially. Suppose that the population is currently 2 million people, and that the population after 5 years is going to be two times the current population. Compute the population after 10 years.

(10 points.)

Answer:

The population grows according to $P(t) = P_0 e^{rt}$, where t is the time in years. From the information we have that

$$P_0 e^{5r} = 2P_0.$$

Simplifying we have that

$$e^{5r} = 2$$

and thus

$$r = \frac{\ln 2}{5}.$$

Thus, after 10 years the population is (in million):

$$P(10) = 2e^{10r} = 2e^{\frac{10 \ln 2}{5}} = 2 \cdot e^{2 \ln 2} = 2 \cdot 2^2 = 8.$$

Problem 3. Compute the sum of the infinite series

$$2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots + \left(\frac{2}{3}\right)^n + \cdots$$

(5 points.)

Answer:

$$\begin{aligned} 2 + \left(\frac{2}{3}\right)^2 + \cdots + \left(\frac{2}{3}\right)^n + \cdots &= 2 + \left(\frac{2}{3}\right)^2 \left[1 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots + \left(\frac{2}{3}\right)^n + \cdots\right] \\ &= 2 + \left(\frac{2}{3}\right)^2 \left[\frac{1}{1 - \frac{2}{3}}\right] = 2 + \frac{4}{3} = \frac{10}{3}. \end{aligned}$$

Problem 4.

1. Use the Integral Test to show that the following series converges:

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

2. Does the infinite series

$$\sum_{n=1}^{\infty} \frac{n^3}{n^5+1}$$

converge? Explain why or why not. (Hint: you might want to use the result of part (1.).)

3. Does the series

$$\sum_{n=1}^{\infty} \frac{2+(-1)^n}{n}$$

converge? Justify your answer.

(5+5+5 = 15 points.)

Answer:

1. We have $\int \frac{1}{x^2}\,dx = -\frac{1}{x} + C$, hence

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{R\to\infty} -\frac{1}{R} + 1 = 1$$

converges. So by the Integral Test, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

2. The infinite series $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges by part (1.) and $0 < \frac{n^3}{n^5+1} \leq \frac{n^3}{n^5} = \frac{1}{n^2}$ for all $n \geq 1$. Hence the given infinite series converges by the Comparison Test.

3. Note that

$$\frac{2+(-1)^n}{n} = \begin{cases} 1/n & \text{if } n \text{ is odd} \\ 3/4 & \text{if } n \text{ is even.} \end{cases}$$

Therefore $\frac{2+(-1)^n}{n} \geq 1/n$ for all n. All terms are positive, so we may apply the Comparison Test. Since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so does $\sum_{n=1}^{\infty} \frac{2+(-1)^n}{n}$ by the Comparison Test.

Problem 5.

1. Find the radius of convergence R of the power series

$$\sum_{n=0}^{\infty} \frac{3^n}{n!} x^n.$$

2. The power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n^2} (x-1)^n$$

has radius of convergence $R = 2$. Determine whether this power series converges for the endpoints of its interval of convergence. (Hint: you might want to use the result of part (1.) of the previous problem.)

(5+5 = 10 points.)

Answer:

1. We have

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}} \right| = \frac{3^{n+1}}{(n+1)!} \frac{n!}{3^n} = \frac{3}{n+1}$$

and thus

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{3}{n+1} = 0.$$

Hence $R = \infty$. That is, this series converges for all x.

2. The endpoints of the interval of convergence of this power series are $x = -1$ and $x = 3$. We test for convergence:

$$x = 3 : \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n^2} (3-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2},$$

which converges by the Leibniz alternating series test or by the fact that this series converges absolutely, namely $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the hint.

$$x = -1 : \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n^2} (-1-1)^n = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

which converges by the hint.

Problem 6. Evaluate

$$I = \int e^x \cos x \, dx.$$

(Hint: integration by parts.) (10 points.)

Answer:

Let $u = e^x$, $v' = \cos x$, then $u' = e^x$, $v = \sin x$, thus

$$I = e^x \sin x - \int e^x \sin x \, dx.$$

Now for the integral $\int e^x \sin x \, dx$, use integration by parts again, set $u = e^x$, $v' = \sin x$, so $u' = e^x$, $v = -\cos x$, then

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx = -e^x \cos x + I.$$

So we have:

$$I = e^x \sin x - \int e^x \sin x \, dx = e^x \sin x + e^x \cos x - I.$$

Solving for I, we have

$$I = \frac{e^x \cos x + e^x \sin x}{2} + C.$$

Problem 7. Consider the following improper integral:

$$I = \int_2^\infty \frac{2x}{x^2-1} dx.$$

1. Use the Comparison Test to determine whether or not I converges.
2. Compute $\displaystyle\int_2^R \frac{2x}{x^2-1}\, dx$, where $R > 2$.
3. By computing $I = \displaystyle\lim_{R\to\infty} \int_2^R \frac{2x}{x^2-1} dx$, justify your conclusion in (1.).

(3+5+2 = 10 points.)

Answer:

1. $\displaystyle\frac{2x}{x^2-1} = \frac{2}{x - \frac{1}{x}} > \frac{2}{x}$ for $x > 2$. Also, $\displaystyle\int_2^\infty \frac{2}{x}\, dx$ diverges. Hence by the Comparison Test, $\displaystyle\int_2^\infty \frac{2x}{x^2-1}\, dx$ also diverges.
2. We use the partial fraction decomposition:

   $$\frac{2x}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1} = \frac{A(x+1)+B(x-1)}{(x-1)(x+1)} = \frac{(A+B)x + A - B}{x^2-1}.$$

   Hence $A + B = 2$ and $A - B = 0$, so $A = B = 1$. This allows us to compute the integral:

   $$\int_2^R \frac{1}{x-1} dx + \int_2^R \frac{1}{x+1} dx = (\ln|x-1| + \ln|x+1|)\Big|_2^R = \ln|(R-1)(R+1)| - \ln 3.$$

3. By (2.),

   $$I = \lim_{R\to\infty} (\ln|(R-1)(R+1)|) - \ln 3 = \infty,$$

   therefore I diverges.

Problem 8.
Let f be a differentiable function and assume that f' is continuous on an interval $[a, b]$.

1. Give a formula for the arc length of the graph of $y = f(x)$ over $[a, b]$.
2. Suppose $f(x) > 0$ for each x and let $g(x) = \ln(f(x))$. Express the arc length of the graph of $y = g(x)$ over $[a, b]$ as an integral depending only on $f(x)$ and its derivatives.

(3+2 = 5 points.)

Answer:

1. The arc length is given by $\int_a^b \sqrt{1 + f'(x)^2}\, dx$.
2. We have $g'(x) = \frac{f'(x)}{f(x)}$ and hence the desired arc length is given by

$$\int_a^b \sqrt{1 + g'(x)^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{f'(x)}{f(x)}\right)^2}\, dx = \int_a^b \frac{\sqrt{f(x)^2 + f'(x)^2}}{f(x)}\, dx.$$

Problem 9. Compute

$$\lim_{x\to 0}(\sin x)^x.$$

(5 points.)

Answer:
Let $f(x) = (\sin x)^x$. Then

$$\ln(f(x)) = x\ln(\sin x) = \frac{\ln(\sin x)}{1/x}.$$

Note that

$$\lim_{x\to 0}\frac{\ln(\sin x)}{1/x} = \lim_{x\to 0}\frac{\frac{\cos x}{\sin x}}{-1/x^2} = \lim_{x\to 0}\left(\frac{x}{\sin x}\right)\lim_{x\to 0}(-x\cos x) = 1\times 0 = 0,$$

where we used l'Hôpital's Rule for the first equality and the fact (also obtain via l'Hôpital's Rule) that

$$\lim_{x\to 0}\frac{x}{\sin x} = \lim_{x\to 0}\frac{1}{\cos x} = 1.$$

Consequently, $\lim_{x\to 0} f(x) = 1$.

Problem 10. Consider the following integral:

$$I = \int_0^1 e^{x^2}\, dx.$$

1. Compute the 4th trapezoidal aproximation T_4 to I.
2. Find a bound for $\mathrm{Error}(T_4) = |T_4 - I|$.
3. Explain graphically whether T_4 is larger or smaller than I.

(5+3+2 = 10 points.)

Answer:

1. $$T_4 = \frac{1}{2}\cdot\frac{1}{4}(e^0 + 2e^{\frac{1}{16}} + 2e^{\frac{1}{4}} + 2e^{\frac{9}{16}} + e) = \frac{1}{8}(1 + 2e^{\frac{1}{16}} + 2e^{\frac{1}{4}} + 2e^{\frac{9}{16}} + e).$$

2. We have
$$f'(x) = 2xe^{x^2}, \qquad f''(x) = 2e^{x^2} + 4x^2e^{x^2},$$
so if $0 \le x \le 1$ then $|f''(x)| \le f''(1) = 2e + 4e = 6e = K_2$, hence
$$\mathrm{Error}(T_4) \le \frac{6e(1-0)^3}{12\cdot 4^2} = \frac{e}{32}$$

3. We have $f''(x) \ge 0$ for $0 \le x \le 1$, thus $f(x)$ is concave up and hence T_4 is too large.

Problem 11. Evaluate

$$I = \int \frac{x}{(1-x^2)^{\frac{3}{2}}} dx.$$

(Hint: substitution $x = \sin\theta$.) (10 points.)

Answer:
Let $x = \sin\theta$, so $dx = \cos\theta d\theta$. Then

$$I = \int \frac{\sin\theta\cos\theta}{\cos^3\theta} d\theta = \int \frac{\sin\theta}{\cos^2\theta} d\theta.$$

Now take $u = \cos\theta$, so $du = -\sin\theta$. Then

$$I = -\int \frac{du}{u^2} = \frac{1}{u} + C = \frac{1}{\cos\theta} + C = \frac{1}{\sqrt{1-x^2}} + C.$$

Problem 12.
Show that the Nth Taylor polynomial for $f(x) = \frac{1}{1+x}$ centered at $c = 1$ is

$$T_N(x) = \sum_{n=0}^{N} \frac{(-1)^n (x-1)^n}{2^{n+1}}.$$

(5 points.)

Answer:
Write

$$f(x) = \frac{1}{1+x} = \frac{1}{2(1-(-1/2)(x-1))}.$$

Using the geometric series expansion, for $|(1/2)(x-1)| < 1$ we have the following power series expansion for $f(x)$:

$$f(x) = \frac{1}{2} \sum_{n=0}^{\infty} \left(-\frac{1}{2}(x-1)\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (x-1)^n.$$

This is the Taylor series for $f(x)$ centered at $c = 1$; its Nth partial sum is the Nth Taylor polynomial for $f(x)$ centered at $c = 1$, so

$$T_N(x) = \sum_{n=0}^{N} \frac{(-1)^n (x-1)^n}{2^{n+1}}$$

as claimed. (This answer can also be obtained by deducing a formula for $f^{(n)}(x)$ and this way computing T_N directly.)