Problem Set 8
Solutions

Mathematical Logic

Math 114L, Spring Quarter 2008

1. No, v_2 is not substitutable for v_0, since v_2 is a quantified variable occurring in the term (namely, v_2) to be substituted.

2. In both parts show by induction on φ simultaneously that the given term t is substitutable in φ, and $\varphi^t_x = \varphi$. We only do it here for (b). If φ is atomic, t is always substitutable for x in φ, and if in addition $x \notin \text{free}(\varphi)$, then x does not occur in φ, so $\varphi^t_x = \varphi$. The inductive steps for $\varphi = (\neg\alpha)$ and $\varphi = (\alpha \to \beta)$ are routine. Suppose now that $\varphi = \forall y\alpha$, and $x \notin \text{free}(\varphi)$. Then by definition, t is substitutable for x in φ. Moreover, if $x = y$, then $\varphi^x_t = \varphi$. If $x \neq y$, then $\varphi^x_t = \forall y(\alpha^x_t)$; since $\text{free}(\varphi) = \text{free}(\alpha) \setminus \{y\}$ and $x \neq y$, we have $x \notin \text{free}(\alpha)$, so t is substitutable for x in α and $\alpha^x_t = \alpha$, by inductive hypothesis, hence $\varphi^x_t = \varphi$.

3. (a) Observe that any formula can built up from prime formulas by use of $\neg$ and $\longrightarrow$. To see this, let Φ be the set of formulas that can be built up from the prime formulas by use of $\neg$ and $\longrightarrow$. Then Φ includes all atomic formulas (which are prime). Φ is closed under quantification (because $\forall\, \boldsymbol{v}_i\, \varphi$ is prime). And Φ is closed under $\neg$ and $\longrightarrow$. So Φ includes all formulas.

   Now suppose we are given any $\mathfrak{A}$ and s, and we define the truth assignment v as specified. We seek to show that for every formula α

$$\overline{v}(\alpha) = T \quad \text{iff} \quad \mathfrak{A} \models \alpha[s]. \qquad (\star)$$

   We do this by induction.

   Basis: α is prime. Then $(\star)$ holds by the definition of v.

   Inductive step for $\neg$:

$$\begin{aligned} \overline{v}(\neg\,\alpha) = T \quad &\Leftrightarrow \quad \overline{v}(\alpha) \neq T \quad \text{by definition of } \overline{v} \\ &\Leftrightarrow \quad \mathfrak{A} \not\models \alpha[s] \quad \text{by the inductive hypothesis} \\ &\Leftrightarrow \quad \mathfrak{A} \models \neg\,\alpha[s] \quad \text{by definition of } \models \end{aligned}$$

   Inductive step for $\longrightarrow$:

$$\begin{aligned} \overline{v}(\alpha \longrightarrow \beta) = T \quad &\Leftrightarrow \quad \overline{v}(\alpha) = F \text{ or } \overline{v}(\beta) = T \quad \text{by definition of } \overline{v} \\ &\Leftrightarrow \quad \mathfrak{A} \not\models \alpha[s] \text{ or } \mathfrak{A} \models \beta[s] \quad \text{by inductive hypothesis} \\ &\Leftrightarrow \quad \mathfrak{A} \models (\alpha \longrightarrow \beta)[s] \quad \text{by definition of } \models \end{aligned}$$

   Hence by induction, $(\star)$ holds for all formulas α.

(b) Assume $\mathfrak{A}$ satisfies every member of Γ with s. Define the truth assignment v as in part (a). By (a), $\overline{v}(\gamma) = T$ for every γ in Γ. So if Γ tautologically implies φ, then $\overline{v}(\varphi) = T$. Now by part (a) again, $\mathfrak{A} \models \varphi[s]$.

Since $\mathfrak{A}$ and s were arbitrary, we conclude that Γ logically implies φ.

4. We seek a deduction of $(\forall x\, \varphi \to \neg \forall x \neg \varphi)$. This is tautologically equivalent to $\neg(\forall x\, \varphi \wedge \forall x \neg \varphi)$. Both $(\forall x\, \varphi \to \varphi)$ and $(\forall x \neg \varphi \to \neg \varphi)$ are axioms, and they tautologically imply what we want.

Let τ be the formula:

$$(\forall x\, \varphi \to \varphi) \to [(\forall x \neg \varphi \to \neg \varphi) \to (\forall x\, \varphi \to \neg \forall x \neg \varphi)]$$

Then τ is a tautology, having the form

$$(\mathbf{A} \to \mathbf{C}) \to [(\mathbf{B} \to \neg \mathbf{C}) \to (\mathbf{A} \to \neg \mathbf{B})].$$

Then one deduction is the following quintuple of formulas:

$$\begin{array}{l} \langle \tau, \\ (\forall x\, \varphi \to \varphi), \\ [(\forall x \neg \varphi \to \neg \varphi) \to (\forall x\, \varphi \to \neg \forall x \neg \varphi)], \\ (\forall x \neg \varphi \to \neg \varphi), \\ (\forall x\, \varphi \to \neg \forall x \neg \varphi)\rangle \end{array}$$

where the third and fifth formulas are obtained by modus ponens from earlier formulas.

5. We want $\vdash Py \leftrightarrow \forall x(x = y \to Px)$. Working backwards, we see that it suffices to obtain lines 4 and 9 below.

1. $\vdash y = x \to (Py \to Px)$ equality axiom
2. $\vdash x = y \to y = x$ (equality is symmetric, proved in class)
3. $\vdash Py \to (x = y \to Px)$ 1,2; rule T
4. $Py \vdash x = y \to Px$ 3; MP
5. $Py \vdash \forall x(x = y \to Px)$ 4; generalization theorem
6. $\vdash Py \to \forall x(x = y \to Px)$ 5; deduction theorem
7. $\forall x(x = y \to Px) \vdash y = y \to Py$ substitution axiom & MP
8. $\vdash y = y$ equality axiom
9. $\forall x(x = y \to Px) \vdash Py$ 7,8; MP
10. $\vdash \forall x(x = y \to Px) \to Py$ 9; deduction theorem
11. $\vdash Py \leftrightarrow \forall x(x = y \to Px)$ 6,10; rule T

6. We want

$$\vdash (\forall x(\neg Px \to Qx) \to \forall y(\neg Qy \to Py)).$$

By the deduction theorem and the generalization theorem, it suffices to show that

$$\forall x(\neg Px \to Qx) \vdash (\neg Qy \to Py).$$

And that we can do.

1. $\forall x(\neg Px \to Qx) \vdash (\neg Py \to Qy)$ substitution axiom, MP
2. $\vdash (\neg Py \to Qy) \to (\neg Qy \to Py)$ tautology
3. $\forall x(\neg Px \to Qx) \vdash (\neg Qy \to Py)$ 1,2; MP
4. $\forall x(\neg Px \to Qx) \vdash \forall y(\neg Qy \to Py)$ 3; generalization theorem
5. $\vdash (\forall x(\neg Px \to Qx) \to \forall y(\neg Qy \to Py))$ 4; deduction theorem