Problem Set 7
Solutions

Mathematical Logic

Math 114L, Spring Quarter 2008

1. This follows by checking the definition of an automorphism for $\alpha \circ \beta$ and α^{-1}. I'll just do it for $\alpha \circ \beta$. Clearly $\alpha \circ \beta$ is a bijection $A \to A$, since the composition of two bijections is a bijection. Moreover, if c is a constant symbol, then
$$(\alpha \circ \beta)(c^{\mathfrak{A}}) = \alpha(\beta(c^{\mathfrak{A}})) = \alpha(c^{\mathfrak{A}}) = c^{\mathfrak{A}}$$
since $c^{\mathfrak{A}} = \alpha(c^{\mathfrak{A}}) = \beta(c^{\mathfrak{A}})$. If f is an n-place function symbol and $a_1, \dots, a_n \in A$, then
$$\begin{aligned}(\alpha \circ \beta)\big(f^{\mathfrak{A}}(a_1, \dots, a_n)\big) &= \alpha\big(\beta\big(f^{\mathfrak{A}}(a_1, \dots, a_n)\big)\big)\\ &= \alpha\big(f^{\mathfrak{A}}\big(\beta(a_1), \dots, \beta(a_n)\big)\big)\\ &= f^{\mathfrak{A}}\big(\alpha(\beta(a_1)), \dots, \alpha(\beta(a_n))\big)\\ &= f^{\mathfrak{A}}\big((\alpha \circ \beta)(a_1), \dots, (\alpha \circ \beta)(a_n)\big).\end{aligned}$$
Here in the second equation we used that β is an automorphism, and in the third equation we used that α is an automorphism. For an n-place relation symbol R and $a_1, \dots, a_n \in A$ we have
$$(a_1, \dots, a_n) \in R^{\mathfrak{A}} \qquad \Longleftrightarrow \qquad \big(\beta(a_1), \dots, \beta(a_n)\big) \in R^{\mathfrak{A}}$$
since β is an automorphism of $\mathfrak{A}$. Since α is an automorphism of $\mathfrak{A}$:
$$\big(\beta(a_1), \dots, \beta(a_n)\big) \in R^{\mathfrak{A}} \quad \Longleftrightarrow \quad \big(\alpha(\beta(a_1)), \dots, \alpha(\beta(a_n))\big) \in R^{\mathfrak{A}}.$$
Hence
$$(a_1, \dots, a_n) \in R^{\mathfrak{A}} \qquad \Longleftrightarrow \qquad \big((\alpha \circ \beta)(a_1), \dots, (\alpha \circ \beta)(a_n)\big) \in R^{\mathfrak{A}}.$$
This shows that $\alpha \circ \beta$ is an automorphism of $\mathfrak{A}$.

2. The automorphisms of $\mathfrak{Z} = (\mathbb{Z}, <^{\mathfrak{Z}})$ are exactly the maps $x \mapsto x+k \colon \mathbb{Z} \to \mathbb{Z}$ (for a constant $k \in \mathbb{Z}$). (So the map $\alpha \mapsto \alpha(0)$ is an automorphism of the automorphism group of $\mathfrak{Z}$ onto the group $(\mathbb{Z}, +)$.)

3. Clearly $\emptyset$ is defined by the formula $\neg v_1 = v_1$. Suppose that φ and ψ with $\mathrm{fr}(\varphi), \mathrm{fr}(\psi) \subseteq \{v_1, \dots, v_k\}$ define D and E, respectively, that is,
$$D = \big\{(a_1, \dots, a_k) \in A^k : \mathfrak{A} \models \varphi[a_1, \dots, a_k]\big\}$$
and
$$E = \big\{(a_1, \dots, a_k) \in A^k : \mathfrak{A} \models \psi[a_1, \dots, a_k]\big\}.$$

(a) We have

$$\begin{aligned}
D \cap E &= \big\{(a_1, \ldots, a_k) \in A^k : \mathfrak{A} \models (\varphi \wedge \psi)[a_1, \ldots, a_k]\big\},\\
D \cup E &= \big\{(a_1, \ldots, a_k) \in A^k : \mathfrak{A} \models (\varphi \vee \psi)[a_1, \ldots, a_k]\big\},\\
A^k \setminus D &= \big\{(a_1, \ldots, a_k) \in A^k : \mathfrak{A} \models \neg\varphi[a_1, \ldots, a_k]\big\},
\end{aligned}$$

showing that $D \cap E$, $D \cup E$ and $A^k \setminus D$ are definable in $\mathfrak{A}$.

(b) We have

$$\begin{aligned}
\pi(D) &= \big\{\pi(a_1, \ldots, a_k) : (a_1, \ldots, a_k) \in D\big\}\\
&= \big\{(a_1, \ldots, a_{k-1}) \in A^{k-1} : (a_1, \ldots, a_{k-1}, a_k) \in D\\
&\qquad\qquad\qquad\qquad\qquad\text{for some } a_k \in A\big\}
\end{aligned}$$

and hence

$$\pi(D) = \big\{(a_1, \ldots, a_{k-1}) \in A^{k-1} : \mathfrak{A} \models \exists v_k\varphi\ [a_1, \ldots, a_{k-1}]\big\}.$$

This shows that $\pi(D)$ is definable in $\mathfrak{A}$.

4. (a) Here is an inductive definition of the set of positive formulas:
   - Every atomic formula is positive;
   - if φ, ψ are positive, then $(\varphi \to \psi)$ is positive;
   - if φ is positive, then $\forall v_i\varphi$ is positive.

   (b) Consider the S-structure $\mathfrak{A}$ whose universe consists of a single element a (so $A^n = \{(a, a, \ldots, a)\}$ for every $n > 0$), and where
   - every n-place function symbol f is interpreted as the function $(a, a, \ldots, a) \mapsto a\colon A^n \to A$;
   - every n-place relation symbol R is interpreted as the relation $R^{\mathfrak{A}} = \{(a, a, \ldots, a)\}$;
   - every constant symbol c is interpreted as $c^{\mathfrak{A}} = a$.

   There is only one assigment s in $\mathfrak{A}$, and clearly every atomic formula holds in $\mathfrak{A}$ with s. By induction on the construction of positive formulas, it follows that $\mathfrak{A} \models \varphi[s]$ for every positive formula φ. (I am leaving out some details here which you were supposed to provide!)

5. Among many possible solutions, here is one: Let φ be the sentence

$$\forall x\forall y(fx = fy \to x = y) \wedge \exists z\forall x(\neg fx = z).$$

A structure $\mathfrak{A} = (A, f^{\mathfrak{A}})$ satisfies φ exactly if the map $f^{\mathfrak{A}}\colon A \to A$ is injective, but not surjective. But a set $A \neq \emptyset$ is infinite if and only if there exists a map $A \to A$ which is injective and not surjective. Hence φ cannot hold in $\mathfrak{A}$ with finite universe A. For a language with a single 2-place relation symbol R, let φ be a sentence which expresses that R is an **ordering without right endpoint**:

$$\varphi = \forall xRxx \wedge \forall x\forall y\forall z(Rxy \wedge Ryz \to Rxz) \wedge \forall x\exists yRxy$$

Any structure $\mathfrak{A} = (A, R^{\mathfrak{A}})$ satisfying φ has infinitely many elements.