Problem Set 2
Solutions

Mathematical Logic

Math 114L, Spring Quarter 2008

1. Using obvious shorthand notation for repeated $\wedge$ and $\vee$:
   (a) $\bigwedge_{i=1}^{m} \bigvee_{j=1}^{m} a_{ij}$
   (b) $\bigvee_{i=1}^{m} \bigvee_{j=1}^{m} a_{ij}$
2. Let v be a truth assignment for a set S of sentence symbols. To show uniqueness of $\bar{v}$, suppose there is another function $\tilde{v}$ satisfying conditions 0–5. One proves by using the induction principle that for every wff α built up from S we get $\bar{v}(\alpha) = \tilde{v}(\alpha)$. To show the existence of the function $\bar{v}$, one also proceeds by induction (or rather, recursion), also using the unique readability theorem. (I'm skipping the details here— this will be discussed in the TA session.)
3. (a) Yes, $(((P \to Q) \to P) \to P)$ is a tautology: Suppose v is a truth assignment. If $v(P) = F$, then $\bar{v}\big((P \to Q)\big) = T$, so
   $$\bar{v}\big(((P \to Q) \to P)\big) = F$$
   and thus
   $$\bar{v}\big((((P \to Q) \to P) \to P)\big) = T.$$
   If $v(P) = T$, then $\bar{v}\big((\cdots \vee P)\big) = T$, in particular
   $$\bar{v}\big((((P \to Q) \to P) \to P)\big) = T.$$
   (b) We claim that σ_k is a tautology precisely if k is positive and even: Clearly, the wff $\sigma_0 = (P \to Q)$ is not a tautology. The wff $\sigma_1 = ((P \to Q) \to P)$ is also not a tautology, since σ_1 is tautologically equivalent to $(P \vee (P \wedge \neg Q))$, so $\bar{v}(\sigma_1) = F$ if $v(P) = F$. Above, we have seen that σ_2 is a tautology. So we are done if we show that if σ is a tautology, then $(\sigma \to P)$ is not a tautology, whereas $((\sigma \to P) \to P)$ is: to see this, note that the former is tautologically equivalent to $(\neg\sigma \vee P)$ (and hence not satisfied if $v(P) = F$), and latter formula is tautologically equivalent to $(\sigma \vee P)$ (which is satisfied for every v, since σ has this property).
4. (a) Suppose $\Sigma \models \alpha$. Let v be a truth assignment satisfying Σ. Then v satisfies α, hence v also satisfies $(\alpha \vee \beta)$. This shows that $\Sigma \models (\alpha \vee \beta)$. So if $\Sigma \models \alpha$, then $\Sigma \models (\alpha \vee \beta)$; similarly one shows that if $\Sigma \models \beta$, then $\Sigma \models (\alpha \vee \beta)$.

(b) It is not true that if $\Sigma \models (\alpha \vee \beta)$, then $\Sigma \models \alpha$ or $\Sigma \models \beta$: to see this, let $\Sigma = \emptyset$ and $\alpha = A_1$, $\beta = (\neg A_1)$. We have $\models (A_1 \vee (\neg A_1))$, but neither $\models A_1$ nor $\models (\neg A_1)$.

5. (a) We use the induction principle to show that for every wff φ one has $\bar{u}(\varphi) = \bar{v}(\varphi^*)$. Suppose first that φ is a sentence symbol: $\varphi = A_n$ for some n. Then $\varphi^* = \alpha_n$, hence

$$\bar{u}(\varphi) = u(A_n) = \bar{v}(\alpha_n) = \bar{v}(\varphi^*).$$

Next suppose $\varphi = (\neg\psi)$ for some wff ψ. By inductive hypothesis we have $\bar{u}(\psi) = \bar{v}(\psi^*)$, and $\varphi^* = (\neg\psi^*)$, hence

$$\bar{u}(\varphi) = T \iff \bar{u}(\psi) = F \iff \bar{v}(\psi^*) = F \iff \bar{v}(\varphi^*) = T.$$

In the case where $\varphi = (\psi \Box \psi')$ for wffs ψ, ψ' and $\Box \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$ one argues similarly.

(b) Suppose φ is a tautology. Let v be a truth assignment. Let u be the truth assignment defined in part (a). Then $\bar{v}(\varphi^*) = \bar{u}(\varphi)$ by (a), and $\bar{u}(\varphi) = T$ since $\models \varphi$; hence $\bar{v}(\varphi^*) = T$. This shows that φ^* is a tautology.

6. Let us introduce three sentence symbols, P (for "the speaker plays tennis"), W (for "the speaker watches tennis") and R (for "the speaker reads about tennis"). Then the translations of the given sentences into sentential logic are

$$\neg P \rightarrow W, \quad \neg W \rightarrow R, \quad \neg((P \wedge W) \vee (P \wedge R) \vee (W \wedge R)).$$

We need to determine the truth assignments satisfying all three wffs. The third wff allows us to restrict attention to only the following three truth assignments:

(a) $v(P) = v(W) = F$, $v(R) = T$;

(b) $v(P) = F$, $v(W) = T$, $v(R) = F$;

(c) $v(P) = T$, $v(W) = v(R) = F$.

For the truth assignment in (a), we have $\bar{v}(\neg P \rightarrow W) = F$, and for the truth assignment in (c), $\bar{v}(\neg W \rightarrow W) = R$, whereas in (b), $\bar{v}(\neg P \rightarrow W) = \bar{v}(\neg W \rightarrow R) = T$. Hence the speaker is watching tennis.