# Averaging of nonlinearity management with dissipation

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Motivated by recent experiments in optics and atomic physics, we derive an averaged nonlinear partial differential equation describing the dynamics of the complex field in a nonlinear Schrödinger model in the presence of a periodic nonlinearity and a periodically varying dissipation coefficient. The incorporation of dissipation in our model is motivated by experimental considerations. We test the numerical behavior of the derived averaged equation by comparing it to the original nonautonomous model in a prototypical case scenario and observe good agreement between the two.

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## I. INTRODUCTION

In the past few years, there has been an intense theoretical interest in the use of nonlinear Schrödinger (NLS) equations to describe both the propagation of optical beams in waveguides and fibers [1,2] and the mean-field evolution of Bose-Einstein condensates (BECs) [3,4]. Within this framework of dispersive equations that support solitary nonlinear waves, one of the particular topics of recent interest has been the effects of spatially and/or temporally (i.e., in the evolution variable) dependent nonlinearities. This subject, often called "nonlinearity management" [5]—by analogy with the topic of "dispersion management" that has been developed (in the context of optics) in far greater depth [6]—was originally proposed in the study of layered optical media [7]. However, it has also garnered considerable attention in the study of Bose-Einstein condensation, where it was reformulated as Feshbach resonance management [8].

Recent experimental work in optics has realized layered media through a concatenation of glass slides and air gaps. This has allowed a more detailed examination of topics such as the breathing of localized pulses [9] and the modulational instability of extended ones [10,11]. Moreover, in the context of BECs, the interatomic interactions (which are the source of the nonlinearity at the mean-field level) can be adjusted experimentally over a very broad range by employing either magnetic [12,13] or optical Feshbach resonances [14]. This has led to a significant number of both theoretical and experimental studies, including the formation (in the laboratory) of bright matter-wave solitons and soliton trains for $^7$Li [15,16] and $^{85}$Rb [17] atoms. On the theoretical side, such a modulation of the interaction scattering length (and hence of the nonlinearity coefficient) has been used, among other things, to stabilize attractive higher-dimensional BECs against collapse [18]. More recently, spatial variations of the nonlinearity have also been considered. In particular, it has been shown that such "collisionally inhomogeneous" condensates lead to a variety of interesting features, including adiabatic compression of matter waves [19,20], Bloch oscillations of matter-wave solitons [19], atomic soliton emission and atom lasers [21], enhancement of transmitivity of matter waves through barriers [22,23], dynamical trapping of matter-wave solitons [22], stable condensates exhibiting both attractive and repulsive interatomic interactions [24], a delocalization transition in matter waves [25], and more. Numerous different types of spatial variations of the nonlinearity have now been considered, including linear [19,22], random [26], periodic [25,27,28], and localized (steplike) [21,29,30] ones. There has also been a number of detailed mathematical studies [31–33] that address aspects such as the effect of a "nonlinear lattice potential" (i.e., a spatially periodic nonlinearity) on the stability of solitary waves and the interplay between drift and diffraction and/or blow-up instabilities.

When the nonlinearity coefficient as a function of the evolution variable (time in BECs and the propagation direction in optics) experiences fast variations, one successful strategy is to average the nonautonomous, nonlinearity managed dynamics to obtain an (averaged) autonomous system [34,35]. The stationary states [36] and collapse properties [37] of the latter can then be analyzed in one and in higher dimensions, respectively. However, the presence of dissipation is a particularly important feature that arises when periodically varying the nonlinearity coefficients in both optics and BEC experiments; to the best of our knowledge, this has not previously been incorporated in such studies. In particular, in the optical case of nonlinearity management, there is a periodic loss (on the order of a few percent of the intensity of the optical beam) every time the beam crosses an interface between the different media (such as glass and air) [9–11]. In BECs, if the Feshbach resonance is crossed, numerous atoms are lost, which results again in dissipative dynamics [3,4] (although it is important to note that it is not always necessary to cross the actual resonance in order to change the sign of the scattering length, which can vanish as a function of the external magnetic field at points away from the resonance [8,18]).

Our goal in this Brief Report is to address the averaging of nonlinearly managed dynamics in the presence of dissipation. The remainder of our presentation is organized as follows. We first present the general setting in which both a periodic nonlinearity and a periodic dissipation are applied. We then apply averaging techniques to this nonautonomous setting and obtain an autonomous partial differential equation (PDE) describing the averaged dynamics. We subsequently test the resulting model against numerical experiments of the original dynamical equations and obtain good agreement between the two. Finally, we summarize our findings and present some suggestions for possible future studies.

## II. ANALYTICAL RESULTS

Motivated by the above physical settings, we consider in our analysis a time-dependent nonlinearity and include a time-dependent dissipation term. We thereby generalize the averaging technique of [34] and obtain a general averaged PDE in arbitrary dimensions. As our derivation hinges on the periodicity of the fast time scale, we require that the length of the period is at least an order of magnitude smaller than that of the slow scale over which we monitor the dynamics.

The primary model used for the physical settings we consider is an NLS equation of the form

$$iu_t + \tfrac{1}{2}\Delta u + \gamma_0|u|^2u + i\zeta_0 u + \frac{1}{\epsilon}\gamma\left(\frac{t}{\epsilon}\right)|u|^2u + \frac{i}{\epsilon}\zeta\left(\frac{t}{\epsilon}\right)u = 0, \quad (1)$$

where $u=u(\mathbf{x},t)$, $\mathbf{x}\in\mathbb{R}^n$, $t\in\mathbb{R}_+$, $\epsilon \ll 1$, and the quantities $\gamma_0$ and $\zeta_0$ are parameters. Letting $\tau = t/\epsilon$, the continuous functions $\gamma(\tau)$, $\zeta(\tau)$ satisfy

$$\gamma(\tau+1) = \gamma(\tau), \quad \int_0^1 \gamma(\tau)d\tau = 0,$$

$$\zeta(\tau+1) = \zeta(\tau), \quad \int_0^1 \zeta(\tau)d\tau = 0,$$

and represent the zero-average, time-dependent parts of the nonlinearity and dissipation, respectively. Note that for the nonconservative term to be dissipative for all time $t$, the condition $\zeta_0 + (1/\epsilon)\zeta(t/\epsilon) > 0$ must be satisfied. The NLS equation (1) describes the envelope of the electric field of light in the context of optics, and it represents the mean-field wave function of the BEC in the context of atomic physics. To better understand the behavior of its solutions, we use a multiple-scale expansion to derive an averaged equation for Eq. (1) in arbitrary dimensions.

Following the notation in Ref. [34], let $f_{-1}$ denote the zero-mean antiderivative of $f$. It is given by

$$f_{-1}(\tau) = \int_0^\tau f(\tau')d\tau' - \int_0^1 \int_0^\tau f(\tau')d\tau' d\tau. \quad (2)$$

Define the transformation

$$u(\mathbf{x},t) = e^{-\zeta_{-1}(\tau)} e^{[ij(\tau)|v(\mathbf{x},t)|^2]} v(\mathbf{x},t), \quad (3)$$

where $j(\tau) = (\gamma e^{-2\zeta_{-1}(\tau)})_{-1}$. Using Eqs. (2) and (3), Eq. (1) can be expressed as

$$\begin{aligned} iv_t - j(\tau)|v|_t^2 v = &-\tfrac{1}{2}\Delta v - \gamma_0 e^{-2\zeta_{-1}(\tau)}|v|^2 v - i\zeta_0 v - \tfrac{1}{2}ij(\tau) \\ &\times[2(\nabla|v|^2\cdot\nabla v) + v\Delta|v|^2] \\ &+ \tfrac{1}{2}j(\tau)^2 v(\nabla|v|^2\cdot\nabla|v|^2), \end{aligned} \quad (4)$$

where $|v|_t^2 = \frac{\partial}{\partial t}(|v|^2)$, $\Delta|v|^2$ stands for $\Delta(|v|^2)$, and $\nabla|v|^2$ stands for $\nabla(|v|^2)$.

We isolate $|v|_t^2$ by considering the expression $\bar{v}(4) - v\overline{(4)}$ [i.e., Eq. (4) times the complex conjugate of $v$ minus $v$ times the complex conjugate of Eq. (4)] to transform Eq. (1) into the standard form

$$\begin{aligned} iv_t = &-\tfrac{1}{2}\Delta v - \gamma_0 e^{-2\zeta_{-1}(\tau)}|v|^2 v - i\zeta_0 v - 2\zeta_0 j(\tau)|v|^2 v - \tfrac{1}{2}ij(\tau) \\ &\times[\Delta(|v|^2 v) - 2|v|^2\Delta v + v^2\overline{\Delta v}] - \tfrac{1}{2}j(\tau)^2[2|v|^2\Delta|v|^2 \\ &+ \nabla|v|^2\cdot\nabla|v|^2]v. \end{aligned} \quad (5)$$

Using the multiple-scale expansion $v(x,t) = w(x,t) + \epsilon v_1(x,t,\tau) + \mathcal{O}(\epsilon^2)$ in Eq. (5), we then obtain

$$\begin{aligned} iw_t = &-iv_{1\tau} - \tfrac{1}{2}\Delta w - \gamma_0 e^{-2\zeta_{-1}(\tau)}|w|^2 w - i\zeta_0 w - 2\zeta_0 j(\tau)|w|^2 w \\ &- \tfrac{1}{2}ij(\tau)[\Delta(|w|^2 w) - 2|w|^2\Delta w + w^2\overline{\Delta w}] \\ &- \tfrac{1}{2}j(\tau)^2[2|w|^2\Delta|w|^2 + \nabla|w|^2\cdot\nabla|w|^2]w. \end{aligned} \quad (6)$$

Integrating Eq. (6) yields an expression that one can reintroduce into the equation to solve for $v_{1\tau}$. Consequently, the averaged equation for Eq. (1) takes the form

$$\begin{aligned} iw_t = &-\tfrac{1}{2}\Delta w - \gamma_0\rho|w|^2 w - i\zeta_0 w \\ &- \frac{\sigma^2}{2}w[\nabla|w|^2\cdot\nabla|w|^2 + 2|w|^2\Delta|w|^2], \end{aligned} \quad (7)$$

where $\sigma^2 = \int_0^1 j(\tau)^2 d\tau$ and $\rho = \int_0^1 e^{-2\zeta_{-1}(\tau)}d\tau$. Observe that the formal expansion $v = w + \epsilon v_1$ yields an equation that no longer depends upon the fast time-scale $\tau$. (That is, we obtain an autonomous PDE.) This approach also enables us to obtain the governing dynamics for the leading-order correction to the averaged behavior,

$$\begin{aligned} v_1 = &-\tfrac{1}{2}[\Delta(|w|^2 w) - 2|w|^2\Delta w + w^2\overline{\Delta w}]j_{-1}(\tau) \\ &+ \tfrac{1}{2}iw[\nabla|w|^2\cdot\nabla|w|^2 + 2|w|^2\Delta|w|^2](j(\tau)^2 - \sigma^2)_{-1} \\ &+ i\gamma_0|w|^2 w(e^{-2\zeta_{-1}(\tau)})_{-1} + 2i\zeta_0 j_{-1}(\tau)|w|^2 w, \end{aligned} \quad (8)$$

which can be compared with Eq. (2.13) in Ref. [34].

## III. NUMERICAL CORROBORATION

In order to test the validity of Eq. (7) for the averaged dynamics, we implement a prototypical two-dimensional realization of the above setting with radial symmetry (following the lines of [18]). In particular, we consider the case of the two-dimensional unstable (against collapse) NLS soliton—the so-called "Townes soliton" [38]—with focusing nonlinearity. It has been shown in the context of BECs that such a solution can be stabilized using a rapidly oscillating nonlinearity coefficient [18]. The resulting dynamics contains a fast time-scale periodicity associated with the nonlinearity management, so that this setting provides an ideal testbed for examining the accuracy of Eq. (7).

Building on the setting of Ref. [18], we consider the equation

$$\left[i\partial_t + \tfrac{1}{2}\Delta + \gamma(t)|u|^2 + i\zeta(t)\right]u = 0, \quad (9)$$

where the time-dependent nonlinearity $\gamma(t)$ and the time-dependent dissipation $\zeta(t)$ (the latter was absent in [18]) are given by

$$\gamma(t) = a_0 + a_1\sin(\omega t),$$

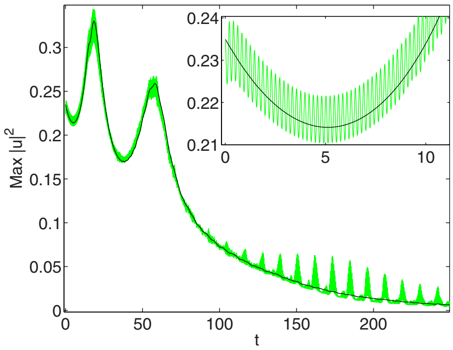


FIG. 1. (Color online) Averaged dynamics [Eq. (7)] versus the full dynamics [Eq. (9)]. The maximum of the intensity of Eq. (9) is shown in gray (green in the online version). We use the parameters $a_0=2\pi$, $a_1=-8\pi$, $\omega=30$, $b=b_0\times10^{-4}$, and $b_0=2$. Overlayed (in black) is the same diagnostic for the averaged equation. It is clear that the averaged dynamical behavior is captured accurately over 250 time units. The inset shows a magnification of the small time-scale oscillations that are responsible for the stabilization and the overlayed average curve over the first 12 time units.

$$\zeta(t) = b[1-\cos(\omega t)]. \tag{10}$$

Let $\epsilon=2\pi/\omega$, $f_0=\frac{1}{\epsilon}\int_0^\epsilon f(t)dt$, and $\tilde{f}(\tau)=\epsilon[f(\tau)-f_0]$, where $\tau$ is defined below Eq. (1), for any function $f$. Applying this operation to the functions $\gamma(\epsilon\tau)$ and $\zeta(\epsilon\tau)$ converts Eq. (9) to the form of Eq. (1) (upon subsequently dropping the tildes) and allows us to apply Eq. (7) to this setting. Note that $\zeta(t)>\zeta_{\rm crit}\approx5.8$ (in the regime of instability) [18,39] for more than half of the period. Figure 1 shows the maximum intensity $|u|^2$ of the field in the full equation (9) in gray (green in the online version) and the intensity $|w|^2$ of the averaged equation (7) in black. It clearly illustrates the strong correlation between the latter and the average (over a fast period) of the former. In Fig. 2, we illustrate this agreement over the initial few periods of the macroscopic dynamics of the equation for several different values of $b=b_0\times10^{-4}$ (using $b_0=1,2,3$). It is clear that as the magnitude of the dissipation is increased, the amplitude of the nonlinear solution sustains a stronger decrease and a weaker oscillatory behavior. Nevertheless, the agreement between the average of the full dynamics (over the fast scale) and the proposed averaged dynamics remains satisfactory in all of the studied cases.

A typical example of parameters that can be used to create a quasi-two-dimensional condensate can be found in Ref. [40]. This can be used to demonstrate trapping frequencies of $\omega_z/(2\pi)=790$ Hz off of the plane and $\omega_\perp/(2\pi)=10$ Hz on the plane. Although we do not use a (weak) radial trap in our work, in the above model one can measure the spatial scale in units of the radial trap oscillator length $d_0=\sqrt{\hbar/(m\omega_\perp)}$, time in units of $1/\omega_\perp$, the wave function in units of $\sqrt{N}/d_0$ (where $N$ denotes the total number of atoms in the BEC), and the dissipation coefficient in units of $\hbar\omega_\perp$.

## IV. CONCLUSIONS

In conclusion, we have considered the physically realistic framework of periodic dissipative dynamics in the setting of

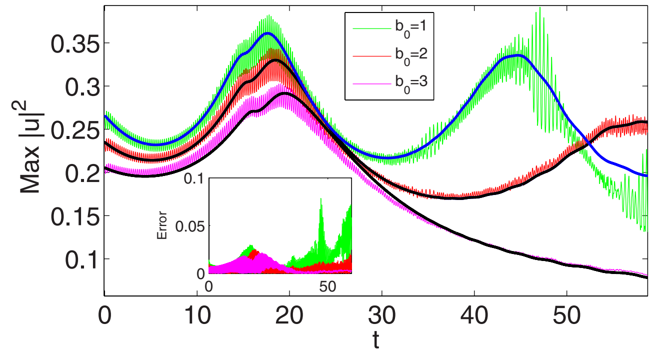


FIG. 2. (Color online) Three comparisons such as that of Fig. 1 over the first 60 time units with $b_0=1$, 2, and 3 for the top, middle, and bottom curves, respectively. All other parameters are the same as in Fig. 1. In particular, observe in the top comparison (showing the example of the smaller-magnitude dissipation term) that the averaged equation (7) begins diverging from the long time-scale dynamics. The inset shows the absolute error between the solution $u_{\rm full}$ of the full equation and the solution $u_{\rm ave}$ of the averaged equation; this error is given by the expression $|\max(|u_{\rm ave}|^2)-\max(|u_{\rm full}|^2)|(t)$. One can see more clearly here that after $t=30$ the error in the case with the smallest dissipation coefficient grows considerably.

periodically managed nonlinearity. We have argued on physical grounds that the inclusion of dissipation is relevant for both optically layered media and for Bose-Einstein condensates crossing Feshbach resonances. We have systematically generalized earlier works by obtaining a PDE that incorporates both the average and the fluctuating parts of the dissipation. We showed that this generalized PDE model, which is valid in both one-dimensional and multidimensional settings, is in good agreement with the average of the original dynamical equation for different dissipation characteristics within a recently proposed, prototypical two-dimensional nonlinearity management setting.

As discussed in the Introduction, in addition to the case of temporally-dependent nonlinearity explicitly considered here, there has been a lot of recent attention on spatially-dependent microstructures in the nonlinearity. It would thus be interesting to extend our considerations to the case of spatially-dependent nonlinearity prefactors and to derive the corresponding "averaged" (homogenized) dynamics when the nonlinearity prefactor varies over a fast spatial scale. Extending such averaging considerations to spatially and spatiotemporally varying nonlinearities is an interesting endeavor under current consideration. We will report relevant results in future studies.

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