HIGHLY NONLINEAR SOLITARY WAVES IN PERIODIC GRANULAR MEDIA

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Summary We use experiments, numerical simulations, and theoretical analysis to investigate the propagation of highly nonlinear solitary waves in periodic arrangements of dimer (two-mass) and trimer (three-mass) cell structures in one dimensional granular media. To vary the composition of the fundamental periodic units in the chains, we utilize particles of different materials (stainless steel, bronze, glass, nylon, PTFE, and rubber). Employing a model with Hertzian interactions between adjacent beads, we find very good agreement between experiments and numerical simulations. Equally good agreement is found between these results and a theoretical analysis of the model in the long-wavelength regime that we derive for heterogeneous environments (dimer chains) and general bead interactions. Our analysis encompasses previously-studied examples as special cases and also provides key insights on the influence of the heterogeneous lattices on the properties (width and propagation speed) of the nonlinear wave solutions of this system. The work paves the road to the study of acoustic band gaps and localization phenomena in heterogeneous randomized granular media.

INTRODUCTION

Granular matter, common in our everyday life, has many known applications but it presents fundamental difficulties in the understanding of its intrinsic dynamic properties due to the strong nonlinearity and complex contact-force distributions between adjacent grains. Lattices of granular media exhibit a highly nonlinear dynamic response, allowing a completely new type of wave propagation [1] that has opened the door to exciting new fundamental physical observations. Such systems provide an ideal setting for investigating phenomena such as the role of "heterogeneous" versus "uniform" lattices, discrete models versus continuum approximations, scale effects and the interplay between nonlinearity and periodicity. One of the unique features of these systems is that they support a new type of wave with a compact support that is independent of the amplitude [1], providing perhaps the most experimentally tractable application of the notion of "compactons" [2].

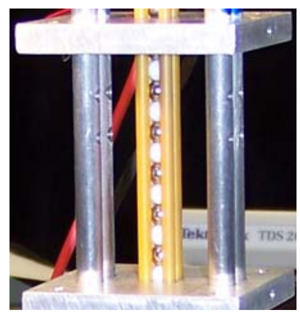

Figure 1. Experimental setup of the 1:1 steel-PTFE dimer.

In this paper we extend the established theory for uniform granular lattices [3] to nonuniform ones studying the effects of defects, such as inhomogeneities, particles with different masses, and disorder. In the past, this has led to the observation of interesting physical responses such as fragmentation [4], anomalous reflections [5], and energy trapping [6]. In our granular setting, we use a variety of soft, hard, heavy, and light materials to examine the effects of different structural properties in the fundamental components of such systems. We also vary the number of beads of a given type in each cell to examine the effects of different unit cell sizes (i.e., different periodicities). In particular, we investigate solitary wave propagation using experiments, numerical simulations, and theoretical analysis. We report very good agreement between experiments and numerics. For the case of dimer chains (Fig.1), we also apply a long-wavelength approximation to the nonlinear lattice model to obtain a quasi-continuum nonlinear partial differential equation (PDE) providing an averaged description of the system. We obtain analytical expressions for wave solutions of this equation and find very good qualitative agreement between the width and propagation speed of these solutions with those obtained from experiments and numerical simulations.

EXPERIMENTAL and NUMERICAL RESULTS

The experimental dimer and trimer chains were composed of vertically aligned beads of different materials (stainless steel, bronze, glass, nylon, PTFE, and rubber) in a delrin guide that contained slots for piezosensors connections or in a guide composed of four vertical garolite rods arranged in a square lattice (Fig. 1) [3]. For the related numerical analysis we model a chain of n spherical beads as a 1D lattice with Hertzian interactions between beads as shown on the left.

$$\ddot{y}_j = \frac{A_{j-1,j}}{m_j}\delta_j^{3/2} - \frac{A_{j,j+1}}{m_j}\delta_{j+1}^{3/2} + g\,,$$

$$A_{j,j+1} = \frac{4E_jE_{j+1}\left(\frac{R_jR_{j+1}}{R_j+R_{j+1}}\right)^{1/2}}{3\left[E_{j+1}\left(1-\nu_j^2\right)+E_j\left(1-\nu_{j+1}^2\right)\right]}\,,$$

Here $j\in\{1, \cdots, n\}$, y_j is the coordinate of the center of the j^{th} particle, $\delta_j \equiv \max\{y_{j-1}-y_j, 0\}$ for $j\in\{2,\ldots,n\}$, $\delta_1\equiv 0$, $\delta_{n+1}\equiv \max\{y_n, 0\}$, g is the gravitational acceleration, E_j is the Young's (elastic) modulus of the j^{th} bead, ν_j is its Poisson ratio, m_j is its mass, and R_j is its radius. Our numerical simulations incorporate the nonuniform gravitational preload due to the vertical orientation of the chains in experiments but do not take dissipation into account.

We begin by discussing our results for 1:1 dimers composed of steel:PTFE, steel:rubber, steel:bronze, PTFE:glass, and PTFE:nylon configurations. In addition we study N:1, 1:N steel:PTFE dimers and various combination of 1:1:1 trimers. In most of these systems the dynamics indicate that the initial excited impulse develops into a solitary wave within the first 10 cells of the chain. Figure 2 shows an example of stationary solitary waves obtained experimentally (Fig. 2(a)) and numerically (Fig. 2(b)) in a chain of steel:bronze:PTFE trimers. We compute the pulse speed using time-of-flight measurements. Two important properties of the pulses observed in these systems are their propagation speeds and widths (measured by the full width at half maximum, or FWHM, Fig. 3). We found both experimentally and numerically that such systems robustly support the formation and propagation of highly nonlinear solitary waves, with widths and pulse propagation speeds that depend on the periodicity of the chain. For dimer chains consisting of cells composed of N_1 particles of massive materials such as steel doped by 1 light particle such as PTFE, we find that the width (expressed as the number of unit cells) decreases, whereas the propagation speed increases with N_1. We also observe a "frustration" phenomenon for $N_1 \geq 4$, although robust localized pulses nevertheless form even in this case.

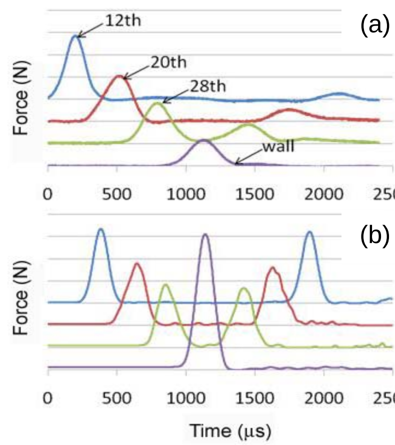

Figure 2. Force versus time response obtained from chains of trimers consisting of 1 steel, 1 bronze and 1 PTFE particle. (a) Experimental and (b) numerical results.

THEORETICAL ANALYSIS

We focus our analysis on the prototypical 1:1 dimer chain with beads of different masses (m_1 and m_2). We consider a general power-law interaction to illustrate the comprehensiveness of our approach and use Hertzian contact to compare our theoretical analysis with our numerical and experimental results. Starting from the equation of motion for the different particles, we apply a long wavelength approximation for an arbitrary mass ratio and a general interaction exponent. We derive coefficients by imposing self-consistency conditions. The resulting PDE [3] bearing the leading-order discreteness corrections is of the form:

$$u_{\tau\tau} = u_x^{n-1}u_{xx} + Gu_x^{n-3}u_{xx}^3 + Hu_x^{n-2}u_{xx}u_{xxx} + Iu_x^{n-1}u_{xxxx}$$

where τ is a rescaled time, n is the nonlinear exponent and the parameters *I, H, G* can be found in [3]. The solution found for this equation, from direct integration is of the form similar to the one found in [1] for homogeneous systems. To test the validity of the derived solutions we used comparison of the amplitude-velocity scaling and the solution width, which depends directly on the mass ratio. Among these, the width is the property that naturally showcases the relevance and novelty of the general dimer theory developed herein. For the comparison refer to Fig. 3 in which line (1) represents the theoretical value for the FWHM with $m_1 \gg m_2$, line (2) the theory for a 1:1 steel:PTFE, and line (3) reports the one for a homogeneous chain.

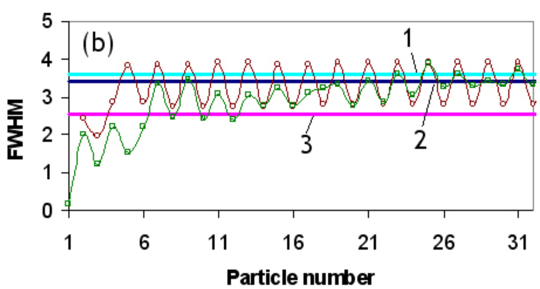

Figure 3. Evolution of solitary wave width (FWHM) as a function of bead number. The experimental values are shown by (green) squares and the numerical values are shown by (red) circles.

CONCLUSIONS

We examined the propagation of solitary waves in heterogeneous, periodic chains of granular media using experiments, numerical simulations, and theory. Using different periodicities, we found that such systems robustly support the formation and propagation of highly localized nonlinear solitary waves. We used force-velocity scaling and solitary-wave width (which depends on periodicity and on the mass ratio of the dimer materials) as relevant benchmarks for the excellent agreement between our three approaches. This qualitative and quantitative understanding of the dimers dynamics paves the way for studies in increasingly heterogeneous media in 1- and higher dimensions and for the presence of acoustic band gaps.

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