Math 168, Networks, UCLA, Spring 2018

Midterm (18 May 2018)

Note: For this exam, you may use hard copies of Newman's book, your notes, and your homework. You may not use anything electronic.

Note 2: Include your reasoning and/or derivations (as appropriate) in your answers to ensure that they are complete.

Note 3: Please include your name and student ID number on each page of the exam.

1. (30 points) Consider the following model of a network. Each of n nodes belongs to one of several groups. (We call each of these groups a 'community'.) The mth community has n_m nodes, and each node in that community is adjacent to other nodes in the same community with independent probability $p_m = R(n_m - 1)^{-\beta}$ for each node, where R and β are constants, but not to any nodes from other communities.

   a. (6 points) Sketch what the adjacency matrix $\mathbf{A}$ looks like.

   b. (6 points) Calculate the expected degree $\langle k \rangle$ of a node in community m.

   c. (6 points) Calculate the expected value $\bar{C}_m$ of the local clustering coefficient for nodes in community m. Hence show that $\bar{C}_m \propto \langle k \rangle^{-\beta/(1-\beta)}$, where "$\propto$" signifies 'is proportional to'.

   d. (6 points) What value would β have to have for the expected value of the local clustering coefficient to fall off with increasing degree as $\langle k \rangle^{-3/4}$.

   e. (6 points) Is this a good model for a social network? Briefly indicate why or why not. Regardless of your prior answer, indicate one precise way that you would adjust this model to make it a better model of a social network.

2. (35 points)

FIG. 1: A 7-node example of a 'line graph'.

Consider a 'line graph' that consists of n nodes in a line. (See Fig. 1 for an example with $n = 7$.)

a. (8 points) Write down the adjacency matrix and combinatorial graph Laplacian matrix of a line graph with $n = 7$.

b. (10 points) For the case $n = 7$, what is the geodesic node betweenness centrality of each of the 7 nodes? If we instead consider a ring with $n = 7$ nodes (with each node adjacent to one neighboring node on each side), now what is the geodesic node betweenness centrality of each node?

c. (17 points) Show, for general n, that if we divide a line graph into two parts — such that one part has r nodes and the other has $n - r$ nodes — that modularity (with the standard configuration-model-like choice of null-model P_{ij}) takes the value

$$Q = \frac{3 - 4n + 4rn - 4r^2}{2(n-1)^2} \,. \tag{1}$$

Hence show that when n is even, the division of the network that achieves the maximum modularity is the one that splits the network exactly down the middle.

3. (35 points) Consider a model that is similar to the Barabási–Albert model, in which undirected edges are added between nodes according to a linear preferential attachment, except that now the network does not grow. Instead, the network starts with a given number n of nodes and neither gains nor loses nodes thereafter. In this model, starting with an initial network of n nodes and some specified arrangement of edges, we add at each step one undirected edge between two nodes, both of which are chosen uniformly at random in proportion to their degree k. Let $p_k(m)$ denote the fraction of nodes with degree k when the network has m total edges.

   a. (5 points) Show that when the network has m edges, the probability that node i will get a new edge upon the addition of the next edge is k_i/m.

   b. (10 points) Write down a master equation that gives $p_k(m+1)$ in terms of $p_{k-1}(m)$ and $p_k(m)$.

   c. (10 points) Eliminate m from the aforementioned master equation in favor of the mean degree $c = 2m/n$ and take the limit $n \to \infty$ (with c held constant) to show that $p_k(c)$ satisfies the differential equation

$$c\frac{dp_k}{dc} = (k-1)p_{k-1} - kp_k \,. \tag{2}$$

   d. (10 points) Define a generating function $g(c,z) = \sum_{k=0}^{\infty} p_k(c)z^k$ and show that it satisfies the partial differential equation

$$c\frac{\partial g}{\partial c} + z(1-z)\frac{\partial g}{\partial z} = 0\,. \tag{3}$$