Borel circle squaring

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Tarski's circle squaring problem

The theory of amenability can be used to show that Lebesgue measure on $\mathbb{R}^2$ can be extended to a finitely additive isometry-invariant measure on $\mathbb{R}^2$. Hence, there is no version of the Banach-Tarski paradox in $\mathbb{R}^2$.

Question (Tarski, 1925)

Are a disc and a square in $\mathbb{R}^2$ (necessarily of the same area) equidecomposable?

Laczkovich (1990) gave a positive answer to this question.

Dubins, Hirsch, and Karush (1963) had shown that Tarski's circle squaring cannot be solved using pieces whose boundaries consist of a single Jordan curve.

A Borel solution to Tarski's circle squaring problem

Theorem (M.-Unger, 2016)

Tarski's circle squaring problem can be solved using Borel pieces. Generally, suppose $k \geq 1$ and $A, B \subseteq \mathbb{R}^k$ are bounded Borel sets such that $\lambda(A) = \lambda(B) > 0$, $\Delta(\partial A) < k$, and $\Delta(\partial B) < k$. Then A and B are equidecomposable by translations using Borel pieces.

Here λ is Lebesgue measure, ∂A is the boundary of A, and Δ is upper Minkowski dimension.

This is a Borel version of a general equidecomposition theorem due to Laczkovich (1992). Grabowski, Máthé, and Pikhurko had proved a measurable/Baire measurable version in 2015.

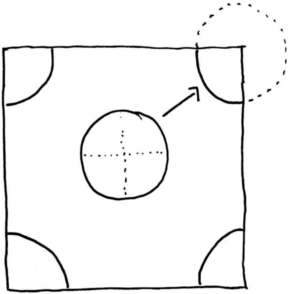
Our equidecomposition of the circle and square uses $\approx 10^{200}$ pieces which are finite boolean combinations of Σ^0_4 sets.

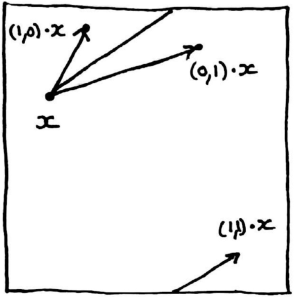
In the remainder of the talk, we sketch a proof of this theorem.

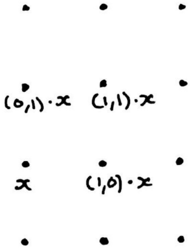
Laczkovich's first idea: work in the torus

We may scale and translate A and B so that they lie in $[0,1)^k$.

View A and B as subsets of the k-torus $\mathbb{T}^k = (\mathbb{R}/\mathbb{Z})^k$ which we identify with $[0,1)^k$. Then A and B are equidecomposable by translations as subsets of the torus iff they are equidecomposable by translations in $\mathbb{R}^k$. (Though perhaps using more pieces).



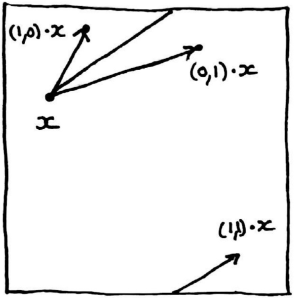


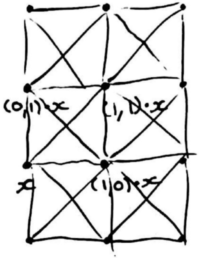


Fix a sufficiently large d, and sufficiently random $u_1, \ldots, u_d \in \mathbb{T}^k$. Obtain an action a of $\mathbb{Z}^d$ on $\mathbb{T}^k$ by letting the ith generator of $\mathbb{Z}^d$ act via u_i.

$$(n_1, \ldots, n_d) \cdot x = n_1 u_1 + \ldots + n_d u_d + x$$

This action will be free. We can visualize each orbit as a copy of $\mathbb{Z}^d$.

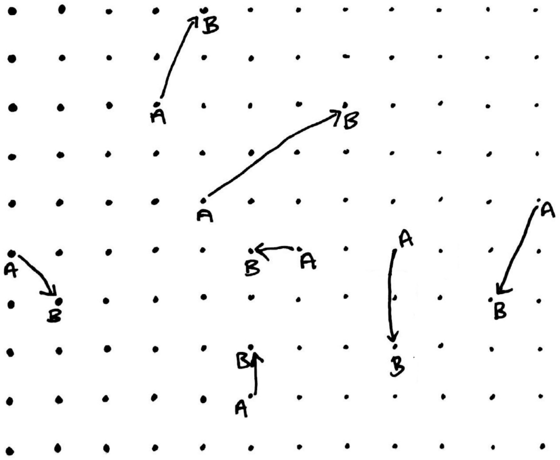




For the rest of the proof, let G be the graph with vertex set $\mathbb{T}^k$ where $x, y \in \mathbb{T}^k$ are adjacent if there is $\gamma \in \mathbb{Z}^d$ such that $\gamma \cdot x = y$ where $|\gamma|_\infty = 1$.

To show A and B are equidecomposable by Borel pieces, it suffices to find a Borel bijection $g \colon A \to B$ so that for some fixed N, for all $x \in A$, $d_G(x, g(x)) \leq N$.

Then if $A_\gamma = \{x : g(x) = \gamma \cdot x\}$, the sets $\{A_\gamma\}_{|\gamma|_\infty \leq N}$ partition A, and the sets $\{\gamma \cdot A_\gamma\}_{|\gamma|_\infty \leq N}$ will partition B.



A picture of an equidecomposition.

Laczkovich's second idea: discrepancy theory

For $x \in \mathbb{T}^k$, let $F_N(x) = \{(n_1, \ldots, n_d) \cdot x \in \mathbb{Z}^d \colon 0 \leq n_i < N\}$, the "square" of side length N in G starting at x. Since $F_N(x)$ has N^d elements, by the ergodic theorem, expect $|F_N(x) \cap A| \approx \lambda(A)N^d$.

Lemma (Laczkovich 1992 building on Schmidt, Niederreiter-Wills)

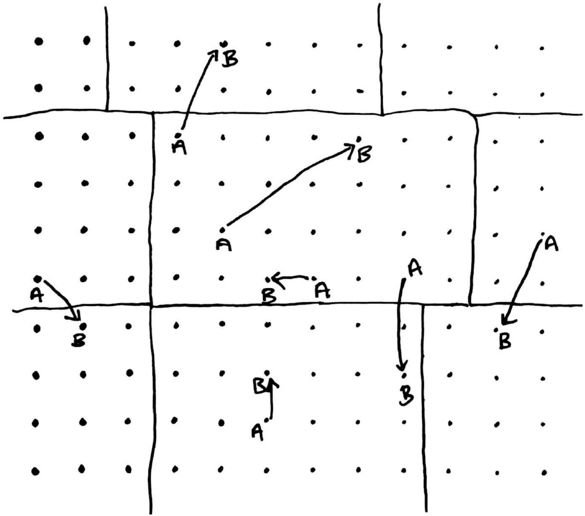
For A, B and the action as above, $\exists \epsilon > 0$ and M such that for every x and N,

$$\left| F_N(x) \cap A - \lambda(A)N^d \right| \leq MN^{d-1-\epsilon}$$

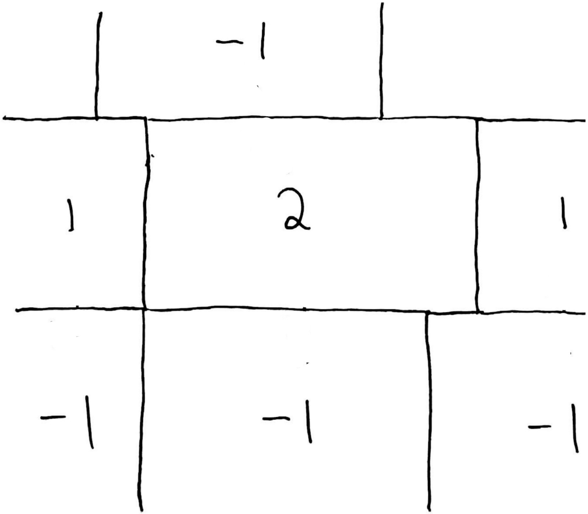
and

$$\left| F_N(x) \cap B - \lambda(B)N^d \right| \leq MN^{d-1-\epsilon}$$

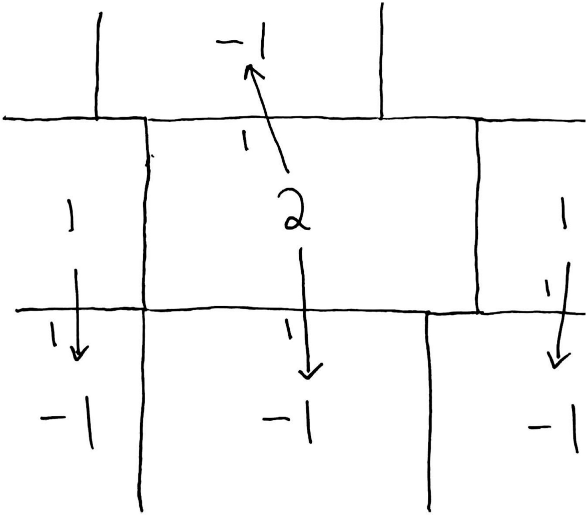
Roughly, every square of side length N contains very close to $\lambda(A)N^d$ many elements of A and B.



Suppose we take an equidecomposition $g\colon A \to B$ and "zoom out" by tiling the action with rectangles in G of side length $\approx N$.



In each tile S, there is some difference in $|A \cap S|$ and $|B \cap S|$.



The equidecomposition tells us how many points of A to move to points of B in adjacent tiles so that the same number of points of A and B remain in each tile afterwards.

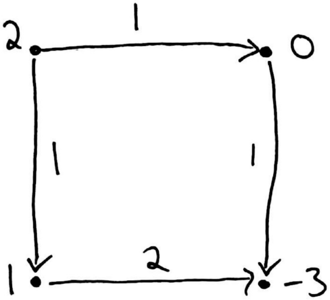
Flows in graphs

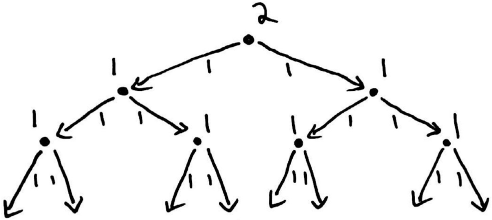
Based on this idea, we introduce the following definition: Suppose G is a graph (symmetric irreflexive relation) on a vertex set X. If $f : X \to \mathbb{R}$ is a function, then an f-**flow** of G is a function ϕ: $G \to \mathbb{R}$ such that

- For every edge $(x, y) \in G$, $\phi(x, y) = -\phi(y, x)$, and
- For every vertex $x \in X$, Kirchoff's law:

$$f(x) = \sum_{(x,y)\in G} \phi(x, y)$$

In finite graph theory, flows are usually studied with a single source and sink (e.g. in the max-flow min-cut theorem). For finite graphs, the above type of flow problem is equivalent to one with a single source and sink (by adding a "supersource" and "supersink" to the graph). For infinite graphs, there is not such an equivalence. E.g. there are "Ponzi schemes" on infinite graphs.





Examples of flows.

Flows and equidecompositions

Proposition

A and B are a-equidecomposable with Borel pieces iff there is a bounded Borel integer-valued $\chi_A - \chi_B$-flow of G.

$\rightarrow$: Suppose $g \colon A \rightarrow B$ is a Borel bijection which moves points a bounded distance in G.

To construct a flow from g, for each $x \in A$ add 1 unit of flow to each edge along the lex-least path from x to $g(x)$.

Constructing an equidecomposition from a flow, I

$\leftarrow$: Suppose now ϕ is a Borel $\chi_A - \chi_B$ flow of G bounded by c.

Find a Borel tiling $T \subseteq [\mathbb{T}^k]^{<\infty}$ of each orbit by rectangles of side length $\approx N$. So T is a partition of T^k, and each $S \in T$ is a rectangle of side length $\approx N$ in G. Let G/T be the graph minor of G formed via T. That is, the vertices of G/T are the tiles in T and two tiles are adjacent if they contain neighbors in G.

Let $F\colon T \to \mathbb{R}$ be $F(R) = \sum_{x \in R} \chi_A(x) - \chi_B(x)$ and let

$$\Phi(R, S) = \sum_{(x,y) \in G\colon x \in R \wedge y \in S} \phi(x, y)$$

Then Φ is an F-flow of G/T.

Constructing an equidecomposition from a flow, II

Each tile $R \in T$ has roughly $\lambda(A)N^d$ points of A and B, and the flow over the boundary of the tile is $\leq O(cN^{d-1})$. Using discrepancy, if N is sufficiently large, there are more points of A and B in every tile than maximum flow out of the boundary of the tile.

Now construct a Borel bijection from A to B witnessing equidecomposability. Suppose R, S are adjacent tiles.

- If $\Phi(R, S) > 0$, then map $\Phi(R, S)$ many points of $A \in R$ to points of $B \in S$.
- If $\Phi(R, S) < 0$, then map $-\Phi(R, S)$ many points of $B \in R$ to $A \in S$.

Since Φ is an F-flow, after doing this the same number of points of A and B remain in each tile. Biject them to finish the construction. $\square$

An aside: how to construct Borel tilings

An **independent set** in a graph G is a set of vertices where no two are adjacent.

Theorem (Kechris, Solecki, Todorcevic, 1999)

If G is a locally finite Borel graph, then there is a Borel maximal independent set for G.

Let $G^{\leq n}$ be the graph on $\mathbb{T}^k$ where x, y are adjacent if $d_G(x, y) \leq n$. Let C be a Borel maximal independent set for $G^{\leq n}$. Use the element of C as center points for "tiles" of G.

If we use these center points to make "Voroni cells", the resulting tiling suffices for our proofs. Gao-Jackson (2015) give a more complicated construction to make rectangular tilings.

Proof overview

1. We construct a real-valued bounded Borel $\chi_A - \chi_B$-flow of G by giving an explicit algorithm for finding such a flow.
   - Relies on Laczkovich's discrepancy estimates.
   - Uses the fact that the average of flows is a flow.
2. We show that given any real-valued Borel $\chi_A - \chi_B$-flow of G, we can find an integer valued Borel $\chi_A - \chi_B$-flow which is "close" to the real-valued one. Uses:
   - the Ford-Fulkerson algorithm in finite combinatorics.
   - a theorem of A. Timár on boundaries of finite sets in $\mathbb{Z}^d$.
   - recent work of Gao, Jackson, Krohne and Seward on hyperfiniteness of free Borel actions of $\mathbb{Z}^d$.
3. We finish by using the proposition we just proved: there's a Borel equidecomposition iff there is a bounded Borel $\chi_A - \chi_B$-flow.

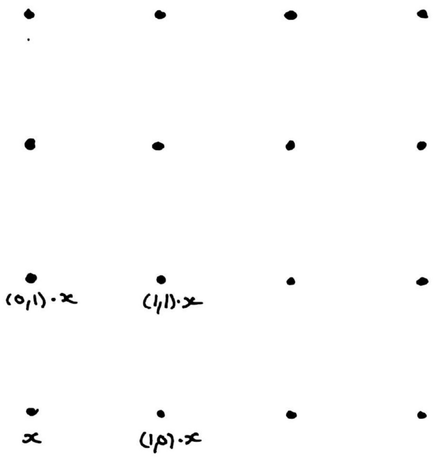
Step 1: Constructing a real-valued flow

Let $f = \chi_A - \chi_B$. Say that a function $\phi\colon G \to \mathbb{R}$ is an f-flow with error ϵ if

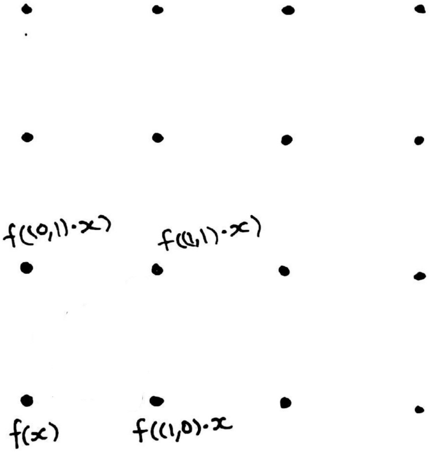
- For every edge $(x, y) \in G$, $\phi(x, y) = -\phi(y, x)$, and
- For every vertex $x \in X$,

$$\left| f(x) - \sum_{(x,y)\in G} \phi(x, y) \right| < \epsilon$$

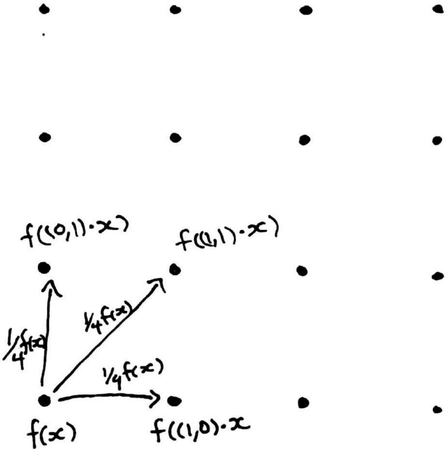
We'll construct our flow as a limit of approximate flows whose error approaches 0.



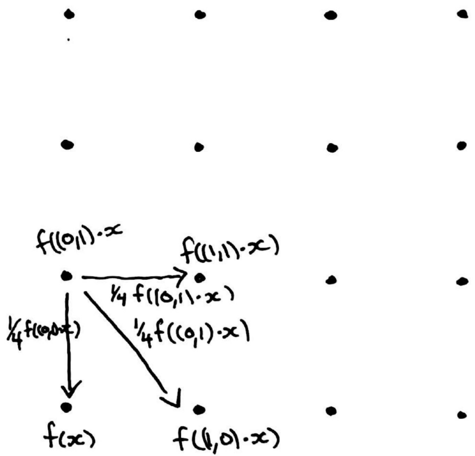
We'll describe an algorithm for constructing a real-valued f-flow where in the connected component of some $x \in \mathbb{T}^k$. We draw pictures with $d = 2$.



Our flow will be constructed in ω many steps. At step n we work in $2^n \times 2^n$ squares. At step 1 we consider 2×2 squares.



Step 1: The idea is to spread out the error in the flow evenly over each 2×2 square. Each point contributes $1/4$ of its f-value to the other 3 points.



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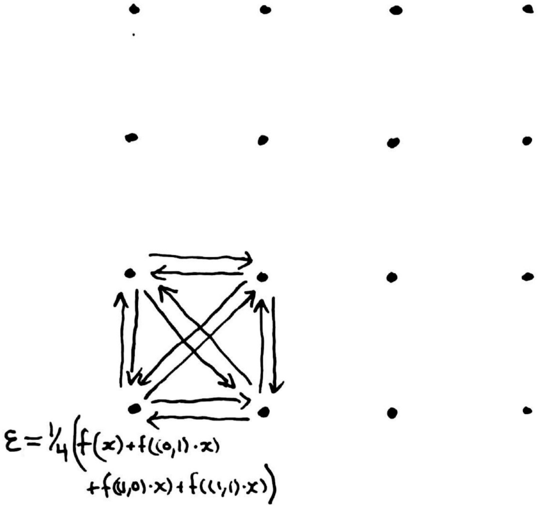
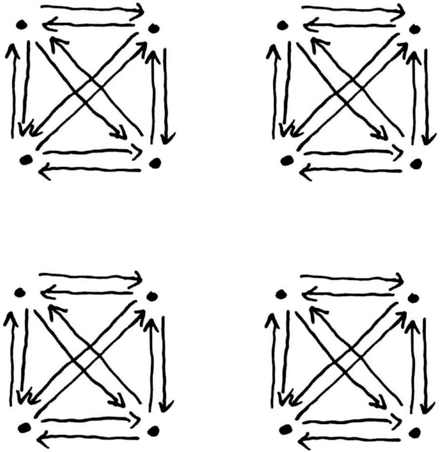
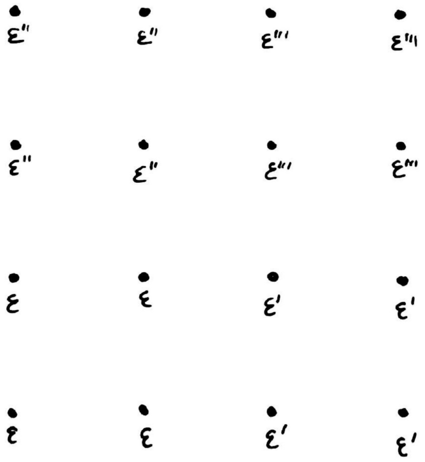


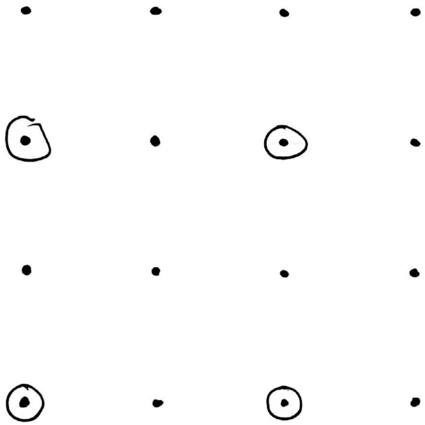
The error in the flow after step 1 is the average of f over the 2×2 square.



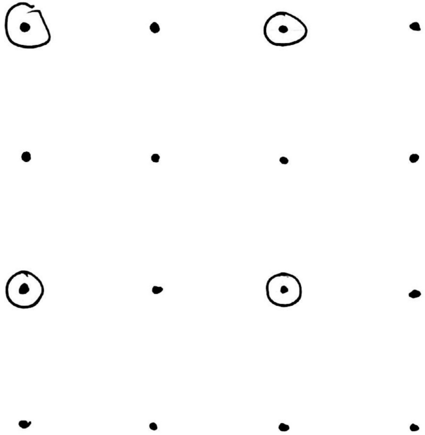
We do this for every 2×2 square in the orbit.



So the error in the flow after step 1 is the average of f on its 2×2 square.



Now we use roughly the same idea in each 4×4 square, but dealing with 4 points at a time in the way given above.



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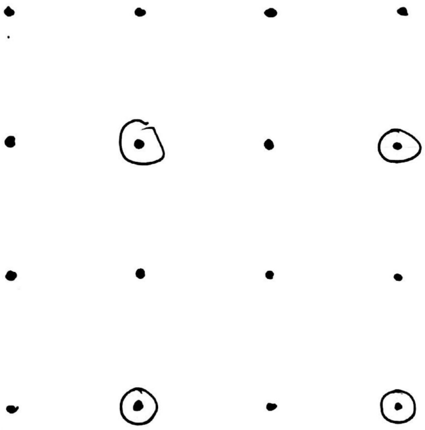

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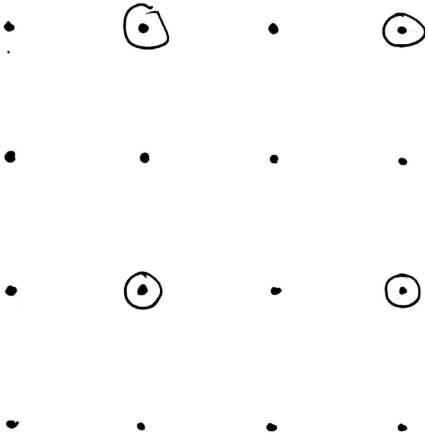

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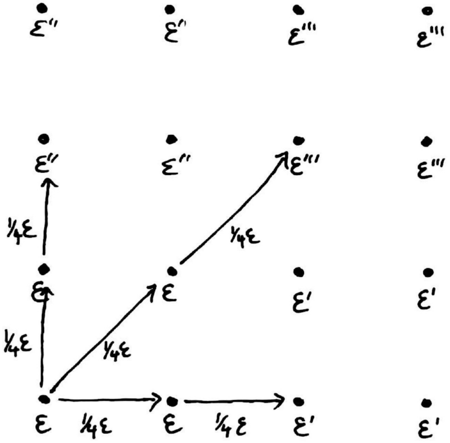
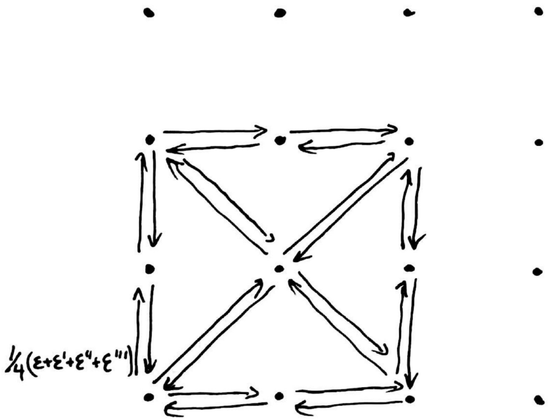
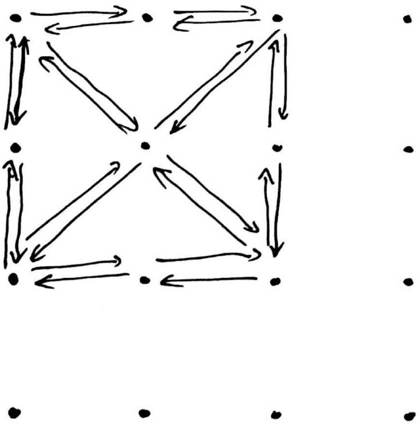


We add to the flow already constructed at the previous step. Once again, each point contributes $1/4$ of its error to the other 3 points.



After this second step, the error at each point will be the average of f over its 4×4 square.



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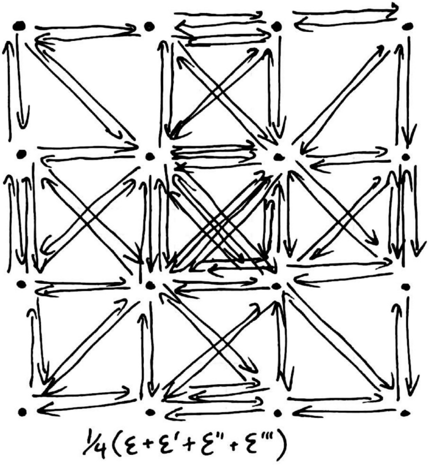


After this second step, the error at each point will be the average of f over its 4×4 square.

Step 1: Constructing a real-valued flow

After step n, the error in our flow at each point will be the average value of f over the $2^n \times 2^n$ square containing the point. Since $f = \chi_A - \chi_B$, and each $2^n \times 2^n$ square contains nearly the same number of points of A and B, this error is very small.

An easy calculation using Laczkovich's discrepancy estimates shows that this construction converges to a bounded f-flow (with error 0 everywhere).

However, we cannot pick a single x in each orbit to be a "starting point" for this construction (since this would be a nonmeasurable Vitali set).

To fix this problem, we use an averaging trick (the average of flows is a flow!).

Step 1: Constructing a real-valued Borel flow

For every $i > 0$, let $\pi_i \colon \mathbb{Z}^d/(2^i\mathbb{Z})^d \to \mathbb{Z}^d/(2^{i-1}\mathbb{Z})^d$ be the canonical homomorphism. This yields the inverse limit

$$\hat{\mathbb{Z}}^d = \varprojlim_{i \geq 0} \mathbb{Z}^d/(2^i\mathbb{Z})^d$$

where elements of $\hat{\mathbb{Z}}^d$ are sequences $(h_0, h_1, \ldots)$ such that $\pi_i(h_i) = h_{i-1}$ for all $i > 0$. Essentially, this describes how to choose a 2×2 grid, 4×4 grid, 8×8 grid, etc. that fit inside each other.

For each $x \in \mathbb{T}^k$ and $h \in \hat{\mathbb{Z}}^d$, our above construction yields a flow $\phi_{(x,h)}$ of the connected component of x, using the grids given by h. The construction is such that if $\gamma \in \mathbb{Z}^d$, then $\phi_{(x,h)} = \phi_{(\gamma \cdot x, -\gamma + h)}$. Hence, the average value of this construction is invariant of our starting point ($h \mapsto -g + h$ is measure preserving):

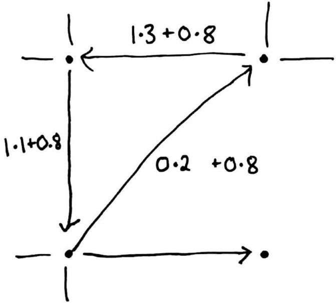
$$\int_h \phi_{(x,h)} \, \mathrm{d}\mu(h) = \int_h \phi_{(\gamma \cdot x, -\gamma + h)} \, \mathrm{d}\mu(h) = \int_h \phi_{(\gamma \cdot x, h)} \, \mathrm{d}\mu(h)$$

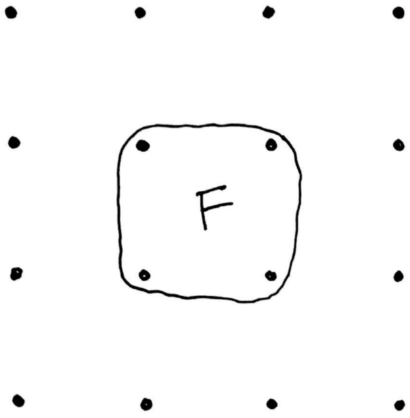
This average value is our real-valued Borel $\chi_A - \chi_B$ flow! (μ is Haar measure on $\hat{\mathbb{Z}}^d$.)

Step 2: modifying to make an integer Borel flow

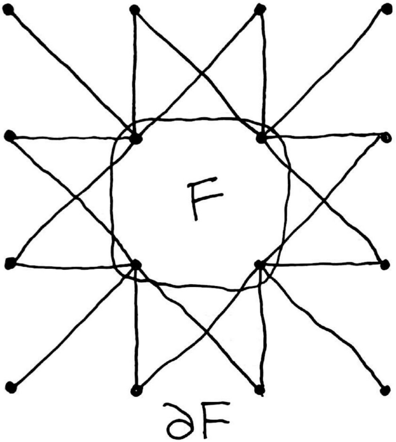
Now we want to modify the flow so that it takes integer values.

Suppose ϕ is an f-flow in G. Given a cycle in G if we add the same real value to every edge in the cycle, this preserves the property of being an f-flow. Hence, we can choose a value in $[0, 1)$ to add to this cycle so that a single edge in the cycle becomes integer.

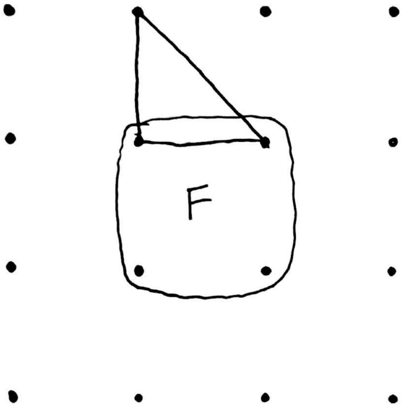




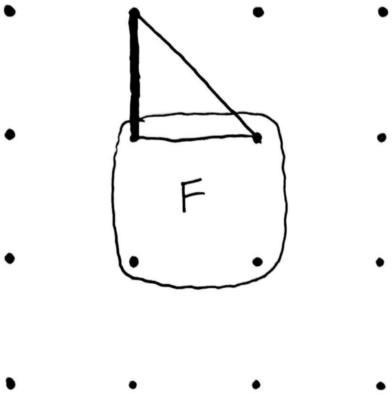
Suppose that F is a finite connected set in G.



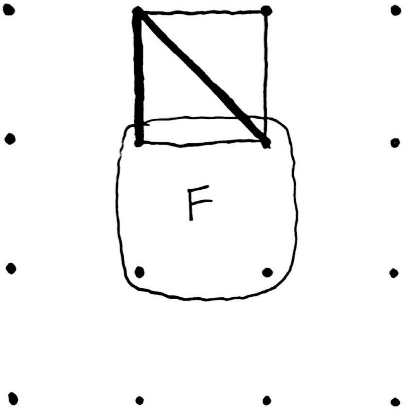
The edge boundary of F is $\partial F = \{(x, y) \in G : x \in F \wedge y \notin F\}$.
I claim we can modify the flow so that it takes integer values on ∂F.



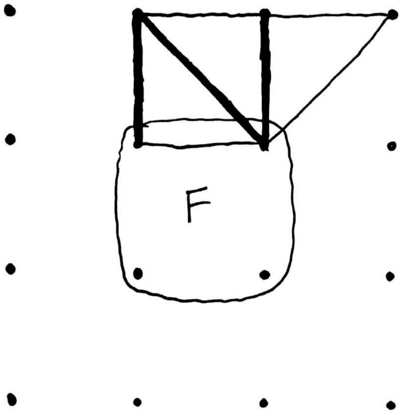
To begin, find a 3-cycle (a triangle) having an edge in ∂F.



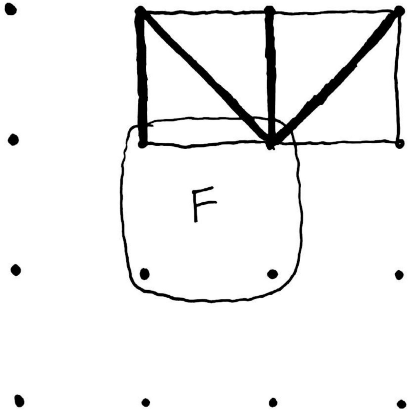
Modify the flow on the cycle to make this edge (the darker one) integer.



Repeat this process.



Repeat this process.



By using work of A. Timár on boundaries of finite sets in $\mathbb{Z}^d$, one can show using Euler's theorem (on the existence of Euler cycles) that for every finite set F, one can find a sequence of triangles that can be used to change the flow to be integer on ∂F.

Step 2: modifying to make an integer Borel flow

Let $[\mathbb{T}^k]^{<\infty}$ be the space of finite subsets of $\mathbb{T}^k$.

Theorem (Gao, Jackson, Krohne, and Seward, 2015)

There is a Borel set $C \subseteq [\mathbb{T}^k]^{<\infty}$ such that

- $\bigcup C = \mathbb{T}^k$
- *Every $S \in C$ is connected in G.*
- *(Boundaries are far apart) all distinct $R, S \in C$ are such that ∂R and ∂S contain no two edges of distance less than 4.*

Use the process described on the previous slides to make the flow integer on ∂S for every $S \in C$. After removing these edges, G has finite connected components. Use the integral flow theorem from finite graph theory (a corollary of the Ford-Fulkerson algorithm) to modify the flow on these components to be integer.

This finishes the proof of Borel circle squaring. □