JUMP OPERATIONS FOR BOREL GRAPHS

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Abstract. We investigate the class of bipartite Borel graphs organized by the order of Borel homomorphism. We show that this class is unbounded by finding a jump operator for Borel graphs analogous to a jump operator of Louveau for Borel equivalence relations. The proof relies on a non-separation result for iterated Fréchet ideals and filters due to Debs and Saint Raymond. We give a new proof of this fact using effective descriptive set theory. We also investigate an analogue of the Friedman-Stanley jump for Borel graphs. This analogue does not yield a jump operator for bipartite Borel graphs. However, we use it to answer a question of Kechris and Marks by showing that there is a Borel graph with no Borel homomorphism to a locally countable Borel graph, but each of whose connected components has a countable Borel coloring.

1. Introduction

The study of Borel graphs and their combinatorial properties is a growing area of research which has developed at the interface of descriptive set theory and combinatorics, and has connections with probability, ergodic theory, and the study of graph limits. The first systematic study of Borel graphs was the paper of Kechris, Solecki and Todorcevic [4], and [5] is a recent survey of the field. We largely follow the conventions and notation from [5]. A *Borel graph* G on X is a graph whose vertex set X is a Polish space, and whose edge relation G is Borel. We will typically abuse notation and identify a graph with its edge relation.

Suppose G and H are graphs on the vertex sets X and Y. Then a *homomorphism* from G to H is a function $h\colon X \to Y$ such that $\forall x, y \in X(xGy \implies h(x)Hh(y))$. That is, f maps adjacent vertices in G to adjacent vertices in H. In classical combinatorics, the book of Hell and Nešetřil [3] is a good introduction to the theory of graph homomorphisms.

If G and H are Borel graphs on X and Y, then we write $G \preceq_B H$ if there is a Borel homomorphism $h\colon X \to Y$ from G to H. Borel homomorphisms play a key role in the study of Borel colorings of Borel graphs. Recall that a *Borel coloring*

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of a Borel graph G on a Polish space X is a Borel function $c\colon X \to Y$ to a Polish space Y such that $\forall x, y \in X(xGy \implies c(x) \neq c(y))$. So for example, if we take a complete graph on n vertices K_n, then $G \preceq_B K_n$ if and only if G has a Borel n-coloring. More generally, if there is a Borel homomorphism from G to H, then composing this homomorphism with a Borel coloring of H yields a Borel coloring of G. Thus, the problem of definably coloring G is no more difficult than coloring H. In an early breakthrough result, Kechris, Solecki and Todorcevic [4] isolated a canonical graph G_0 having no Borel ω-coloring such that for every Borel (indeed, analytic) graph G, either G has a Borel coloring with countably many colors, or $G_0 \preceq_B G$. Hence, G_0 is the canonical obstruction to an analytic graph having countable Borel chromatic number.

In this paper, we study the class of Borel graphs organized under $\preceq_B$. We can view $\preceq_B$ as a general way of organizing all Borel graphs by the relative difficulty of definably solving problems on them, such as coloring, where solutions can be pulled back under homomorphisms.

An important subclass of the Borel graphs is the bipartite graphs, or equivalently, the graphs with no odd cycles. Such graphs are of particular interest in descriptive set theory since they isolate the descriptive-set-theoretic difficulties that occur in definably coloring Borel graphs, from the difficulties that arise if the graph has high classical chromatic number (a graph has classical chromatic number > 2 if and only if it is not bipartite). For example, the graph G_0 is acyclic and hence it is classically 2-colorable.

Note that there may be no Borel witness to the bipartiteness of a bipartite Borel graph. Indeed, Borel graphs G that do possess such a witness are trivial in the since that both $G \preceq_B K_2$ and $K_2 \preceq_B G$ where K_2 is the complete graph on two vertices.

One of our main theorems is that the class of bipartite Borel graphs is unbounded under $\preceq_B$. In contrast, note that among all Borel graphs, there is a trivial example of a maximal Borel graph under $\preceq_B$, specifically, the complete graph on any uncountable Polish space.

Our proof of this theorem is inspired by a jump operator of Louveau which comes from the theory of Borel reducibility between Borel equivalence relations [7]. If G is a Borel graph on a Polish space X, define G^* to be the Borel graph defined on X^ω where

$$xG^*y \leftrightarrow \exists n \forall i \geq n(x(i)Gy(i)).$$

We show this induces a jump operation on bipartite Borel graphs organized under $\preceq_B$. Note that if G is bipartite, then so is G^*.

Theorem 1. *Let G be a nontrivial bipartite analytic graph. Then $G \preceq_B G^*$ and $G^* \not\preceq_B G$.*

By nontrivial, we mean that G has at least one edge. Our theorem here in fact generalizes to G which are triangle-free.

Our proof of Theorem 1 requires an analysis of iterated Fréchet filters that differs from that in [7]. In particular, our proof uses the following theorem due to Debs and Saint Raymond.

Theorem 2 ([1, Theorem 3.2, proof of Theorem 6.5])**.** *Let $F_\alpha, I_\alpha \subseteq \mathcal{P}(\omega)$ be the α^{th} iterate of the Fréchet ideal and Fréchet filter. Then I_α and F_α cannot be separated by $\mathbf{\Pi}^0_{\alpha+1}$ sets (but they can be separated by disjoint $\mathbf{\Sigma}^0_{\alpha+1}$ sets).*

In Section 4, we give a new proof of this theorem which uses different methods than [1]. In particular, it uses effective descriptive set theory and an analysis of suitably chosen generic reals.

A more well-known jump operation on Borel equivalence relations is the Friedman-Stanley jump [2]. We can also define an operation on Borel graphs similar to the Friedman-Stanley jump. Suppose G is a Borel graph on a Polish space X. Define $X^+ \subseteq X^\omega$ by $x \in X^+$ if $i_1 \neq i_2$ implies $x(i_1)$ and $x(i_2)$ are in different connected components of G. Now we define the Borel graph G^+ on X^+ by:

$$xG^+y \leftrightarrow \forall i \exists j (x(i) G y(j)) \wedge \forall i \exists j (y(i) G x(j)).$$

Now it is easy to see that if G is a Borel graph with countably many connected components, then there is a Borel homomorphism from G^+ to G; take a Borel ordering $<$ on the connected components of G of ordertype ω, and then define a homomorphism h from G^+ to G by letting $h(x)$ be the $<$-least $x(j)$ in the sequence $(x(i))_{i \in \omega}$. Similarly, for such a G, there is a Borel homomorphism from G^{++} (which has continuum many connected components) to G^+. Characterizing the Borel graphs G such that $G^+ \not\preceq_B G$ remains an open question. However, we show that in at least one special case $G^+ \not\preceq_B G$: when G is a nontrivial locally countable bipartite Borel graph with meager connectedness relation. More generally,

Proposition 3. *Suppose G is a Borel graph on a Polish space X whose connectedness relation is meager and there is a Borel homeomorphism $T \colon X \to X$ such that $T(x)\ G\ x$ for every $x \in X$. Then for every locally countable Borel graph H, $G^+ \not\preceq_B H$.*

We use this proposition to prove the following theorem, which answers a question of Kechris and Marks [5, Section 3.(H)]:

Theorem 4. *There is a Borel graph G such that for any connected component C of G, $G \restriction C$ has a Borel ω-coloring. However, there is no Borel graph homomorphism from G to a locally countable Borel graph.*

The paper is organized as follows. In Section 2, we use a Baire category argument to prove a lemma about the Friedman-Stanley jump for Borel graphs, and then use this to prove Theorem 4. In Section 3, we prove Theorem 1 about the Louveau jump, modulo Theorem 2. Finally, in Section 4, we prove Theorem 2 concerning how hard it is to separate the iterated Fréchet ideal and filter, using effective descriptive set theory.

2. A Friedman-Stanley Jump for Borel Graphs

We begin by considering the jump $G \mapsto G^+$ for Borel graphs defined above. First, we show that Proposition 3 implies Theorem 4.

Proof of Theorem 4: Let $T\colon X \to X$ be a fixed-point free Borel homeomorphism of a perfect Polish space X, which induces the Borel graph G_T on X where $x \mathrel{G_T} y$ if $T(x) = y$ or $T(y) = x$. Let $G = (G_T)^+$. Now if C is a connected component of G, then fixing some $y \in C$, map each $x \in C$ to the unique $x(i)$ such that $x(i)$ is in the same G-component as $y(0)$. This is a countable Borel coloring, since connected components of G are countable. However, there is no Borel homomorphism from G to a locally countable Borel graph H by Proposition 3. □

We now use a Baire category argument to prove Proposition 3.

Proof of Proposition 3: A basis for X^ω consists of the basic open sets $N_{U_0,\ldots,U_{k-1}} = \{x \in X^\omega : \forall i < k(x(i) \in U_i)\}$, where $U_0, \ldots, U_{k-1} \subseteq X$ are open. Here we say $N_{U_0,\ldots,U_{k-1}}$ *restricts coordinates less than* k. Let H be a locally countable Borel graph on a Polish space Y. By changing the topology on Y, we may assume that H is generated by countably many Borel homeomorphisms $S_0, S_1, \ldots$ (see [6, Exercise 13.5]).

Suppose that $p\colon \omega \to \omega$ is bijection. Then define the function $T_p\colon X^\omega \to X^\omega$ by

$$T_p(x)(n) = T(x(p(n)))$$

for all $n \in \omega$. Note that since T is a homeomorphism, T_p is a homeomorphism which sends basic open sets to basic open sets, and $T_p(x) \mathrel{G^+} x$ for every $x \in X^+$.

Suppose h is a Borel function from X^+ to Y. We will use a Baire category argument to show that h is not a homomorphism from G^+ to H. Note that X^+ is a comeager subset of X^ω, since the connectedness relation of G is meager. Fix a comeager set $C \subseteq X^+$ such that $h \restriction C$ is continuous ([6, Theorem 8.38]). Our proof breaks down into three cases.

Case 1: Assume that $h \restriction C$ is constant on some open set U. We will find $x, y \in U$ such that $x \mathrel{G^+} y$, but $h(x) = h(y)$. We may assume U is basic open and restricts coordinates less than k. Let $p\colon \omega \to \omega$ be the permutation which swaps the interval $[0, k)$ with the interval $[k, 2k)$:

$$p(n) = \begin{cases} n + k & \text{if } n < k \\ n - k & \text{if } k \leq n < 2k \\ n & \text{if } n > 2k. \end{cases}$$

Then $T_p^{-1}(U)$ is a basic open set which restricts coordinates in $[k, 2k)$, and so $V = T_p^{-1}(U) \cap U \neq \emptyset$. We claim that comeagerly many $x \in V$ have $h(x) = h(T_p(x))$. This is because $h \restriction C$ is constant on U, and since $C \cap U$ is comeager in U and T_p is a homeomorphism, $T_p^{-1}(C \cap U)$ is comeager in $T_p^{-1}(U)$. Hence comeagerly many $x \in V$ have $x \in C$ and $T_p(x) \in C$, and so $h(x) = h(T_p(x))$.

Case 2: There is a basic open set $U = N_{U_0,\dots,U_{k-1}}$, such that the value of $h \restriction C$ on $x \in N_{U_0,\dots,U_{k-1}}$ depends only on the first k coordinates of x, but $h \restriction C$ is not constant on any open set. That is, if we take any two basic open neighborhoods $N_{V_0,\dots,V_n}, N_{V_0',\dots,V_n'} \subseteq N_{U_0,\dots,U_{k-1}}$ where $V_i = V_i'$ for $i < k$, then $h(C \cap N_{V_0,\dots,V_n}) \cap h(C \cap N_{V_0',\dots,V_n'}) \neq \emptyset$. In this case, we will find $x, y \in U$ such that $y = T_p(x)$ and so $x \, G^+ \, y$, but $h(x) \not\!\!H \, h(y)$.

Let p be the permutation above which swaps the interval $[0,k)$ with $[k,2k)$. Let $V = U \cap T_p^{-1}(U)$ which is a nonempty open set. Let $A_m = \{x \in X^+ : S_m(h(x)) \neq h(T_p(x))\}$. It suffices to show that each A_m is comeager in V, since then the set of x such that $h(x) \not\!\!H \, h(T_p(x))$ is comeager in V. Fix $m \in \omega$. Given an arbitrary open subset $W \subseteq V$, it suffices to show that A_m is nonmeager in W. Since $h \restriction C$ is continuous and not constant on any open set and S_m is a homeomorphism, we can find open sets $W_0, W_1 \subseteq W$ such that $S_m(h(C \cap W_0)) \cap h(C \cap T_p(W_1)) = \emptyset$. By refining, we may assume that W_0 and W_1 are basic open sets. But since $h \restriction C$ depends only on the first k coordinates, we may assume W_0 and $T_p(W_1)$ restrict only coordinates less than k, and so W_1 only restricts coordinates in $[k, 2k)$. Thus, $W_0 \cap W_1$ is open and nonempty, and comeagerly many x in $W_0 \cap W_1$ are in A_m.

Case 3: Assume Case 1 and Case 2 do not occur. In this case, instead of specifying a particular permutation $p \colon \omega \to \omega$ and considering h applied to a generic x and $T_p(x)$, we will consider a generic permutation p. More precisely, let $Y \subseteq \omega^\omega$ be the closed set of injective functions from $\omega \to \omega$. A basis for Y consists of the sets $N_q = \{p \in Y : p \supseteq q\}$ for some finite partial injection q. Now the set of $p \in Y$ such that p is a bijection from ω to ω is dense G_δ. To complete the proof, we will show that comeagerly many $(x, p) \in X^+ \times Y$ have the property that $h(x) \not\!\!H \, h(T_p(x))$.

For each n, let $B_n = \{(x,p) \in X^+ \times Y : S_n(h(x)) \neq h(T_p(x))\}$. It suffices to show that B_n is nonmeager in every open set $U \times V \subseteq X^+ \times Y$. By refining, we may assume there is some k so that U is a basic open set which restricts coordinates less than k, and V is a basic open set of the form $V = N_q$ where q is a finite partial injection with $\mathrm{dom}(q) = \mathrm{ran}(q) = k$.

Because U only restricts coordinates less than k, and $\mathrm{dom}(q) = \mathrm{ran}(q) = k$ we have that for all $p, p' \in N_q$, $T_p(U) = T_{p'}(U)$. Hence we can define $T_q(U)$ to be this common value (though q is only a finite partial function). This can be done in any situation where the range and domain of q include all coordinates restricted by a basic open set U.

Let $W = T_q(U)$. We claim that there are basic open sets $U' \subseteq U$ and $W' \subseteq W$, $q' \supseteq q$ and k' such that:

(i) $\mathrm{dom}(q') = \mathrm{ran}(q') \geq k'$.
(ii) U' and W' only restrict coordinates less than k'.
(iii) $T_{q'}(U') \cap W' \neq \emptyset$.
(iv) $S_n(h(U' \cap C)) \cap h(W' \cap C) = \emptyset$.

We establish this claim as follows. Since we are not in Case 1 or Case 2, we can find basic open $U_1 = N_{V_0,\dots,V_j}$ and $U_2 = N_{V'_0,\dots,V'_j}$ with $U_1, U_2 \subseteq U$ such that $V_i = V'_i$ for all $i < k$, and $h(U_1 \cap C) \cap h(U_2 \cap C) = \emptyset$. Hence, $S_n(h(U_1 \cap C)) \cap S_n(h(U_2 \cap C)) = \emptyset$ since S_n is a homeomorphism. Let $U^* = N_{V_0,\dots,V_{k-1}}$. We can find a basic open $W' \subseteq T_q(U^*)$ such that $h(W' \cap C)$ is disjoint from $S_n(h(U_i \cap C))$ for some $i \in \{0,1\}$. Let $U' = U_i$ and so part (iv) of the claim holds. As $U^* \subseteq U$, this definition of W' ensures that $W' \subseteq W$. Let k' be such that W' and U' do not restrict coordinates less than k'.

Let $c = k' - k$. Define q' as follows

$$q'(i) = \begin{cases} q(i) & \text{if } i < k \\ i + c & \text{if } k \leq i < k + c \\ i - c & \text{if } k + c \leq i < k + 2c. \end{cases}$$

It remains to show part (iii) of the claim. If $T_{q'}(U') \cap W' = \emptyset$ then these sets must place incompatible requirements on some coordinate. By construction of q', this coordinate must be less than k (as W' restricts coordinates in $[0,\dots,k+c)$ and $T_{q'}(U')$ restricts coordinates in $[0,\dots,k) \cup [k+c,\dots,k+2c)$). However this is not the case as $W' \subseteq T_q(U^*)$ and $T_q(U^*)$ places the same restrictions on coordinates less than k as does $T_q(U')$.

Now using the claim we will complete the proof. Let r be the inverse of q'. Take any $p \in N_{q'}$. Take any $x \in U' \cap T_r(W')$. If x is also in the comeager set $C \cap T_p^{-1}(C)$, then $T_p(x) \in W' \cap C$ and $x \in U' \cap C$. Hence $S_n(h(x)) \neq h(T_p(x))$. Thus for comeagerly many $x \in U' \cap T_r(W')$ we have that $(x,p) \in B_n$. Thus $B_n \cap ((U' \cap T_r(W')) \times N_{q'})$ is comeager and so B_n is nonmeager in $U \times V$.

□

3. The Louveau jump for Borel graphs

In this section, we consider the Louveau jump $G \mapsto G^*$ defined in the introduction, and we prove Theorem 1, that if G is a bipartite Borel graph, there is no Borel homomorphism from G^* to G. Note that among all graphs, there is not always a homomorphism from G^* to G (consider the G that is the disjoint union of the complete graphs K_n for every n). However, if G is bipartite, then using the axiom of choice we can always find a homomorphism from G^* to G. This is because if G^* has a cycle of length n, then G clearly has a cycle of length n. Hence, if G is bipartite, then G^* is also bipartite, so using choice we can find a homomorphism of G^* into any two vertices of G connected by an edge.

Like Louveau's argument in [7], our proof will involve potential complexity in a key way. Recall that if Γ is a class of subsets of Polish spaces, X is a Polish space with topology τ, and $A \subseteq X \times X$, then A is said to be *potentially* Γ if there is a Polish topology τ' on X inducing the same Borel σ-algebra as τ such that A is Γ as a subset of $X \times X$ with topology $\tau' \times \tau'$. Louveau's proof relies on the fact that if E and F are equivalence relations on Polish spaces X and Y and Γ is a class of sets closed under continuous preimages, then if $E \leq_B F$ and F is potentially Γ, then so is E. However, the analogous fact fails for Borel graphs under Borel homomorphism: there are Borel graphs of arbitrarily high potential complexity which admit Borel homomorphisms to Borel graphs of low potential complexity (e.g. to K_2). So instead, we will consider the complexity of separating pairs of points of distance one and distance two.

Definition 5. If G is a graph on X, let $D_{G,n}$ be the set of pairs $(x, y) \in X \times X$ such that there is a path from x to y of length n in G.

For example, if G is an analytic triangle-free graph on a Polish space X, then $D_{G,1}$ and $D_{G,2}$ are disjoint analytic sets which can therefore be separated by a Borel set.

Proposition 6. *Suppose G and H are triangle-free graphs on the Polish spaces X and Y, and there is a Borel homomorphism from G to H. Then if $D_{H,1}$ and $D_{H,2}$ can be separated by a potentially $\mathbf{\Sigma}^0_\alpha$ (resp. $\mathbf{\Pi}^0_\alpha$) set, then $D_{G,1}$ and $D_{G,2}$ can also be separated by a potentially $\mathbf{\Sigma}^0_\alpha$ (resp. $\mathbf{\Pi}^0_\alpha$) set.*

Proof. Let h be a Borel homomorphism from G to H. By changing topology, we may assume that h is continuous (see [6, Theorem 3.11]). Since h is a homomorphism, if there is a path from x to y in G of length n, then there is a path from $h(x)$ to $h(y)$ in H of length n. Hence, the preimage of a set separating $D_{H,1}$ and $D_{H,2}$ under the function $(x, y) \mapsto (h(x), h(y))$ will separate $D_{G,1}$ and $D_{G,2}$. The result follows since the classes $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$ are closed under taking continuous preimages. □

Next, we will define a transfinite way to iterate the operation $G \mapsto G^*$. In what follows, for each countable limit ordinal λ, let $\pi_\lambda : \omega \to \alpha$ be an increasing and cofinal function. For each successor ordinal $\alpha + 1$, let $\pi_{\alpha+1} : \omega \to \alpha + 1$ be the constantly α function. These functions will allow us to simultaneously handle limit and successor cases in a uniform way.

Suppose $(G_i)_{i\in\omega}$ is a countable sequence of graphs. Define $(G_i)^*$ to be the graph whose vertices are elements of $\prod_i G_i$ and such that $x(G_i)^*y$ if there exists an n such that for all $m > n$ we have $x(m)G_m y(m)$. Now we define G^α for countable ordinals α as follows.

$$G^0 = G$$
$$G^\alpha = (G^{\pi_\alpha(n)})^* \quad \text{if } \alpha > 0$$

Note that under this definition, $G^1 = G^*$.

Lemma 7. *Let G and H be graphs. If there is a Borel homomorphism from G to H, then for all countable ordinals α, there is a Borel homomorphism from G^α to H^α.* □

Lemma 8. *Let $(G_i)_{i\in\omega}$ be a countable sequence of graphs. Assume for each $i \in \omega$ that $h_i : G_i \to G$ is a Borel homomorphism. Then there is a Borel homomorphism from $(G_i)^*$ to G^*.*

Proof. Let X be the vertex set of G and X_i be the vertex set of G_i. Then define the Borel homomorphism $g : \prod_i X_i \to \prod_i X$ from $(G_i)^*$ to G^* by $g(x)(n) = h_n(x(n))$ i.e. apply to the n^{th} vertex in the sequence the mapping h_n. Now assume that $x\ (G_i)^*\ y$. Then for some n for all $m > n$ we have that $x(m)\ G_m\ y(m)$. Hence $h_m(x(m))\ G\ h_m(y(m))$ and so $h(x)\ G^*\ h(y)$. □

Lemma 9. *If there is a Borel homomorphism from G^* into G, then for all countable ordinals α, there is a Borel homomorphism from G^α into G.*

Proof. If for all $\beta < \alpha$ there is a Borel homomorphism from G^β to G, then by the previous lemma, there is a Borel homomorphism from G^α to G^*. Now this homomorphism can be composed with the homomorphism from G^* to G to obtain a Borel homomorphism from G^α to G. □

As usual, let us denote by K_2 the complete graph on two vertices. The graphs K_2^α for countable ordinals α will play a key part in the main theorem of this section. We will investigate these graphs further. First we will define the spaces on which these graphs live. Inductively define X^α as follows.

$$X^0 = 2$$
$$X^\alpha = \prod_n X^{\pi_\alpha(n)} \quad \text{if } \alpha > 0$$

It should be clear from the definition of G^α that K_2^α is a graph on X^α. If $\alpha > 0$, then the space X^α is clearly homeomorphic to 2^ω. We want to define a homeomorphism from X^α to 2^ω that respects the manner in which X^α has been inductively constructed. We define $\gamma^\alpha : X^\alpha \to 2^\omega$ inductively as follows.

(i) γ^1 is the identity map.
(ii) For $\alpha > 1$, we define $\gamma^\alpha(x)$ to be the sequence z such that for all m and n, $z(\langle m, n\rangle) = (\gamma^{\pi_\alpha(m)}(x(m)))(n)$.

The edge relation on K_2^α can be thought of in terms of the iterated Fréchet filter. The α iterate of the Fréchet filter is a subset of X^α. It will be denoted by F^α and it is defined inductively as follows.

$$\mathrm{F}^0 = \{1\}$$
$$\mathrm{F}^\alpha = \{x \in X^\alpha \colon \exists n\, \forall m > n (x(m) \in \mathrm{F}^{\pi_\alpha(m)})\} \quad \text{if } \alpha > 0.$$

Note that F^1 is the standard Fréchet, or cofinite, filter. Similarly we can define the iterated Fréchet ideals.

$$\mathrm{I}^0 = \{0\}$$
$$\mathrm{I}^\alpha = \{x \in X^\alpha \colon \exists n\, \forall m > n (x(m) \in \mathrm{I}^{\pi_\alpha(m)})\} \quad \text{if } \alpha > 0.$$

If x, y are elements of 2^ω then we denote by $x\Delta y$ the binary sequence z where $z(n) = 0$ if $x(n) = y(n)$ and $z(n) = 1$ otherwise. The notation Δ is used because $x\Delta y$ is the symmetric difference of x and y when x and y are considered as subsets of ω under the standard bijection between 2^ω and $\mathcal{P}(\omega)$.

We can use the fact that X^α is homeomorphic to 2^ω in order to induce a symmetric difference operation on X^α when $\alpha > 0$. Define the operation Δ^α on X^α by $x\Delta^\alpha y = z$ if and only if $\gamma^\alpha(x)\Delta\gamma^\alpha(y) = \gamma^\alpha(z)$. For the case $\alpha = 0$, for $x, y \in X^0$, define $x\Delta^0 y = 0$ if $x = y$ and $x\Delta^0 y = 1$ if $x \neq y$.

Since the homeomorphism γ^α is defined simply by permuting indices from a product of infinitely many copies of 2 to yield 2^ω, we have that symmetric difference operation commutes with γ^α.

We will make use of the following properties of Δ^α.

(i) The operation Δ^α is associative and commutative.
(ii) $x\Delta^\alpha x\Delta^\alpha y = y$.
(iii) For all $\alpha > 0$, $(x\Delta^\alpha y)(n) = x(n)\Delta^{\pi_\alpha(n)}y(n)$.
(iv) For all α, if $x \in \mathrm{I}^\alpha$, then $x\Delta^\alpha y \in \mathrm{F}^\alpha$ if and only if $y \in \mathrm{F}^\alpha$.

The following lemma explains how to define the edge relation in K_2^α using F^α.

Lemma 10. *For all α and $x, y \in X^\alpha$, $xK_2^\alpha y$ if and only if $x\Delta^\alpha y \in \mathrm{F}^\alpha$.*

Proof. This result holds trivially for the case $\alpha = 0$. Take $\alpha > 0$ and assume the result holds for all smaller ordinals. Consider $x, y \in X^\alpha$. We have that

$$\begin{aligned} xK_2^\alpha y &\Leftrightarrow \exists n\, \forall m > n(x(m)K_2^{\pi_\alpha(m)}y(m)) \\ &\Leftrightarrow \exists n\, \forall m > n(x(m)\Delta^{\pi_\alpha(m)}y(m)) \in \mathrm{F}^{\pi_\alpha(m)} \\ &\Leftrightarrow \exists n\, \forall m > n(x\Delta^\alpha y)(m) \in \mathrm{F}^{\pi_\alpha(m)} \\ &\Leftrightarrow x\Delta^\alpha y \in \mathrm{F}^\alpha. \end{aligned}$$ □

Similarly, we have the following:

Lemma 11. *For all α and $x, y \in X^\alpha$, $x\Delta^\alpha y \in \mathrm{I}^\alpha$ if and only if there is a path of length 2 from x to y in K_2^α.*

Proof. Suppose $xK_2^\alpha z$. By the above lemma, this is true if and only if $x\Delta^\alpha z \in \mathrm{F}^\alpha$. Letting $w = x\Delta^\alpha z$, we have $zK_2^\alpha y$ if and only if $z\Delta^\alpha y \in \mathrm{F}^\alpha$. Finally, note that $z\Delta^\alpha y = (w\Delta^\alpha x)\Delta^\alpha y = w\Delta^\alpha(x\Delta^\alpha y)$ which is in F^α if and only if $x\Delta^\alpha y \in \mathrm{I}^\alpha$, since $w \in \mathrm{F}^\alpha$. □

Lemma 12. *Let $C \subseteq 2^\omega$ be a comeager set. Then there is a continuous map $f : 2^\omega \to 2^\omega \times 2^\omega$ such that for all x:*

(i) $f_0(x) \in C$.
(ii) $f_1(x) \in C$.
(iii) $f_0(x)\Delta f_1(x) = x$.

Proof. Let (E_n) be a sequence of dense open sets such that $\bigcap_n E_n \subseteq C$. We will use p_n and q_n to denote finite strings.

Fix x. Let p_0 and q_0 be the empty string $\emptyset$. Now given p_n, take $p'_{n+1} < p_n$, such that $N_{p'_{n+1}} \subseteq E_n$. Define $q'_{n+1} < q_n$ so that for all $i < |q'_{n+1}|$, $x(i) = 1$ if and only if $p'_{n+1}(i) \neq q'_{n+1}(i)$. Now take $q_{n+1} < q'_{n+1}$, such that $N_{q_{n+1}} \subseteq E_n$. Define $p_{n+1} < p'_{n+1}$ so that for all $i < |q_{n+1}|$, $x(i) = 1$ if and only if $p_{n+1}(i) \neq q_{n+1}(i)$.

By induction, for all n, we have that $p_{n+1} < p_n$, and $q_{n+1} < q_n$. Let $f_0(x) = \lim_n p_n$ and $f_1(x) = \lim_n q_n$. Because we only used finitely many bits of x at each stage of the construction, this construction gives a continuous map with the desired properties. $\square$

Lemma 13. *For any α and any comeager set C in X^α, there is a continuous map $f : X^\alpha \to X^\alpha \times X^\alpha$ such that for all x:*

(i) $f_0(x) \in C$.
(ii) $f_1(x) \in C$.
(iii) $f_0(x)\Delta^\alpha f_1(x) = x$.

Proof. This follows from the previous lemma and the fact that there is a homeomorphism $\gamma^\alpha : X^\alpha \to 2^\omega$ such that for $x, y \in X^\alpha$, $x\Delta^\alpha y$ if and only if $\gamma^\alpha(x)\Delta\gamma^\alpha(y)$. $\square$

We now have the following lemma:

Lemma 14. *Suppose $\alpha < \omega_1$. Then $D_{K_2^\alpha,1}$ and $D_{K_2^\alpha,2}$ can be separated by a potentially $\mathbf{\Sigma}^0_\alpha$ set, but not by a potentially $\mathbf{\Delta}^0_\alpha$ set.*

Proof. By the above lemmas, we have $D_{K_2^\alpha,1} = \{(x, y)\colon x\Delta^\alpha y \in \mathrm{F}^\alpha\}$, and $D_{K_2^\alpha,2} = \{(x, y)\colon x\Delta^\alpha y \in \mathrm{I}^\alpha\}$. Now the function $(x, y) \mapsto x\Delta^\alpha y$ is continuous and so by Theorem 2, $D_{K_2^\alpha,1}$ and $D_{K_2^\alpha,2}$ can be separated by a potentially $\mathbf{\Sigma}^0_\alpha$ set.

Suppose there was a topology τ' on X^α so that $D_{K_2^\alpha,1}$ and $D_{K_2^\alpha,2}$ were separable by a potentially $\mathbf{\Delta}^0_\alpha$ set. Let τ be the usual topology on X^α. Then there is a comeager set C on which the identity function $id\colon (X^\alpha, \tau) \to (X^\alpha, \tau')$ is continuous. Let f_0 and f_1 be as in Lemma 13. Then $id \circ f_0$ and $id \circ f_1$ are continuous, and taking the preimage of the separating set under $x \mapsto (id \circ f_0(x), id \circ f_1(x))$ would separate I^α and F^α by a $\mathbf{\Delta}^0_\alpha$ set, contradicting Theorem 2. $\square$

We are now ready to prove (a slight generalization of) Theorem 1 from the introduction.

Theorem 15. *Let G be a triangle-free analytic graph on a Polish space, with at least one edge. Then there is no Borel homomorphism from G^* to G.*

Proof. Since $D_{G,1}$ and $D_{G,2}$ are analytic sets, they can be separated by a Borel set which is $\mathbf{\Delta}^0_\beta$ for some countable ordinal β. Let α be such that F^α and I^α are not separable by a $\mathbf{\Delta}^0_\beta$ set.

Assume that there is a Borel homomorphism from G^* into G. By Lemma 9, there is a Borel homomorphism from G^α to G. By Lemma 7, there is Borel homomorphism from K_2^α to G^α and hence a Borel homomorphism from K_2^α to G. By Proposition 6, this implies $D_{K_2^\alpha,1}$ and $D_{K_2^\alpha,2}$ can be separated by a Borel set which is potentially $\mathbf{\Delta}^0_\beta$, contradicting our choice of α. □

4. Separating the Iterated Fréchet Filters and Ideals

In this section, we give a new proof of Theorem 2.

We begin by proving that F^α and I^α can be separated by disjoint $\mathbf{\Sigma}^0_{\alpha+1}$ sets. The idea here is based on a trick for switching quantifier order which is encapsulated in the following lemma.

Lemma 16. *For each $n, k \in \mathbb{N}$ let $A_{n,k}$ and $B_{n,k}$ be disjoint $\mathbf{\Sigma}^0_\alpha$ subsets of a Polish space $X_{n,k}$. Let*

$$A = \{x \in \textstyle\prod X_{n,k} \colon (\exists m)(\forall n > m)(\exists j)(\forall k > j)(x(n,k) \in A_{n,k})\},$$
$$B = \{x \in \textstyle\prod X_{n,k} \colon (\exists m)(\forall n > m)(\exists j)(\forall k > j)(x(n,k) \in B_{n,k})\}.$$

Then there are A^*, B^*, disjoint $\mathbf{\Sigma}^0_{\alpha+2}$ subsets of $\prod X_{n,k}$ such that $A \subseteq A^*$ and $B \subseteq B^*$.

Proof. Let us define the sets

$$A' = \{x \in \textstyle\prod X_{n,k} \colon (\exists m)(\forall n > m)(\forall j)(\exists k > j)(x(n,k) \in A_{n,k})\},$$
$$B' = \{x \in \textstyle\prod X_{n,k} \colon (\exists m)(\forall n > m)(\forall j)(\exists k > j)(x(n,k) \in B_{n,k})\}.$$

The sets A' and B' are $\mathbf{\Sigma}^0_{\alpha+2}$. Now $A \subseteq A'$ because if for some n there are cofinitely many k such that $x(n,k) \in A_{n,k}$ then there are infinitely many k such that $x(n,k) \in A_{n,k}$. Also $B \cap A' = \emptyset$ because if there are infinitely many k such that $x(n,k) \in A_{n,k}$, then there cannot be cofinitely many k such that $x(n,k) \in B_{n,k}$ (as $A_{n,k}$ and $B_{n,k}$ are disjoint). Similarly we have that $B \subseteq B'$ and $A \cap B' = \emptyset$. Finally apply separation to obtain disjoint $\mathbf{\Sigma}^0_{\alpha+2}$ sets A^* and B^* with $A^* \subseteq A'$, $B^* \subseteq B'$, and $A^* \cup B^* = A' \cup B'$. Hence $A \subseteq A^*$ and $B \subseteq B^*$. □

Lemma 17. *For all α, F^α and I^α can be separated by disjoint $\mathbf{\Sigma}^0_{\alpha+1}$ sets.*

Proof. If α is 0, 1, or a limit ordinal then there is nothing to prove because the sets F^α and I^α already have the required complexity.

If $\alpha = \beta + 1$ where β is a limit ordinal, then

$$\mathrm{F}^\alpha = \{x \in X^\alpha \colon (\exists m)(\forall n > m)(\exists j)(\forall k > j)(x(n)(k) \in \mathrm{F}^{\pi_\beta(k)})\},$$
$$\mathrm{I}^\alpha = \{x \in X^\alpha \colon (\exists m)(\forall n > m)(\exists j)(\forall k > j)(x(n)(k) \in \mathrm{I}^{\pi_\beta(k)})\}.$$

As for all k, the sets $\mathrm{F}^{\pi_\beta(k)}$ and $\mathrm{I}^{\pi_\beta(k)}$ are disjoint $\mathbf{\Sigma}^0_\beta$, we can apply Lemma 16, to find disjoint $\mathbf{\Sigma}^0_{\alpha+1}$ sets that separate F^α and I^α.

Finally, we need to consider the case that α is $\beta+n$ where $n \geq 2$ and β is a limit ordinal or 0. In this situation we have that

$$\begin{aligned} \mathrm{F}^\alpha &= \{x \in X^\alpha \colon (\exists m)(\forall n > m)(\exists j)(\forall k > j)(x(n)(k) \in \mathrm{F}^{\beta+n-2})\}, \\ \mathrm{I}^\alpha &= \{x \in X^\alpha \colon (\exists m)(\forall n > m)(\exists j)(\forall k > j)(x(n)(k) \in \mathrm{I}^{\beta+n-2})\}. \end{aligned}$$

By induction we can assume that we have disjoint $\mathbf{\Sigma}^0_{\beta+n-1}$ sets A and B such that $\mathrm{F}^{\beta+n-2} \subseteq A$ and $\mathrm{I}^{\beta+n-2} \subseteq B$. Hence

$$\begin{aligned} \mathrm{F}^\alpha &\subseteq \{x \in X^\alpha \colon (\exists m)(\forall n > m)(\exists j)(\forall k > j)(x(n)(k) \in A)\}, \\ \mathrm{I}^\alpha &\subseteq \{x \in X^\alpha \colon (\exists m)(\forall n > m)(\exists j)(\forall k > j)(x(n)(k) \in B)\}. \end{aligned}$$

Thus again by Lemma 16 we have that F^α and I^α can be separated by disjoint $\mathbf{\Sigma}^0_{\alpha+1}$ sets. □

The next goal is to show that if $\alpha \geq 1$, the separation obtained in the previous lemma is optimal. The case for $\alpha = 1$ is simple.

Lemma 18. *There do not exist disjoint $\mathbf{\Pi}^0_2$ sets A and B such that $\mathrm{F}^1 \subseteq A$ and $\mathrm{I}^1 \subseteq B$.*

Proof. If $\mathrm{F}^1 \subseteq A$ then A must be comeager (as any dense $\mathbf{\Pi}^0_2$ set is comeager). If $\mathrm{I}^1 \subseteq B$ then B is comeager. Hence $A \cap B \neq \emptyset$. □

For $\alpha > 1$, our plan is to take $x \in X^1$ and continuously encode x into an element $\rho(x)$ of X^α such that if $x \in \mathrm{F}^1$, then $\rho(x) \in \mathrm{F}^\alpha$ and if $x \in \mathrm{I}^1$, then $\rho(x) \in \mathrm{I}^\alpha$. We want to do this in such a way that, relative to some parameter p, x can uniformly compute the α^{th} iterate of the Turing jump of $\rho(x)$ (or the $(\alpha-1)^{\text{th}}$ iterate of the jump if $\alpha < \omega$). From this it will follow that if F^α and I^α are separable by $\Delta^0_{\alpha+1}$ sets, then F^1 and I^1 are separable by $\Delta^0_2(p)$ sets which we know by Lemma 18 is impossible.

We will introduce some of the main ideas needed with an example. Let T be the full ω-branching tree of height 2. Hence the nodes of T are all strings of natural numbers of length 0, 1, or 2. We will denote the root of the tree by $\emptyset$. Given x we are now going to label all nodes of T except $\emptyset$ with either 0 or 1. This is a function $f : T \setminus \{\emptyset\} \to 2$. We will encode x at the first level of the tree so for all $i \in \mathbb{N}$, $f(i) = x(i)$. At the second level of the tree, we will label nodes so that for all i, $\lim_j f(i \frown j)$ exists and is equal to $f(i)$. In order to complete our definition of f we need some additional information as to what values $f(i \frown j)$ should take before the limit is reached. For this, we will take an additional function g that maps from the nodes of T, that are neither the root nor a leaf, to $2^{<\omega}$. For our example tree, g is a function from nodes of length 1 to $2^{<\omega}$. Now we can complete our definition of f as follows. Fix i.

(i) If $j < |g(i)|$, then $f(i \frown j) = g(i)(j)$.
(ii) If $j = |g(i)|$, then $f(i \frown j) = 1 - f(i)$.

(iii) If $j > |g(i)|$, then $f(i \frown j) = f(i)$.

Defining $f(i \frown j) = 1 - f(i)$ for $j = |g(i)|$ allows us to recover $g(i)$ from the function $j \mapsto f(i \frown j)$.

Now consider f, restricted to the leaf nodes. We can regard f as an element of X^2 by mapping f to $z \in X^2$ where $z(i)(j) = f(i \frown j)$. Further, if $x \in \mathrm{F}^1$, then $z \in \mathrm{F}^2$, and if $x \in \mathrm{I}^1$, then $z \in \mathrm{I}^2$. Hence if we fix g, we have a mapping $\rho : X^1 \to X^2$ with the desired properties. We will show that if g is sufficiently generic, then *for any* x, $\rho(x)'$ is uniformly Turing reducible to $x \oplus g \oplus 0'$. Given this, assume that F^2 and I^2 are separable by disjoint Δ^0_3 sets A and B with $\mathrm{F}^2 \subseteq A$ and $\mathrm{I}^2 \subseteq B$. This means that the double jump of $\rho(x)$ can uniformly determine whether $\rho(x)$ is in A or B, and hence $(g \oplus x \oplus 0')'$ can uniformly determine if $\rho(x)$ is in A or B. As g is independent of x, this means that the sets $\{x \in X^1 : \rho(x) \in A\}$ and $\{x \in X^1 : \rho(x) \in B\}$ are $\Delta^0_2(g \oplus 0')$. But these sets separate F^1 from I^1, a contradiction. By relativization, we get that F^2 and I^2 are not separable by $\boldsymbol{\Delta}^0_3$ sets. As separation by disjoint $\boldsymbol{\Pi}^0_\alpha$ sets implies separation by disjoint $\boldsymbol{\Delta}^0_\alpha$ sets, we conclude that F^2 and I^2 are not separable by $\boldsymbol{\Pi}^0_3$ sets.

Our proof of Theorem 2 is written in the language of effective descriptive set theory. We will establish a lightface version of this theorem, and then obtain the full result by relativization. Let us start by defining a wellfounded tree $T_\alpha \subseteq \omega^{<\omega}$ for each computable ordinal α. Let T_0 be the tree consisting of just the empty string, so $T_0 = \{\emptyset\}$. For $\alpha > 0$, let T_α be the tree having ω many branches at the root with the tree above the n^{th} node being $T_{\pi_\alpha(n)}$, so $n \frown \sigma \in T_\alpha$ if $\sigma \in T_{\pi_\alpha(n)}$. We can assume that the functions π_β, for $\beta \leq \alpha$, are all uniformly computable. Note that for each α, the tree T_α has rank α.

In what follows, fix a computable ordinal γ and take T to be T_γ. For all $\alpha \leq \gamma$ we define the following subsets of T.

$$\begin{aligned} N_\alpha &= \{\sigma \in T : \operatorname{rank}(\sigma) = \alpha\} \\ L_\alpha &= \{\sigma \in T : \operatorname{rank}(\sigma) < \alpha\} \\ A_\alpha &= \{\sigma \in T : \operatorname{rank}(\sigma) < \alpha \wedge \operatorname{rank}(\sigma^-) \geq \alpha\} \end{aligned}$$

Here by σ^- we mean σ without its last bit.

Note that $N_0 = A_1$ and is the set of all leaf nodes of T. However, if $\gamma \geq \omega$, then $N_1 \subsetneq A_2$.

The following lemma follows immediately from these definitions.

Lemma 19.

(i) $N_\alpha \subseteq A_{\alpha+1}$.
(ii) $A_{\alpha+1} \setminus N_\alpha \subseteq A_\alpha$.
(iii) For each $0 < \alpha \leq \gamma$ *the set* A_α *is a maximal anti-chain.*
(iv) A_γ *is the set of successors of the root of* T_γ. □

Because T_γ is constructed using the same functions π_α, for $\alpha \leq \gamma$ as X^γ, there is a natural homeomorphism from 2^{N_0} to X^γ. This is defined as follows. We map $f \in 2^{N_0}$ to $x \in X^\gamma$ where for all sequences $i_1 \frown i_2 \frown \ldots \frown i_k \in N_0$ we have that

$$x(i_1)(i_2)\ldots(i_k) = f(i_1 \frown i_2 \frown \ldots \frown i_k).$$

To simplify the exposition, let us identify X^γ with 2^{N_0} and so we can regard both F^γ and I^γ as subsets of 2^{N_0}.

We will now define a family of continuous maps ρ_g from X^1 to X^γ. Suppose $x \in X^1$ and g is a function from $T \setminus (N_0 \cup \{\emptyset\})$ to $2^{<\omega}$. Then there is a unique mapping $f_{g,x} : T \setminus \{\emptyset\} \to 2$ with the following properties.

(i) For all $i \in \mathbb{N}$, $f_{g,x}(i) = x(i)$.
(ii) For all nodes $\sigma \in T \setminus (N_0 \cup \{\emptyset\})$, if $|g(\sigma)| = n$, then
   (a) For all $i < n$, $f_{g,x}(\sigma \frown i) = g(\sigma)(i)$
   (b) $f_{g,x}(\sigma \frown n) = 1 - f_{g,x}(\sigma)$
   (c) For all $i > n$, $f_{g,x}(\sigma \frown i) = f_{g,x}(\sigma)$.

Our idea is for that each σ, the values of $f_{g,x}(\sigma \frown i)$ for $i \in \omega$ should eventually reach a limit equal to $f_{g,x}(\sigma)$. The finitely many values before the limit is reached are specified by $g(\sigma)$

If we fix g, we now have a continuous mapping $\rho_g : X^1 \to X^\gamma$ defined by $\rho_g(x) = f_{g,x} \restriction_{N_0}$. This mapping is not surjective, but it does have the following important property.

Lemma 20. *If $x \in \mathrm{F}^1$, then $\rho_g(x) \in \mathrm{F}^\gamma$ and if $x \in \mathrm{I}^1$, then $\rho_g(x) \in \mathrm{I}^\gamma$.* □

We will soon examine what happens to the mapping ρ_g if we take g to be sufficiently generic. In the following, by generic, we will always mean for the Cohen partial order of finite partial functions ordered by inclusion. In particular, suppose f is a function between computable sets A and B (for example, from $T \setminus (N_0 \cup \{\emptyset\})$ to $2^{<\omega}$), and C is a computable subset of A. Then we say that $f \restriction_C$ is *arithmetically generic relative to* $z \in 2^\omega$, if for every arithmetically definable dense open set D of finite partial functions from C to B, there is a finite partial function $p \subseteq f \restriction_C$ such that p meets D. We say that f, g are mutually z-generic, if $f \oplus g$ is z-generic, where $f \oplus g$ is defined on the disjoint union of the domains of f and g.

Suppose h is a function from ω to $2^{<\omega}$, and $y \in 2^\omega$. Fix a computable pairing function $\langle \cdot, \cdot \rangle \colon \omega^2 \to \omega$. Define $h_y \in 2^\omega$ as follows. For all i, j,

$$h_y(\langle i, j \rangle) = \begin{cases} h(i)(j) & \text{if } j < |h(i)| \\ 1 - y(i) & \text{if } j = |h(i)| \\ y(i) & \text{otherwise.} \end{cases}$$

Once again, here we are coding the real y into h_y where each bit $y(i)$ is the limit $\lim_{j \to \infty} h_y(\langle i, j \rangle)$, and the finitely many values before this limit is reached are specified by $h(i)$. We now have the following lemma from [8, Lemma 2.3], where x from that paper corresponds to our h, y from that paper corresponds to our y, $J(x, y)$ is h_y, and we state our lemma only for the case $n = 1$ and relativized to a real z.

Recall finally that if $x \in 2^\omega$, we use x' to denote the Turing jump of x, and if $\alpha < \omega_1^{ck}$, we use $x^{(\alpha)}$ to denote the α^{th} iterate of the Turing jump relative to x.

Lemma 21. *Suppose $y, z \in 2^\omega$, and $g, h\colon \omega \to 2^{<\omega}$ are mutually arithmetically generic relative to z. Then $(g \oplus h_y \oplus z)' \equiv_T g \oplus h \oplus y \oplus z'$ uniformly.*

We emphasize that for the previous lemma to hold, g and h do not need to be generic relative to x. The proof of the next lemma is standard.

Lemma 22. *Suppose $x, z \in 2^\omega$, and $g\colon T \setminus (N_0 \cup \{\emptyset\}) \to 2^{<\omega}$ is arithmetically generic relative to z. Let E be an infinite computable subset of T such that*

(i) $E \cap (A_\gamma \cup \{\emptyset\}) = \emptyset$.
(ii) No infinite subset of E consists of elements sharing a common predecessor

Let A be an infinite computable subset of $T \setminus (N_0 \cup \{\emptyset\})$ such that no element of A is a predecessor of an element of E. Then $g \restriction_A$ and $f_{g,x} \restriction_E$ are mutually arithmetically z-generic.

Proof. Suppose D is an dense open set in the partial order for building $g \restriction_A \oplus f_{g,x} \restriction_E$ which is arithmetically definable from z. Let $\mathbb{P}$ be the partial order for building g: the order of finite partial functions from $(N_0 \cup \{\emptyset\})$ to $2^{<\omega}$. We will define a dense open set D^* in $\mathbb{P}$ so that if g meets D^*, then $g \restriction_A \oplus f_{g,x} \restriction_E$ meets D.

Suppose $p \in \mathbb{P}$. Let P be the set of elements of E that have a predecessor in $\mathrm{dom}(p)$. Note that if $\sigma \in P$, then we cannot effect the value of $f_{g,x}(\sigma)$ by extending p. However, P is finite and for any string $\sigma \in E \setminus P$ with $\sigma = n \frown \tau$, if we extend p to some p^* where $p^*(\tau)$ has length $\geq n$, then $f_{g,x}(\sigma) = \tau(n)$. By considering every possible value of $f_{g,x} \restriction_P$ and iteratively finding extensions for each of them, we can find an extension p^* of p so that $g \restriction_A \oplus f_{g,x} \restriction_E$ meets D, no matter what the value of $f_{g,x} \restriction P$ is. Now define D^* to be the set union over all $p \in \mathbb{P}$ of such strings p^*. $\square$

It does not matter for the purposes of our proof, but a more precise calculation shows that if g is 2-generic relative to z, then $g \restriction_A$ and $f_{g,x} \restriction_E$ will be mutually 1-generic relative to z. Similarly, in Lemma 21, g and h need only be 2-generic for the conclusion to hold.

Following is our main technical lemma.

Lemma 23. *Suppose $x, z \in 2^\omega$, $\alpha < \gamma$, $g\colon T \setminus (N_0 \cup \{\emptyset\}) \to 2^{<\omega}$ is arithmetically generic relative to z. Then uniformly*

$$(f_{g,x} \restriction_{A_\alpha} \oplus g \restriction_{L_\alpha} \oplus z)' \leq_T f_{g,x} \restriction_{A_{\alpha+1}} \oplus g \restriction_{L_{\alpha+1}} \oplus z'.$$

Proof. Separate A_α into two sets D and E. Let D be those elements of A_α who have predecessors in N_α and let E be the rest. Note that if $\sigma \frown i \in E$, then $\operatorname{rank}(\sigma) > \alpha \geq \operatorname{rank}(\sigma \frown i) + 1$ and hence $\operatorname{rank}(\sigma)$ is a limit ordinal. Thus for some m, for all n greater than m, $\operatorname{rank}(\sigma \frown m) > \alpha$ (as the tree is defined to have strictly increasing ranks at limits). It follows that for any $\sigma \in T$, there are not infinitely many successors of σ in E. By considering ranks, it follows that no element of $L_{\alpha+1}$ is a predecessor of an element of E. Hence by Lemma 22, $g \restriction_{L_{\alpha+1}}$ and $f_{g,x} \restriction_E$ are mutually arithmetically z-generic. As $L_{\alpha+1}$ is the disjoint union of L_α and N_α, it follows that $g \restriction_{N_\alpha}$ and $g \restriction_{L_\alpha} \oplus f_{g,x} \restriction_E$ are mutually arithmetically z-generic.

If we let $h = g \restriction_{N_\alpha}$ and $y = f_{g,x} \restriction_{N_\alpha}$, then $h_y = f_{g,x} \restriction_D$ (under some uniform computable bijection) using the notation of Lemma 21. Hence applying Lemma 21, we have that,

$$\begin{aligned}(f_{g,x} \restriction_{A_\alpha} \oplus g \restriction_{L_\alpha} \oplus z)' &\equiv_T ((f_{g,x} \restriction_E \oplus g \restriction_{L_\alpha}) \oplus f_{g,x} \restriction_D \oplus z)' \\ &\leq_T f_{g,x} \restriction_E \oplus g \restriction_{L_\alpha} \oplus g \restriction_{N_\alpha} \oplus f_{g,x} \restriction_{N_\alpha} \oplus z'.\end{aligned}$$

Now as $L_{\alpha+1} = L_\alpha \cup N_\alpha$, and $A_{\alpha+1} = N_\alpha \cup E$. We obtain the desired result. □

Lemma 24. *Suppose $x, z \in 2^\omega$, and $g \colon T \setminus (N_0 \cup \{\emptyset\}) \to 2^{<\omega}$ is arithmetically generic relative to z. If $1 \leq n < \omega$ and $n \leq \gamma$, then uniformly*

$$f_{g,x} \restriction_{A_{n+1}} \oplus g \restriction_{L_{n+1}} \oplus 0^{(n)} \geq_T (f_{g,x} \restriction_{N_0})^{(n)}.$$

Proof. The result holds if $n = 0$ because $A_1 = N_0$. Consider the case $n+1$. We have, by the previous lemma, and the induction hypothesis, that uniformly

$$\begin{aligned}f_{g,x} \restriction_{A_{n+2}} \oplus g \restriction_{L_{n+2}} \oplus 0^{(n+1)} &\geq_T (f_{g,x} \restriction_{A_{n+1}} \oplus g \restriction_{L_{n+1}} \oplus 0^{(n)})' \\ &\geq_T (f_{g,x} \restriction_{N_0})^{(n+1)}.\end{aligned}$$

□

Lemma 25. *Suppose $x, z \in 2^\omega$, and $g \colon T \setminus (N_0 \cup \{\emptyset\}) \to 2^{<\omega}$ is arithmetically generic relative to $0^{(\alpha)}$. If $\omega \leq \alpha \leq \gamma$, then uniformly*

$$f_{g,x} \restriction_{A_\alpha} \oplus g \restriction_{L_\alpha} \oplus 0^{(\alpha)} \geq_T (f_{g,x} \restriction_{N_0})^{(\alpha)}.$$

Proof. Note that if $\alpha > \beta$ then uniformly in α and β we have that $f_{g,x} \restriction_{A_\beta}$ is computable from $f_{g,x} \restriction_{A_\alpha} \oplus g \restriction_{L_\alpha}$. Consider the case that $\alpha = \omega$. By the previous remark, $f_{g,x} \restriction_{A_\omega} \oplus g \restriction_{L_\omega} \oplus 0^{(\omega)}$ uniformly computes $f_{g,x} \restriction_{A_n} \oplus g \restriction_{L_n} \oplus 0^{(n)}$ for all n. Hence by Lemma 23, $f_{g,x} \restriction_{A_\omega} \oplus g \restriction_{L_\omega} \oplus 0^{(\omega)}$ uniformly computes $(f_{g,x} \restriction_{N_0})^{(n)}$ and so $(f_{g,x} \restriction_{N_0})^{(\omega)}$. For successors, just last repeat the previous lemma, and at limits repeat the argument for ω. □

The difference between the previous two lemmas corresponds to a slight difference between the indexing of light-face Borel sets and the iterates of the Turing jump. This discrepancy also occurs in the following standard lemma.

Lemma 26. *A set $X \subseteq 2^\omega$ is Δ^0_α if and only if there is an e such that*

(i) For all x, $\Phi_e(x^{(\beta)}; 0) \downarrow$.

(ii) For all x, $\Phi_e(x^{(\beta)}; 0) = 1$ if and only if $x \in X$.
Where $\beta = \alpha$ if $\alpha \geq \omega$ and $\beta = \alpha - 1$ if $1 \leq \alpha < \omega$.

Lemma 27. *If $q \leq \gamma \leq \omega_1^{ck}$, then the sets F^γ and I^γ are not separable by $\Delta^0_{\gamma+1}$ sets.*

Proof. Assume that A and B are disjoint $\Delta^0_{\gamma+1}$ sets with $\mathrm{F}^\gamma \subseteq A$ and $\mathrm{I}^\gamma \subseteq B$. We can additionally assume that $A \cup B = X^\gamma$. Fix g which is arithmetically generic relative to $0^{(\gamma)}$. We have the associated function $\rho_g \colon X^1 \to X^\gamma$ as above.

Suppose first that $\gamma \geq \omega$. Then from $(\rho_g(x))^{(\gamma+1)}$ we can compute whether $\rho_g(x) \in A$ or $\rho_g(x) \in B$. Hence we can compute this from $(x \oplus g \restriction_{L_\gamma} \oplus 0^{(\gamma)})'$. Note that $g \restriction_{L_\gamma} \oplus 0^{(\gamma)}$ is fixed for all x. Hence F^1 and I^1 are separable by $\Delta^0_2(g \restriction_{L_\gamma} \oplus 0^{(\gamma)})$ sets. But this is a contradiction. If $1 \leq \gamma < \omega$ then the same argument works after replacing occurrences of γ with $\gamma - 1$. □

Theorem 2 follows immediately from relativizing this lemma and observing that if F^γ and I^γ could be separated by $\mathbf{\Pi}^0_{\gamma+1}$ sets, then they could be separated by $\mathbf{\Delta}^0_{\gamma+1}$ sets.

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