

BAIRE MEASURABLE PARADOXICAL DECOMPOSITIONS VIA MATCHINGS

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ABSTRACT. We show that every locally finite bipartite Borel graph satisfying a strengthening of Hall's condition has a Borel perfect matching on some comeager invariant Borel set. We apply this to show that if a group acting by Borel automorphisms on a Polish space has a paradoxical decomposition, then it admits a paradoxical decomposition using pieces having the Baire property. This strengthens a theorem of Dougherty and Foreman who showed that there is a paradoxical decomposition of the unit ball in \mathbb{R}^3 using Baire measurable pieces. We also obtain a Baire category solution to the dynamical von Neumann-Day problem: if a is a nonamenable action of a group on a Polish space X by Borel automorphisms, then there is a free Baire measurable action of \mathbb{F}_2 on X which is Lipschitz with respect to a .

1. INTRODUCTION

The Banach-Tarski paradox states that the unit ball in \mathbb{R}^3 is equidecomposable with two unit balls in \mathbb{R}^3 by rigid motions. In 1930, Marczewski asked whether there is such an equidecomposition where each piece has the Baire property [20]. Using an intricate construction, Dougherty and Foreman gave a positive answer to this question [6, 7]. The key result used in their proof is that every free action of \mathbb{F}_2 on a Polish (separable, completely metrizable) space by homeomorphisms has a paradoxical decomposition using pieces with the Baire property, where \mathbb{F}_2 is the free group on two generators.

Recall that a group Γ is said to *act by Borel automorphisms* on a Polish space X if for every $\gamma \in \Gamma$, the function $x \mapsto \gamma \cdot x$ is Borel. We generalize Dougherty and Foreman's theorem to completely characterize which group actions by Borel automorphisms have paradoxical decompositions using pieces with the Baire property.

Theorem 1.1. *Suppose a group acts on a Polish space by Borel automorphisms, and this action has a paradoxical decomposition. Then the action has a paradoxical decomposition where each piece has the Baire property.*

This also yields a new proof of Dougherty and Foreman's theorem that the unit ball B in \mathbb{R}^3 is equidecomposable with two unit balls using pieces with the Baire property. The action of the group of rotations on the ball $B \setminus \{\mathbf{0}\}$ without the origin has a paradoxical decomposition. Since this action is continuous (and hence Borel) and $B \setminus \{\mathbf{0}\}$ is Polish, by Theorem 1.1, this action has a paradoxical decomposition using pieces with the Baire property. To finish, we combine this with the easy fact that B is equidecomposable with $B \setminus \{\mathbf{0}\}$ using pieces which are Borel (see [20, Corollary 3.10]).

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In contrast to Theorem 1.1, we also show that there is an action of a finitely generated group on a Polish space X by Borel automorphisms, and two Borel sets $B_0, B_1 \subseteq X$ which are equidecomposable with respect to this action, but not equidecomposable using pieces with the Baire property.

An action of a group Γ on a space X is said to be *amenable* if X admits a finitely additive probability measure which is invariant under the action of Γ . A discrete group Γ is said to be *amenable* if the left translation action of Γ on itself is amenable. The notion of amenability was introduced by von Neumann in response to the Banach-Tarski paradox as an obstruction to having a paradoxical decomposition. A theorem of Tarski states that an action of a group is nonamenable if and only if the action admits a paradoxical decomposition (see [20, Corollary 9.2]).

Since every subgroup of an amenable group is amenable and \mathbb{F}_2 is easily seen to be nonamenable, it is natural to ask whether a group is nonamenable if and only if it contains \mathbb{F}_2 as a subgroup. This problem was first posed in print by Day [3] and is sometimes known as the von Neumann conjecture. It was answered in the negative by Ol'shanskii [19]. Despite this negative answer, interesting positive results have been proven for variants of this question where the role of subgroup is replaced by more general notions. Whyte has given a positive solution to the von Neumann-Day problem in the setting of geometric group theory [22], and Gaboriau and Lyons have given a positive solution in the setting of measurable group theory [8]. We give the following Baire category solution to the dynamical von Neumann-Day problem. If a and b are actions of Γ and Δ respectively on a space X , then the action b is said to be *a-Lipschitz*, if for every $\delta \in \Delta$, there is a finite set $S \subseteq \Gamma$ such that $\forall x \in X \exists \gamma \in S (\delta \cdot_b x = \gamma \cdot_a x)$. Recall that a function is said to be *Baire measurable* if the preimage of every Borel set has the Baire property.

Theorem 1.2. *Suppose a is a nonamenable action of a group Γ on a Polish space X by Borel automorphisms. Then there is a free a -Lipschitz action of \mathbb{F}_2 on X by Baire measurable automorphisms.*

An important tool we use to prove both Theorems 1.1 and 1.2 is a connection between paradoxical decompositions and perfect matchings. As we describe in Section 4, if a is an action of a group Γ on X , then to each finite symmetric set $S \subseteq \Gamma$ we can associate a graph $G_p(a, S)$ so that $G_p(a, S)$ has a perfect matching if and only if a has a paradoxical decomposition using group elements from S . We exploit this connection by showing that we can generically construct perfect matchings for Borel graphs satisfying a strengthening of Hall's condition, and we can amplify Hall's condition in the graph $G_p(a, S)$ to this stronger condition in $G_p(a, S^2)$. Recall that a bipartite graph G with bipartition $\{B_0, B_1\}$ satisfies *Hall's condition* if for every finite subset F of B_0 or B_1 , $|N_G(F)| \geq |F|$, where $N_G(F)$ is the set of vertices adjacent to elements of F that are not contained in F . Hall's theorem states that a locally finite bipartite graph has a perfect matching if and only if it satisfies Hall's condition.

Theorem 1.3. *Suppose G is a locally finite bipartite Borel graph on a Polish space with bipartition $\{B_0, B_1\}$ and there exists an $\epsilon > 0$ such that for every finite set F with $F \subseteq B_0$ or $F \subseteq B_1$, $|N_G(F)| \geq (1 + \epsilon)|F|$. Then there is a Borel perfect matching of G restricted to some G -invariant comeager Borel set.*

This theorem is optimal in the sense that the ϵ cannot be made equal to 0. Laczkovich has given an example of a 2-regular Borel bipartite graph which satisfies

Hall's condition but has no Borel perfect matching restricted to any comeager set [15]. Conley and Kechris have also extended this example to show that for every even d there is a d -regular Borel bipartite graph which satisfies Hall's condition but has no Borel perfect matching restricted to any comeager set [1]. We note that while both the theorems of Laczkovich and Conley-Kechris are stated in the measure theoretic context, their use of ergodicity can be replaced with generic ergodicity to prove these results as we have stated them.

In contrast to Theorem 1.3, Kechris and Marks have observed that for every $k \geq 1$, there is a bipartite Borel graph G of bounded degree on a standard probability space (X, μ) with Borel bipartition $\{B_0, B_1\}$ such that for every finite subset F of B_0 or B_1 , $|N_G(F)| \geq k|F|$, yet G has no Borel perfect matching restricted to any μ -conull set [12].

The papers [2], [17], and [18] also contain some related work on Borel matchings. See [12] for a recent survey of work on matchings in the setting of Borel graph combinatorics, which also contains a different proof of Dougherty and Foreman's result.

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2. DEFINITIONS

Our conventions in graph theory follow [5]. A *perfect matching* M of a graph G is a set of edges from G such that each vertex of G is incident to exactly one edge of M . If F is a set of vertices in a graph G , the set of neighbors of points in F that are not contained in F is denoted $N_G(F) = \{y \notin F : \exists x \in F(x G y)\}$. A subset of the vertices of a graph G is said to be *G -invariant* if it is a union of connected components of G . A graph is *locally finite* (*respectively countable*) if the degree of every vertex is finite (*respectively countable*), and is *d -regular* if the degree of every vertex is d . The *closed r -ball* around a vertex x in G is the set $\{y : d_G(x, y) \leq r\}$, where $d_G(x, y)$ is the distance from x to y in G . If G is a graph with vertex set X and $A \subseteq X$, then we use $G \upharpoonright A$ to denote the induced graph on A . So $G \upharpoonright A$ is the graph with vertex set A whose edges are all the edges in G between elements of A . If G is a graph on X , then we let $G^{\leq n}$ be the graph on X where $x G^{\leq n} y$ if $1 \leq d_G(x, y) \leq n$ and G^n be the graph on X where $x G^n y$ if $d_G(x, y) = n$.

The descriptive combinatorics of Borel graphs was first systematically studied by Kechris, Solecki and Todorcevic [14]. A *Borel graph* G on a Polish space X (or a standard Borel space X) is a symmetric irreflexive relation on X that is Borel as a subset of $X \times X$. That is, we will identify the graph with its edge relation. By [14, Proposition 4.10] (which is a corollary of the proof of the Feldman-Moore theorem [13, Theorem 1.3]), if G is a locally countable Borel graph on a Polish space X , then there is a countable set $\langle T_i \mid i \in \mathbb{N} \rangle$ of Borel automorphisms of X such that $x G y$ if and only if $x \neq y$ and there is an i such that $T_i(x) = y$. (Indeed, the T_i may be chosen to be involutions, so $T_i^2(x) = x$). From this, it follows that any property of vertices of a locally finite Borel graph which only depends on the isomorphism class of its r -balls is Borel. All the Borel graphs used to prove the theorems in the introduction will come equipped with natural functions generating their edges. A standard reference for descriptive set theory is [11].

A *paradoxical decomposition* of an action of a group Γ on a space X is a partition of X into finitely many sets $\{A_1, \dots, A_n, B_1, \dots, B_m\}$ such that there exist group elements $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m \in \Gamma$ such that the sets $\alpha_i A_i$ are pairwise disjoint, the sets $\beta_i B_i$ are pairwise disjoint, and $X = \alpha_1 A_1 \cup \dots \cup \alpha_n A_n = \beta_1 B_1 \cup \dots \cup \beta_m B_m$. A basic reference on paradoxical decompositions is [20]. Note that the action of a group Γ on a space X is paradoxical if and only if there exists a finitely generated subgroup $\Delta \leq \Gamma$ so that the restriction of the action to Δ is paradoxical. So while we typically state our results for actions of arbitrary groups, it generally suffices to consider actions of finitely generated groups.

If a is an action of a group Γ on a space X , then $A, B \subseteq X$ are said to be *a-equidecomposable* if there is a partition $\{A_1, \dots, A_n\}$ of A into finitely many sets and group elements $\alpha_1, \dots, \alpha_n \in \Gamma$ such the sets $\alpha_i A_i$ are pairwise disjoint, and $B = \alpha_1 A_1 \cup \dots \cup \alpha_n A_n$. It is easy to see that equidecomposability is an equivalence relation. In the language of equidecomposability, a has a paradoxical decomposition if and only if X can be partitioned into two sets $\{A, B\}$ such that X is *a-equidecomposable* with both A and B .

3. BAIRE MEASURABLE MATCHINGS

We begin with a lemma which says roughly that modulo a meager set, the vertices of a Borel graph can be decomposed into countably many Borel sets where pairs of distinct vertices in each set have large pairwise distance.

Lemma 3.1. *Let G be a locally finite Borel graph on a Polish space X and $f : \mathbb{N} \rightarrow \mathbb{N}$ be a function. Then there is a sequence $\langle A_n \mid n \in \mathbb{N} \rangle$ of Borel sets such that $A = \bigcup_{n \in \mathbb{N}} A_n$ is comeager and G -invariant and distinct $x, y \in A_n$ have $d_G(x, y) > f(n)$.*

Proof. Let $\langle U_i \mid i \in \mathbb{N} \rangle$ enumerate a basis of open sets for X . We define sets $B_{i,r}$ by setting $x \in B_{i,r}$ if and only if $x \in U_i$ and for all $y \neq x$ in the closed r -ball around x , we have $y \notin U_i$. These sets are Borel since we can identify the closed r -ball around x in a Borel way. Note that distinct $x, y \in B_{i,r}$ have $d_G(x, y) > r$. Finally, we claim that for each fixed r , $X = \bigcup_i B_{i,r}$. This is because for each $x \in X$, since G is locally finite, the elements of the closed r -ball around x which are not equal to x form a finite set. Hence, x can be separated from this finite set by some U_i .

We are now ready to construct our sets A_n . Choose a countable set of Borel automorphisms $S_i : X \rightarrow X$ generating G , and let $\langle T_i \mid i \in \mathbb{N} \rangle$ be the closure of the automorphisms S_i under composition and inverses. Hence, x and y are in the same connected component of G if and only if there is an i such that $T_i(x) = y$. Fix an enumeration of \mathbb{N}^2 . We will choose each A_n such that $T_i(A_n)$ is nonmeager in U_j , where (i, j) is the n th pair of natural numbers. For a fixed n and i , since $X = \bigcup_k B_{k, f(n)}$ and T_i is an automorphism, we have that $X = \bigcup_k T_i(B_{k, f(n)})$. Thus, we can apply the Baire Category theorem to find a k such that $T_i(B_{k, f(n)})$ is nonmeager in U_j . Let A'_n be this $B_{k, f(n)}$ and let $A' = \bigcup_n A'_n$. Note that A'_n has the property that distinct $x, y \in A'_n$ have $d_G(x, y) > f(n)$. Now for every i , the set $T_i(A')$ is comeager since it is nonmeager in every open set. Thus, the intersection $A = \bigcap_i T_i(A')$ is comeager since it is a countable intersection of comeager sets. Finally, A is a G -invariant set since the T_i generate the connectedness relation of G ; if $x \notin A$, then $T_i(x) \notin A$ for all i . Our desired sets are $A_n = A \cap A'_n$. \square

We remark that we could give a more compact but less elementary proof of the above theorem as follows. For each r , fix a Borel \mathbb{N} -coloring $c_r: X \rightarrow \mathbb{N}$ of $G^{\leq r}$ by [13, Proposition 4.5]. Let $B_{i,r} = c_r^{-1}(i)$. (Indeed, the first paragraph of the proof of Lemma 3.1 essentially repeats the proof from [13, Proposition 4.5] that the graphs $G^{\leq r}$ have Borel \mathbb{N} -colorings). For each $x \in X$, the set of parameters $p \in \mathbb{N}^{\mathbb{N}}$ such that $x \in X_p = \bigcup_{n \in \mathbb{N}} B_{p(n),f(n)}$ is comeager (in fact, it is dense open). Now let Y_p be the set of $x \in X$ so that every y in the connected component of x is contained in X_p . So the set of $p \in \mathbb{N}^{\mathbb{N}}$ for which $x \in Y_p$ is also comeager. By the Kuratowski-Ulam Theorem [11, Theorem 8.41], there are comeagerly many $p \in \mathbb{N}^{\mathbb{N}}$ for which Y_p is comeager. Fixing such a p , we let our sets A_n be $Y_p \cap B_{p(n),f(n)}$.

Throughout the paper, Lemma 3.1 will be the only way in which we discard meager sets.

We are now ready to prove Theorem 1.3. We begin by introducing some notation. If G is a graph and M is a partial matching of G , then $G - M$ is the graph obtained by removing the vertices incident on edges of M and any edges incident on removed vertices. We note that if G is a Borel graph and M is Borel, then $G - M$ is a Borel graph.

If G is a bipartite graph on X , then every finite set $F \subseteq X$ can be partitioned into G^2 -connected sets having disjoint sets of neighbors in G . Thus, to show that Hall's condition holds for a graph, it suffices to show it holds just for G^2 -connected sets. We will use the following strengthening of Hall's condition where we restrict our attention to G^2 -connected sets of size at least n .

Definition 3.2. A bipartite graph G with bipartition $\{B_0, B_1\}$ satisfies $\text{Hall}_{\epsilon,n}$ if G satisfies Hall's condition and additionally for every finite G^2 -connected set F with $|F| \geq n$ and $F \subseteq B_0$ or $F \subseteq B_1$, $|\mathbb{N}_G(F)| \geq (1 + \epsilon)|F|$.

If $n = 1$, by decomposing into G^2 -connected sets with disjoint neighborhoods, this definition is equivalent to the simpler requirement that for all finite sets F with $F \subseteq B_0$ or $F \subseteq B_1$, $|\mathbb{N}_G(F)| \geq (1 + \epsilon)|F|$. This is the hypothesis of Theorem 1.3. However, for $n > 1$ it becomes important that we only consider G^2 -connected sets in our proof.

Proof of Theorem 1.3. Let G be as in the theorem with ϵ witnessing that G satisfies $\text{Hall}_{\epsilon,1}$. Note we do not require that B_0 and B_1 are Borel. Let $f: \mathbb{N} \rightarrow \mathbb{N}$ be an increasing function such that for all n , $f(n) \geq 8$ and $\sum_{n \in \mathbb{N}} 8/f(n) < \epsilon$. Apply Lemma 3.1 with the function f to obtain an invariant comeager $A = \bigcup_{n \in \mathbb{N}} A_n$, which is a union of Borel sets A_n of elements of pairwise distance greater than $f(n)$. We will find a Borel perfect matching of $G \upharpoonright A$. Indeed, our argument will show that any bipartite Borel graph satisfying $\text{Hall}_{\epsilon,1}$ has a Borel perfect matching provided the vertex set of the graph can be written as a union of Borel sets A_n of pairwise distance greater than $f(n)$, for f sufficiently large as above.

We construct by induction an increasing sequence of Borel partial matchings M_n for $n \in \mathbb{N}$ such that the set of vertices incident to edges in M_n includes $\bigcup_{m \leq n} A_m$. We will also ensure that $G - M_n$ satisfies $\text{Hall}_{\epsilon_n, f(n)}$, where $\epsilon_n = \epsilon - \sum_{i \leq n} 8/f(i)$.

For ease of notation we let $M_{-1} = \emptyset$ and $\epsilon_{-1} = \epsilon$. Note that the hypothesis of the theorem gives us $\text{Hall}_{\epsilon_{-1},1}$. Suppose that we have constructed M_{n-1} as above. We will now construct M_n . Let $X_{n-1} \subseteq X$ be the vertex set of $G - M_{n-1}$. For each $x \in A_n \cap X_{n-1}$, we can find an edge e incident to x such that $(G - M_{n-1}) - \{e\}$ satisfies Hall's condition. This is because $G - M_{n-1}$ satisfies Hall's condition,

and any edge e from a perfect matching of $G - M_{n-1}$ has this property. Let M'_n be a Borel set containing one such edge e for each $x \in A_n \cap X_{n-1}$ and let $M_n = M_{n-1} \cup M'_n$. For instance, let $\langle T_i \mid i \in \mathbb{N} \rangle$ be a Borel set of automorphisms generating G , and for each $x \in A_n \cap X_{n-1}$ put the edge $\{x, T_i(x)\} \in M'_n$ if i is least such that $T_i(x) \neq x$ and $(G - M_{n-1}) - \{\{x, T_i(x)\}\}$ satisfies Hall's condition.

To finish the proof we will show $G - M_n$ satisfies $\text{Hall}_{\epsilon_n, f(n)}$ assuming $G - M_{n-1}$ satisfies $\text{Hall}_{\epsilon_{n-1}, f(n-1)}$. Let F be a finite $(G - M_n)^2$ -connected subset of $B_0 \cap X_n$ or $B_1 \cap X_n$, where $X_n \subseteq X$ is the vertex set of $G - M_n$. Our rough idea is that if F is large, since A_n is sparse, we lose only a tiny fraction of the neighbors of F in passing from $G - M_{n-1}$ to $G - M_n$. If F is small, we use the fact that the removal of any single edge in M_n from $G - M_{n-1}$ preserves Hall's condition.

Let $D = N_{G-M_{n-1}}(F) - N_{G-M_n}(F)$. Suppose first that $|D| \geq 2$. Each $x \in D$ must be a neighbor of some element y_x of F . Each $x \in D$ must also be an element of A_n or a neighbor of an element of A_n (which it is matched to in M_n). Fix distinct $x, x' \in D$. Then x and x' are associated to distinct elements of A_n since M_n is a matching and F is a subset of one half of the bipartition of G . Further, since elements of A_n have pairwise distance greater than $f(n)$, we have $d_G(y_x, y_{x'}) > f(n) - 4$. Since F is connected in $(G - M_n)^2$, there must be at least $\lfloor (f(n) - 4)/4 \rfloor \geq f(n)/8$ elements of F of distance at most $(f(n) - 4)/2$ from each y_x . Since the closed $(f(n) - 4)/2$ -balls around each y_x are disjoint, it follows that $|D| \leq (8/f(n))|F|$ and so

$$\begin{aligned} |N_{G-M_n}(F)| &= |N_{G-M_{n-1}}(F)| - |D| \\ &\geq (1 + \epsilon_{n-1})|F| - (8/f(n))|F| \\ &\geq (1 + \epsilon_n)|F|. \end{aligned}$$

Suppose now that $N_{G-M_{n-1}}(F)$ contains at most one vertex that is not an element of $N_{G-M_n}(F)$. First we establish $|N_{G-M_n}(F)| \geq |F|$. Observe that either for some $e \in M'_n$, we have $N_{G-M_n}(F) = N_{G-M_{n-1}-\{e\}}(F)$ or we have $N_{G-M_n}(F) = N_{G-M_{n-1}}(F)$. In the first case we have Hall's condition from the choice of M'_n and in the second we have it by the induction hypothesis. Now if additionally $|F| \geq f(n)$, then since $|N_{G-M_{n-1}}(F)| - |N_{G-M_n}(F)| \leq 1$, we have $|N_{G-M_n}(F)| \geq |N_{G-M_{n-1}}(F)| - (1/f(n))|F|$, and we can conclude $|N_{G-M_n}(F)| \geq (1 + \epsilon_n)|F|$ as above. \square

We remark here that an analogue of Theorem 1.3 for one-sided matchings is also true. Suppose G is a Borel graph with Borel bipartition $\{B_0, B_1\}$ satisfying a one-sided version of $\text{Hall}_{\epsilon, 1}$. Then one can use the same type of argument as Theorem 1.3, inductively constructing one-sided matchings M_n and verifying that $G - M_n$ satisfies the one-sided $\text{Hall}_{\epsilon_n, f(n)}$ condition.

4. PARADOXICAL DECOMPOSITIONS

We now exploit a well-known (see for instance [4]) connection between matchings and paradoxical decompositions to prove Theorem 1.1. This connection has been previously used in the setting of descriptive graph combinatorics in [9] and [12].

Definition 4.1. Suppose a is an action of a group Γ on a space X and $S \subseteq \Gamma$ is a finite symmetric set. Let $G_p(a, S)$ be the bipartite graph with vertex set $\{0, 1, 2\} \times X$ where there is an edge from (i, x) to (j, y) if exactly one of i and j is equal to 0, and there is a $\gamma \in S$ such that $\gamma \cdot x = y$.

Now $G_p(a, S)$ has a perfect matching M if and only if a has a paradoxical decomposition using group elements from S . To see this, suppose $S = \{\gamma_1, \dots, \gamma_n\}$ and M is a perfect matching of $G_p(a, S)$. Now put $x \in A_i$ if $(0, x)$ is matched to $(1, \gamma_i \cdot x)$ and $\gamma_j \cdot x \neq \gamma_i \cdot x$ for all $j < i$. Similarly, put $x \in B_i$ if $(0, x)$ is matched to $(2, \gamma_i \cdot x)$ and $\gamma_j \cdot x \neq \gamma_i \cdot x$ for all $j < i$. Then we see $\{A_1, \dots, A_n, B_1, \dots, B_n\}$ partitions the space, as does $\gamma_i A_i$ and also $\gamma_i B_i$. The converse is proved via the same identification.

Proof of Theorem 1.1. Assume Γ acts by Borel automorphisms on a Polish space X and the action has some paradoxical decomposition using group elements from a finite symmetric set S , so $G_p(a, S)$ satisfies Hall's condition. Now by enlarging S to $S^2 = \{\gamma\delta : \gamma, \delta \in S\}$, we claim that $G_p(a, S^2)$ satisfies Hall_{1,1}. This is trivial for finite sets of the form $F = \{0\} \times F'$ since $N_{G_p(a, S)}(\{0\} \times F') = \{1, 2\} \times F''$ for some F'' where $|F''| \geq |F'|$.

To finish, it suffices to check sets of the form $F = \{1, 2\} \times F'$, since

$$N_{G_p(a, S)}(\{1\} \times F_1 \cup \{2\} \times F_2) = N_{G_p(a, S)}(\{1, 2\} \times (F_1 \cup F_2))$$

and $|(\{1\} \times F_1) \cup (\{2\} \times F_2)| \leq |\{0, 1\} \times (F_1 \cup F_2)|$. So consider $F = \{1, 2\} \times F'$. If we let $\{0\} \times F'' = N_{G_p(a, S)}(\{1, 2\} \times F')$, then $|F''| \geq |F|$, so

$$\begin{aligned} |N_{G_p(a, S^2)}(F)| &= |N_{G_p(a, S^2)}(\{1, 2\} \times F')| \\ &\geq |N_{G_p(a, S)}(\{1, 2\} \times F')| \\ &\geq |\{1, 2\} \times F''| \\ &\geq 2|F|. \end{aligned}$$

Thus, by Theorem 1.3, since $G_p(a, S^2)$ is a bipartite Borel graph satisfying Hall_{1,1}, it has a Borel perfect matching on a $G_p(a, S^2)$ -invariant comeager Borel set A . We can use the axiom of choice to extend to a perfect matching M defined on the whole graph $G_p(a, S^2)$. Now the paradoxical decomposition we defined above associated to this matching will have pieces with the Baire property. This is because the set of $x \in X$ such that $(0, x) \in A$ is Borel and comeager, and hence the pieces of the paradoxical decomposition are relatively Borel in a comeager Borel set. \square

Note that we needed to increase the number of pieces in our paradoxical decomposition in the proof above to obtain a paradoxical decomposition using pieces with the Baire property; this occurs when we enlarge the set S of group elements to S^2 . This is known to be necessary in general by a result of Wehrung [21]. In particular, Wehrung has shown that a Baire measurable paradoxical decomposition of the 2-sphere by isometries must use at least six pieces, while there are paradoxical decompositions using pieces without the Baire property with only four pieces.

5. EQUIDECOMPOSABILITY

Suppose a is an action of a group Γ on a space X and A and B are subsets of X . Then for any finite symmetric $S \subseteq \Gamma$ we may form a bipartite graph $G(a, S, A, B)$ on the space $\{0\} \times A \cup \{1\} \times B$ where $(i, x) G(a, S, A, B) (j, y)$ if $i \neq j$ and there is a $\gamma \in S$ such that $\gamma \cdot x = y$. It is obvious that A and B are a -equidecomposable if and only if there is some finite symmetric $S \subseteq \Gamma$ such that $G(a, S, A, B)$ has a perfect matching. From this perspective, the graph $G_p(a, S)$ used in Section 4 is really the graph $G(b, S, \{0\} \times X, \{1, 2\} \times X)$ where b is the action of $\mathbb{Z}/3\mathbb{Z} \times \Gamma$ on

$\{0, 1, 2\} \times X$ via $(n, \gamma) \cdot_b (i, x) = (n + i, \gamma \cdot_a x)$. Of course, a has a paradoxical decomposition if and only if $\{0\} \times X$ is b -equidecomposable with $\{1, 2\} \times X$.

We begin by proving a lemma relating matchings in 2-regular acyclic Borel bipartite graphs with certain graphs containing them. If G is a 2-regular acyclic graph on X and $n \geq 1$, let G_n be the Borel graph on X where $x G_n y$ if there is a G -path of odd length at most $2n - 1$ from x to y . Note that $G_1 = G$, and G_n is a $2n$ -regular graph.

Lemma 5.1. *Suppose G is a 2-regular acyclic Borel bipartite graph with Borel bipartition $\{B_0, B_1\}$ and $n \geq 1$. Then if G_n has a Borel perfect matching on a G_n -invariant Borel set Y , then G has a Borel perfect matching on Y .*

Proof. Since G is 2-regular and acyclic, using the axiom of choice, we may linearly order each connected component of G so that each vertex x has one neighbor greater than x and one neighbor less than x . Note that a perfect matching of G is characterized by the property that for each connected component C of G , either every $x \in C \cap B_0$ is matched to its unique neighbor less than x , or every $x \in C \cap B_0$ is matched to its unique neighbor greater than x . The idea of the proof is that from a matching in G_n we can select this unique direction in G by “averaging” the direction of matched edges in G_n in a ball around each point.

Suppose that $G_n \upharpoonright Y$ has a Borel perfect matching given by the Borel function $M: B_0 \cap Y \rightarrow B_1 \cap Y$. We will use M to define a Borel perfect matching M' of $G \upharpoonright Y$. If $x \in B_0$, let $D_n(x) = \{y \in B_0 : d_G(x, y) \leq 2n - 2\}$. Note that $|D_n(x)| = 2n - 1$. Define $M': B_0 \cap Y \rightarrow B_1 \cap Y$ by setting $M'(x)$ to be the unique G -neighbor y of x such that the connected component of y in $G \upharpoonright X \setminus \{x\}$ contains at least n elements of $M(D_n(x))$. In terms of our order, $M'(x)$ will be the neighbor of x that is less than x if and only if $M(D_n(x))$ contains at least n elements less than x . We now claim that for every $x, y \in B_0$ in the same connected component of G , $M'(x) < x$ if and only if $M'(y) < y$. Hence, M' is a perfect matching of G . Our claim is easy to see when $D_n(x)$ and $D_n(y)$ are disjoint since if $z \in B_0$ is in the same connected component as x and y but $z \notin D_n(x) \cup D_n(y)$, then $x < M(z) < y$ if and only if $x < z < y$. Our claim then follows from the fact that M is a bijection. To finish, note that when $D_n(x)$ and $D_n(y)$ are not disjoint, then we can find some y' where $D_n(y')$ is disjoint from both $D_n(x)$ and $D_n(y)$. \square

We now prove a result which shows we cannot generalize Theorem 1.1 to say that equidecomposable Borel sets have Baire measurable equidecompositions.

Theorem 5.2. *There is an action of a group Γ on a Polish space X by Borel automorphisms and Borel sets $B_0, B_1 \subseteq X$ that are equidecomposable, but not equidecomposable using pieces with the Baire property.*

Proof. Let G be the graph of Laczkovich [15], which is a 2-regular acyclic bipartite Borel graph on a Polish space X with a bipartition into Borel sets $\{B_0, B_1\}$ so that G has no Borel perfect matching restricted to any comeager Borel set. For this graph, we may find finitely many Borel involutions $\langle T_i \mid i < n \rangle$ generating the edges of G . Such T_i are easy to define explicitly for Laczkovich’s graph, and more generally, such involutions always exist for any bounded degree Borel graph (see the remarks after [14, Proposition 4.10]). These T_i define an action a of the group \mathbb{F}_n on X by Borel automorphisms. G has a perfect matching since it is 2-regular and acyclic. Hence, B_0 and B_1 are a -equidecomposable.

Suppose S is a finite set of group elements in \mathbb{F}_n of word length at most $2m - 1$, and B_0 and B_1 are a -equidecomposable using pieces with the Baire property and group elements from S . Then the graph $G(a, S, B_0, B_1)$ defined above must have a perfect matching M which is Borel on a G -invariant comeager Borel set. This is because every set with the Baire property differs from an open set by a meager set, and every comeager set contains a Borel comeager (indeed, G_δ) set. Finally, every comeager Borel subset of X contains a comeager G -invariant Borel set since G is generated by homeomorphisms and the image of a meager set under a homeomorphism is meager. Since $G(a, S, B_0, B_1)$ is a subset of G_m , this would mean G_m has a Borel perfect matching on an comeager invariant Borel set. But by Lemma 5.1, this would mean G has a Borel perfect matching on a comeager invariant Borel set which is a contradiction. \square

Laczkovich has solved Tarski's circle squaring problem, showing that a circle and square of the same area in the plane are equidecomposable by rigid motions [16]. It is an open problem whether the pieces used in such a decomposition may have the Baire property.¹

6. A BAIRE CATEGORY SOLUTION TO THE DYNAMICAL VON NEUMANN-DAY PROBLEM

Suppose that a is an action of a group Γ on a space X . Say that a function $f: X \rightarrow X$ is a -Lipschitz if there is a finite set of group elements $S \subseteq \Gamma$ such that for every $x \in X$ there is a $\gamma \in S$ such that $f(x) = \gamma \cdot x$. Say a graph G on X is a -Lipschitz if there is a finite set of group elements $S \subseteq \Gamma$ such that for every edge $\{x, y\} \in G$ there is a $\gamma \in S$ such that $y = \gamma \cdot x$.

We now prove a lemma which closely parallels Whyte's geometric solution to the von Neumann-Day problem.

Lemma 6.1. *Suppose that a is a nonamenable action of a group Γ on a Polish space X by Borel automorphisms. Then there exists a 4-regular acyclic a -Lipschitz graph G on X and a comeager G -invariant Borel set $A \subseteq X$ so that $G \upharpoonright A$ is Borel.*

Proof. Since the action of a is nonamenable and hence paradoxical, we may find a finite symmetric set $S \subseteq \Gamma$ so that $G_p(a, S)$ satisfies Hall's condition and $G_p(a, S^2)$ satisfies $\text{Hall}_{1,1}$ as in the proof of Theorem 1.1. Let $G'_p(a, S^2)$ be the graph on $\{0, 1, 2, 3\} \times X$ where there is an edge between (i, x) and (j, y) if exactly one of i and j is equal to 0 and there is a $\gamma \in S^2$ such that $\gamma \cdot x = y$. Since $G_p(a, S^2)$ satisfies $\text{Hall}_{1,1}$, the new graph $G'_p(a, S^2)$ satisfies $\text{Hall}_{1/3,1}$. Thus, by Theorem 1.3, we may find a perfect matching of $G'_p(a, S^2)$ which is Borel on a comeager invariant Borel set A' . Now define functions $f_0, f_1, f_2: X \rightarrow X$ by setting $f_i(x)$ to be the unique y such that $(i + 1, x)$ is matched to $(0, y)$. These functions are a -Lipschitz, injective, and have disjoint ranges which partition X . Let $A = \{x : (0, x) \in A'\}$ which is comeager and Borel. It is invariant under f_0, f_1 , and f_2 since A' is $G'_p(a, S^2)$ -invariant and S^2 contains the identity and so $(0, x) \in A'$ if and only if $(n, x) \in A'$ for every n . Note that the functions f_i are Borel when restricted to A . We will define a 4-regular tree in terms of these functions.

¹This question has now been resolved by in the affirmative by Grabowski, Máthé, and Pikhurko [10].

Consider the graph H generated by the three functions f_0, f_1 and f_2 , so $x H y$ if there is an i such that $f_i(x) = y$ or $f_i(y) = x$. Each connected component of H contains at most one cycle. To see this, observe that since the f_i have disjoint ranges and generate H , any cycle must arise from some x and i_0, \dots, i_n where $f_{i_n} \circ \dots \circ f_{i_0}(x) = x$. Further, for such an x , every other element in the same connected component is in the forward orbit of x under the f_i . Hence, another cycle of this form would contradict the fact that the f_i are injective and have disjoint ranges.

We may assume that any cycle in H is of the form $f_0 \circ \dots \circ f_0(x) = x$, so all the edges in the cycle arise from the function f_0 . This is because given any f_i as above, we can replace them with the functions g_i , for $i \in \{0, 1, 2\}$ which are defined as follows. If $\{x, f_j(x)\}$ is an edge in a cycle in H , then define $g_0(x) = f_j(x)$, $g_1(x) = f_{j+1 \bmod 3}(x)$ and $g_2(x) = f_{j+2 \bmod 3}(x)$. If x is not contained in any cycle, then set $g_i(x) = f_i(x)$. Then g_0, g_1 , and g_2 will be injective, have disjoint ranges that partition X , and any cycle in the graph generated by the g_i will contain edges generated only by the function g_0 .

Now it is easy to define a 4-regular acyclic a -Lipschitz graph G on X with the same connected components as H and so that if $x T y$, then x and y have distance at most 2 in H . We give one such construction.

Define functions f'_0, f'_1 and f'_2 on X as follows. Fix a Borel linear ordering of X . Suppose x is in a connected component containing a unique cycle $x_0, x_1, \dots, x_n, x_0$, with $f_0(x_i) = x_{i+1}$ and $f_0(x_n) = x_0$, and where x_0 is the least element of x_0, \dots, x_n under the Borel linear ordering. Now if x is of the form $f_1^k(x_0)$ or $f_1^k(x_n)$ for some $k \geq 0$, define $f'_0(x) = f_0 \circ f_1(x)$, $f'_1(x) = f_1(x)$, and $f'_2(x) = f_2(x)$. For x not of this form, or in connected components not containing cycles, define $f'_0(x) = f_0(x)$, $f'_1(x) = f_1(x)$, and $f'_2(x) = f_2(x)$. Let G be the graph where $x G y$ if there is an i such that $f'_i(x) = y$ or $f'_i(y) = x$, or if x and y are contained in the unique cycle $x_0, x_1, \dots, x_n, x_0$ of a connected component of H , and there is an i such that $\{x, y\} = \{x_i, x_{i+1}\}$.

Finally, our construction of G is Borel on A , since the f_i are Borel on A . \square

To finish the proof of Theorem 1.2 it suffices to prove the following lemma.

Lemma 6.2. *Suppose that G is an acyclic 4-regular Borel graph on a comeager Borel subset C of a Polish space X . Then there is a free Borel action of $\mathbb{F}_2 = \langle a, b \rangle$ on a G -invariant comeager Borel set $A \subseteq C$ generating $G \upharpoonright A$.*

Proof. Apply Lemma 3.1 to obtain $A = \bigcup_{n \in \mathbb{N}} A_n \subseteq C$, which is invariant and comeager and where points in A_n are pairwise of distance greater than $16 \cdot 4^n$. It suffices to construct two Borel automorphisms f_1, f_2 of A so that for every $x \in A$, $f_1(x) \neq f_2(x)$ and $\{x, f_1(x)\}, \{x, f_2(x)\} \in G$. We will construct f_1 and f_2 and their inverse functions $f_{-1} = f_1^{-1}$ and $f_{-2} = f_2^{-1}$ as increasing unions of injective Borel partial functions $\langle f_{i,n} \mid n \in \mathbb{N} \rangle$, so $f_i = \bigcup_{n \in \mathbb{N}} f_{i,n}$ for each $i \in \{-2, -1, 1, 2\}$. For each n , these partial functions $f_{i,n}$ will all have the same domain D_n , where

- (1) $A_n \subseteq D_n$.
- (2) If $x, y \in D_n$ and $d(x, y) \leq 4$, then x and y are connected in $G \upharpoonright D_n$.
- (3) $G^{\leq 8} \upharpoonright D_n$ has finite connected components of diameter at most 4^n , where $x G^{\leq 8} y$ if and only if $1 \leq d_G(x, y) \leq 8$.

For ease of notation, let $f_{i,-1} = \emptyset$ for all $i \in \{-2, -1, 1, 2\}$.

Given Borel partial functions $f_{i,n}$ with domain D_n , we construct the functions $f_{i,n+1}$ as follows. For each $x \in A_{n+1} \setminus D_n$, we define the set $D_{n+1,0}(x) = \{x\}$, and recursively let $D_{n+1,k+1}(x)$ be the set of points y that are adjacent to $D_{n+1,k}(x)$ and not contained in D_n or $\bigcup_{j \leq k} D_{n+1,j}(x)$ such that y is distance at most 3 from D_n . Let $D_{n+1}(x) = \bigcup_j D_{n+1,j}(x)$. We can think of $D_{n+1}(x)$ as being comprised of (overlapping) paths which start at x and end at elements of D_n . We will define D_{n+1} to be the union of D_n and $\bigcup_{x \in A_{n+1} \setminus D_n} D_{n+1}(x)$. It is clear that D_{n+1} is Borel and satisfies (1).

We first check that D_{n+1} satisfies properties (2) and (3). Suppose $x \in A_{n+1} \setminus D_n$. Since x cannot be of G -distance less than or equal to 4 from two different connected components of $G^{\leq 8}$, there can be most one connected component C of $G^{\leq 8} \upharpoonright D_n$ with $d_G(x, C) \leq 4$. If there is such a C , then inductively, all elements of $D_{n+1}(x)$ must be G -distance at most 4 from this C . From this, we can see that the connected component C' of x in the graph $G^{\leq 8} \upharpoonright D_n \cup D_{n+1}(x)$ is a subset of the union of $N_{G^{\leq 8}}(C)$ with all connected components of $G^{\leq 8}$ that are $G^{\leq 8}$ -adjacent to $N_{G^{\leq 8}}(C)$. Thus, the $G^{\leq 8}$ -diameter of C' is at most $3 \cdot 4^n + 4 \leq 4^{n+1}$, since connected components of $G^{\leq 8} \upharpoonright D_n$ have diameter at most 4^n . If there is no such C , then the connected component of x in the graph $G^{\leq 8} \upharpoonright D_n \cup D_{n+1}(x)$ consists of x together with all connected components of $G^{\leq 8}$ that are $G^{\leq 8}$ -adjacent to x . The $G^{\leq 8}$ -diameter of this connected component is therefore at most $2 \cdot 4^n + 2 \leq 4^{n+1}$. Since distinct elements of A_{n+1} are of $G^{\leq 8}$ -distance greater than $2 \cdot 4^{n+1}$ we see that each $x \in A_{n+1} \setminus D_n$ is an element of a distinct connected component of $G^{\leq 8} \upharpoonright D_{n+1}$. Thus, the set D_{n+1} satisfies (2) and (3).

We will extend $f_{i,n}$ to $f_{i,n+1}$ in finitely many steps by letting $f_{i,n+1,0} = f_{i,n}$ and iteratively extending $f_{i,n+1,k}$ to $f_{i,n+1,k+1}$ so that its domain includes $D_{n+1,k} = \bigcup_{x \in A_{n+1}} D_{n+1,k}(x)$. In particular, we will let $f_{i,n,k+1}$ be a partial Borel function extending $f_{i,n,k}$ such that for each $x \in D_{n+1,k}$:

- (1) If there is a neighbor y of x such that $f_{-i,n,k}(y) = x$, then $f_{i,n,k+1}(x) = y$.
- (2) The values of $f_{i,n,k+1}(x)$ over $i \in \{-2, -1, 1, 2\}$ are distinct neighbors of x such that $f_{i,n,k+1}(x) \notin \text{ran}(f_{i,n,k})$.

There will be at most one neighbor y of x from $D_{n+1,k}$ for which $f_{i,n,k}(y) = x$ for some i . Further, there will at most one point in D_n of distance 1 from x by condition (2) on D_n . If there is no point of D_n of distance 1 from x , then likewise there is at most one point in D_n of distance 2 from x by condition (2) on D_n . Thus, it is possible to define the $f_{i,n,k+1}(x)$ and still satisfy our requirements (1) and (2) for the $f_{i,n,k+1}(x)$. This is because there are at most two values of $i \in \{-2, -1, 1, 2\}$ for which $f_{i,n,k}(x)$ is determined by requirement (1). And if there is a $z \in D_n$ of distance 2 from x , there is only one j with $f_{j,n,k}(z)$ of distance 1 from x , so at least one remaining $i' \in \{-2, -1, 1, 2\}$ with $i' \neq j$ for which we can set $f_{i',n,k+1}(x) = f_{j,n,k}(z)$ and so satisfy requirement (2). \square

It is not the case that every acyclic Borel graph of degree 4 is generated by a free Borel action of \mathbb{F}_2 . This follows from results in [18]. Hence, discarding a meager set is necessary.

We may conclude Theorem 1.2 by combining Lemma 6.1 and Lemma 6.2. Indeed, we obtain the stronger result that there is a Borel comeager set $B \subseteq X$ such that the action of \mathbb{F}_2 is invariant and Borel on B .

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