A SHORT PROOF OF THE CONNES-FELDMAN-WEISS THEOREM

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Abstract. In this note, we give a short proof of the theorem due to Connes, Feldman, and Weiss that a countable Borel equivalence relation on a standard probably space (X, μ) is μ-hyperfinite if and only if it is μ-amenable.

1. Introduction

Recall that a Borel equivalence relation E on a standard Borel space X is said to be *finite* if every equivalence class of E is finite, and *countable* if every equivalence class of E is countable.

Definition 1.1. A countable Borel equivalence relation E on a standard Borel space X is *hyperfinite* if there is an increasing sequence $F_0 \subseteq F_1 \subseteq \ldots$ of finite Borel equivalence relations such $E = \bigcup E_i$. A countable Borel equivalence relation E on a standard probability space (X, μ) is said to be *μ-hyperfinite* if there is a μ-conull Borel set A such that $E \restriction A$ is hyperfinite.

Note that throughout this note we do not assume that E is μ-invariant (or even μ-quasi-invariant).

By the Feldman-Moore theorem, every countable Borel equivalence relation arises as the orbit equivalence relation of a Borel action of a countable group.

Theorem 1.2 (Feldman-Moore, see [KM, Theorem 1.3])**.** *Suppose E is a countable Borel equivalence relation on a standard Borel space X. Then there is a Borel action $\Gamma \curvearrowright^a X$ of a countable discrete group Γ on X such that E is equal to the orbit equivalence relation E_a of this action where $x \mathrel{E_a} y$ if there exists $\gamma \in \Gamma$ such that $\gamma \cdot x = y$.*

Analogously to the theory of amenable groups, one can develop a theory of amenable equivalence relations. Roughly, a countable Borel equivalence relation E on standard Borel space X is said to be *amenable* if there is a Borel way of assigning Reiter functions to each equivalence class of E. Note that in the literature (see for instance [JKL]) what we call amenability is sometimes called 1-amenability. We use $\ell^1(X)^+$ to denote the set of functions from $X \to [0, \infty)$ that are summable.

Definition 1.3. Suppose E is a countable Borel equivalence relation on X. Then E is *amenable* if there is a Borel sequence $\lambda^n \colon E \to [0,1]$ of functions assigning each pair of E-related points $(x, y) \in E$ to some $\lambda^n(x, y) \in [0,1]$ such that if we let $\lambda^n_x(y) = \lambda^n(x, y)$, then

(1) $\lambda^n_x \in \ell^1([x]_E)^+$ and $\|\lambda^n_x\| = 1$.
(2) For all $(x, y) \in E$, we have $\|\lambda^n_x - \lambda^n_y\|_1 \to 0$ as $n \to \infty$.

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If E is a countable Borel equivalence relation on a standard probability space (X, μ), then we say E is *μ-amenable* if there is a conull Borel set A such that $E \upharpoonright A$ is amenable.

In this note, we give a short proof of the following fundamental theorem due to Connes, Feldman, and Weiss:

Theorem 1.4 ([CFW])**.** *Suppose E is a countable Borel equivalence relation on a standard probability space (X, μ). Then E is μ-hyperfinite iff E is μ-amenable.*

The main difference between our proof and other arguments in the literature is that we avoid the use of Radon-Nikodym derivatives, following a suggestion of Robin Tucker-Drob. In Section 2 we give some examples of amenable equivalence relations. In Sections 3 and 4 we review some well-known facts about μ-hyperfinite equivalence relations. In Section 5 we give our proof of the Connes-Feldman-Weiss theorem.

2. Examples of amenable equivalence relations

We briefly give several examples of amenable Borel equivalence relations.

Proposition 2.1. *Suppose E is a countable Borel equivalence relation which admits a Borel transversal $A \subseteq X$ (that is, E is smooth). Then E is amenable.*

Proof. This is witnessed by

$$\lambda^n(x, y) = \begin{cases} 0 & \text{if } y \notin A \\ 1 & \text{if } y \in A. \end{cases}$$

□

Proposition 2.2. *Suppose Γ is a countable amenable group and $\Gamma \curvearrowright^a X$ is a Borel action of Γ. Then E_a is amenable.*

Proof. Let $f_n \colon \Gamma \to [0, 1]$ be a sequence of Reiter functions for Γ. Then the amenability of E_a is witnessed by the functions

$$\lambda^n(x, y) = \sum_{\{\gamma \in \Gamma : \gamma \cdot x = y\}} f_n(\gamma)$$

□

Proposition 2.3. *Suppose E is a hyperfinite Borel equivalence relation. Then E is amenable*

Proof. Let they hyperfiniteness of E be witnessed by $F_0 \subseteq F_1 \subseteq \ldots$. Then the amenability of E is witnessed by the functions

$$\lambda^n(x, y) = \begin{cases} 0 & \text{if } y \notin [x]_{F_n} \\ \frac{1}{|[x]_{F_n}|} & \text{if } y \in [x]_{F_n} \end{cases}$$

□

The converse of Proposition 2.3 is an important open problem:

Open Problem 2.4 (See [KM, page 30])**.** *Suppose E is a countable Borel equivalence relation. If E is amenable, is E hyperfinite?*

3. Characterizing μ-hyperfiniteness with finite subequivalence relations

One of the tools we will need to prove that Connes-Feldman-Weiss theorem is the theorem of Dye and Krieger that an increasing union of μ-hyperfinite Borel equivalence relations is μ-hyperfinite. This follows from the fact that a Borel equivalence relation E is μ-hyperfinite if it can be approximated well by finite Borel equivalence relations.

Lemma 3.1. *Suppose E is a countable Borel equivalence relation on a standard probability space (X, μ) generated by a Borel action of a countable group $\Gamma \curvearrowright^a X$. Then E is μ-hyperfinite iff for every $\epsilon > 0$ and every finite set $S \subseteq \Gamma$, there is a Borel equivalence relation $F \subseteq E$ with finite classes such that*

$$\mu(\{x \in X : \forall \gamma \in S(\gamma \cdot x\ F\ x)\}) > 1 - \epsilon$$

Proof. $\Rightarrow$: Suppose that E is μ-hyperfinite as witnessed by $F_0 \subseteq F_1 \subseteq \ldots$. Pick a finite $S \subseteq \Gamma$ and let

$$A_k = \{x \in X : \forall \gamma \in S(\gamma \cdot x\ F\ x)\}$$

Since the F_k are increasing and their union is E modulo a nullset, the A_k are also increasing and their union is conull. Hence there must be some A_k with $\mu(A_k) > 1 - \epsilon$ and hence F_k is as desired.

$\Rightarrow$: Let $\gamma_0, \gamma_1, \ldots$ enumerate Γ. For each k, find a Borel equivalence relation F_k with finite classes as above for the finite set $S = \{\gamma_0, \ldots, \gamma_k\}$ and $\epsilon = 1/2^k$. Let $F'_n = \bigcap_{k \geq n} F_k$. Now the F'_n clearly have finite classes since the F_k do, and are increasing by definition. It suffices to show that there is a conull Borel set A so that $E \restriction A = \bigcup_n F'_n \restriction A$.

To see this, it suffices to show that for each $\gamma_i \in \Gamma$, the set $A_i = \{x : \exists n \gamma_i \cdot x F'_n x\}$ is conull, since we can take $A = \bigcap_i A_i$. But for each n,

$$\mu(\{x : \neg(\gamma_i \cdot x\ F'_n\ x)\}) \leq \sum_{k \geq n} \mu(\{x : \neg(\gamma_i \cdot x\ F_n\ x)\}) \leq \sum_{k \geq n} \frac{1}{2^k} \leq \frac{1}{2^{n-1}}$$

So we are done. $\square$

Now we show that an increasing union of μ-hyperfinite Borel equivalence relations is μ-hyperfinite. The rough idea of the proof is that an increasing union of hyperfinite Borel equivalence relations is approximated well by hyperfinite equivalence relations which themselves are approximated well by finite equivalence relations. The only technicality is arranging the right way of generating these equivalence relations so that we can apply the above lemma.

Theorem 3.2 (Dye [D] and Krieger [Kr])**.** *Suppose $E_0 \subseteq E_1 \subseteq \ldots$ are countable Borel equivalence relations on a standard probability space (X, μ) and for every i, E_i is μ-hyperfinite. Then $\bigcup_i E_i$ is μ-hyperfinite.*

Proof. By the Feldman-Moore Theorem, we can choose Borel actions $\Gamma_i \curvearrowright^{a_i} X$ so the ith action generates E_i. If we then let the free product $*_{i \in \mathbb{N}} \Gamma_i$ act on X in the natural way where each $\gamma \in \Gamma_i$ acts via a_i, then this action clearly generates $\bigcup_i E_i$, and the restriction of the action to $*_{i \leq n} \Gamma_n$ generates E_n.

Now we use Lemma 3.1. Pick any finite $S \subseteq *_{i \in \mathbb{N}} \Gamma_i$ and $\epsilon > 0$. Since S is contained in $*_{i \leq n} \Gamma_n$ for some n, we can find our desired F by applying the lemma to E_n. $\square$

4. Characterizing μ-hyperfiniteness with subgraphs

If Γ is a finitely generated group, then it is standard to show that Γ is amenable if and only if the Cayley graph of Γ has isoperimetric constant 0. Analogously, we give a proof of a result due to Kaimanovich characterizing when a locally finite Borel graph has a μ-hyperfinite connectedness relation in terms of a measure-theoretic version of isoperimetric constant.

Recall that a Borel graph G on a standard Borel space X is a graph whose set of vertices is X, and where the edge relation of G is Borel as a subset of $X \times X$. A graph is *locally finite* (*locally countable*) if every vertex has finitely many (countably many) neighbors. Let $N_G(x)$ denote the set of neighbors of x in G.

Definition 4.1. Suppose G is a locally finite Borel graph on a Polish space X. We say G is *hyperfinite* if its connectedness relation E_G is hyperfinite. Similarly, we say that G is μ-hyperfinite with respect to some Borel probability measure μ if E_G is μ-hyperfinite.

An important example of Borel graphs is the following class of "Cayley graphs" of group actions. If $\Gamma \curvearrowright^a X$ is a Borel action of the countable group Γ on a standard Borel space X, and $S \subseteq \Gamma$ is symmetric set, then $G(a, S)$ is the Borel graph on X where distinct $x, y \in X$ are adjacent iff $\exists \gamma \in S(\gamma \cdot x = y)$.

Definition 4.2. If G is a locally finite Borel graph on a standard probability space (X, μ), then the μ-isoperimetric constant of G is the infimum of $\mu(\partial_G A)/\mu(A)$ over all Borel subsets $A \subseteq X$ of positive measure such that the induced subgraph $G \restriction A$ of G on A has finite connected components. Here, $\partial_G A$ denotes the set of vertices in $X \setminus A$ which are G-adjacent to a vertex in A.

For example, suppose Γ is a countable amenable group, $S \subseteq \Gamma$ is finite symmetric, and $\Gamma \curvearrowright^a (X, \mu)$ is a free μ-measure preserving Borel action of Γ. Then it is easy to check that $G(a, S)$ has μ-isoperimetric constant 0.

Now we give a proof of the theorem of Kaimanovich characterizing when a locally finite graph G is μ-hyperfinite in terms of isoperimetric constant. Robin Tucker-Drob pointed out that Elek's proof [E] of Kaimanovich's theorem works in the greater generality of when the graph G is not assumed to be μ-invariant. The following presentation of Theorem 4.3 is due to Tucker-Drob.

Theorem 4.3 (Kaimanovich [Ka], see also Elek [E])**.** *Let G be a locally finite Borel graph on a standard probability space (X, μ). Then G is μ-hyperfinite if and only if for every positive measure Borel subset $X_0 \subseteq X$, the isoperimetric constant of the induced subgraph $G \restriction X_0$ of G on X_0 is 0.*

Proof. Suppose first that G is μ-hyperfinite. Let $X_0 \subseteq X$ be a Borel set of positive measure and let $H = G \restriction X_0$. Then H is μ-hyperfinite, so after ignoring a null set we can find finite Borel subequivalence relations $F_0 \subseteq F_1 \subseteq \cdots$ with $E_H = \bigcup_n F_n$. Since H is locally finite, for each $x \in X_0$ there exists an n such that $N_H(x) \subseteq [x]_{F_n}$, where $N_H(x)$ denotes the set of H-neighbors of x.

Given $\epsilon > 0$, we may find n large enough so that $\mu(A_n) > \mu(X_0)(1 - \epsilon)$, where $A_n = \{x \in X_0 : N_H(x) \subseteq [x]_{F_n}\}$. Then $H \restriction A_n \subseteq R_n$ so $H \restriction A_n$ has finite connected components. Finally, $\mu(\partial_H A_n)/\mu(A_n) < \epsilon/(1 - \epsilon)$. Since $\epsilon > 0$ was arbitrary this shows the isoperimetric constant of G is 0.

Assume now that for every positive measure Borel subset $X_0 \subseteq X$ the isoperimetric constant of $G \restriction X_0$ is 0. To show that G is μ-hyperfinite it suffices to show that for any $\epsilon > 0$ there exists a Borel set $Y \subseteq X$ with $\mu(Y) \geq 1 - \epsilon$ such that $G \restriction Y$ has finite connected components.

This is because then we can find a sequence of such sets Y_n, $n \in \mathbb{N}$, with $\mu(Y_n) \geq 1 - 2^{-n}$. Let $Z_n = \bigcap_{k>n} Y_k$. So $Z = \bigcup_n Z_n$ is a Borel set of μ-measure 1. Then E_G is μ-hyperfinite since $E_G \restriction Z$ is the increasing union $\bigcup_n E_{G \restriction Z_n}$. (This argument is essentially a graph-theoretic version of Lemma 3.1).

Given $\epsilon > 0$, by measure exhaustion we can find a maximal countable collection $\mathcal{A}$ of pairwise disjoint non-null Borel subsets of X such that

(i) $G \restriction \bigcup \mathcal{A}$ has finite connected components;
(ii) $\mu(\partial_G(\bigcup \mathcal{A})) \leq \epsilon\mu(\bigcup \mathcal{A})$;
(iii) If $A, B \in \mathcal{A}$ are distinct, then no vertex in A is adjacent to a vertex in B.

Let $Y = \bigcup \mathcal{A}$. We now claim that the set $X_0 = X - (Y \cup \partial_G Y)$ is null. Otherwise, by hypothesis we may find a Borel set $A_0 \subseteq X_0$ of positive measure such that $G \restriction A_0$ has finite connected components and $\mu(\partial_{G \restriction X_0} A_0) < \epsilon\mu(A_0)$. But then the collection $\mathcal{A}_0 = \mathcal{A} \cup \{A_0\}$ satisfies (i)-(iii) in place of $\mathcal{A}$, ((i) and (iii) are clear and (ii) follows from the equality $\partial_G(\bigcup \mathcal{A}_0) = \partial_G Y \cup \partial_{G|X_0} A_0$) contradicting maximality of $\mathcal{A}$. Thus, $\mu(Y) = 1 - \mu(\partial_G Y) \geq 1 - \epsilon\mu(Y) \geq 1 - \epsilon$, and $G \restriction Y$ has finite connected components, which finishes the proof. □

Remark 4.4. It is an open question whether every Δ^1_1 hyperfinite equivalence relation has a Δ^1_1 witness to its hyperfiniteness (see [JKL, 6.1(B)]). Theorems 3.2 and 4.3 are both easily seen to be effective, modulo an effective nullset. This can be used to show that if E is a μ-hyperfinite Δ^1_1 equivalence relation on $(2^{\mathbb{N}}, \mu)$, where μ is a Δ^1_1 Borel probability measure, then there is a Δ^1_1 μ-conull set $A \subseteq 2^{\mathbb{N}}$ and a Δ^1_1 sequence $F_0 \subseteq F_1 \subseteq \ldots$ of equivalence relations on A witnessing that $E \restriction A$ is hyperfinite. This result is due originally to M. Segal [S].

5. Characterizing μ-amenability

Namioka's trick is the standard way of transforming Reiter functions into Følner sets in amenable groups. We will use it as part of our proof of the Connes-Feldman-Weiss theorem.

Lemma 5.1 (Namioka's trick). *For $a > 0$, let $1_{(a,\infty)}$ be the characteristic function of (a, ∞). Then if $f, g \in \ell^1(X)^+$, then*

$$\int_0^\infty \|1_{(a,\infty)}(f) - 1_{(a,\infty)}(g)\|_1 \, da = \|f - g\|_1$$

Proof. For arbitrary $s, t > 0$,

$$\int_0^\infty |1_{(a,\infty)}(s) - 1_{(a,\infty)}(t)| \, \mathrm{d}a = |s - t|$$

Hence, for $f, g \in \ell^1(X)^+$ and every $x \in X$ we have

$$\int_0^\infty |1_{(a,\infty)}(f(x)) - 1_{(a,\infty)}(g(x))| \, \mathrm{d}a = |f(x) - g(x)|$$

and hence

$$\int_0^\infty \|1_{(a,\infty)}(f) - 1_{(a,\infty)}(g)\|_1 \, \mathrm{d}a = \|f - g\|_1$$

by Fubini's theorem. □

By Theorem 4.3, to show that every μ-amenable countable Borel equivalence relation is μ-hyperfinite, it suffices to show the following.

Theorem 5.2. *Suppose that E is a μ-amenable countable Borel equivalence relation on a standard probability space (X, μ), B is a Borel subset of X of positive measure, and G is a bounded degree Borel graph on B. Then G has μ-isoperimetric constant 0.*

Proof. By discarding a nullset we may assume that E is amenable (instead of μ-amenable). Let $\lambda^n \colon E \to [0,1]$ be Reiter functions witnessing the amenability of E. Fix $\epsilon > 0$. The rough idea of the proof is that we can use Namioka's trick to change the Reiter functions into Følner sets. Having chosen an appropriate Følner set $\Pi \subseteq E$, we will find a set A witnessing that G has isoperimetric constant less than ϵ. The set A will be a union of sets of the form Π^z. These sets will have small boundary since the Følner sets are almost invariant.

To begin,

$$\lim_{n\to\infty} \int_B \sum_{y \in N_G(x)} \|\lambda^n_x - \lambda^n_y\|_1 \, \mathrm{d}\mu(x) = 0$$

by the dominated convergence theorem; since G has degree bounded by some d, for each n the integral on the LHS is bounded by d. Hence, we can find a large enough n so that

$$\int_B \sum_{y \in N_G(x)} \|\lambda^n_x - \lambda^n_y\|_1 \, \mathrm{d}\mu(x) < \epsilon\mu(B)$$

Let $\Lambda = \lambda^n$. As usual, let $1_{(a,\infty)}$ be the characteristic function of the interval (a, ∞). Now by Namioka's trick, Fubini's theorem, and the fact that $\|\Lambda_x\|_1 = 1$, applied to the displayed equation above, we have

$$\int_0^\infty \int_B \sum_{y \in N_G(x)} \|1_{(a,\infty)}(\Lambda_x) - 1_{(a,\infty)}(\Lambda_y)\|_1 \, \mathrm{d}\mu(x) \, \mathrm{d}a < \epsilon\mu(B)$$

$$= \epsilon \int_B \|\Lambda_x\|_1 \, \mathrm{d}\mu(x) = \epsilon \int_0^\infty \int_B \|1_{(a,\infty)}(\Lambda_x)\|_1 \, \mathrm{d}\mu(x) \, \mathrm{d}a$$

Hence there is an $a > 0$ so that

$$\int_B \sum_{y \in N_G(x)} \|1_{(a,\infty)}(\Lambda_x) - 1_{(a,\infty)}(\Lambda_y)\|_1 \, \mathrm{d}\mu(x) < \epsilon \int_B \|1_{(a,\infty)}(\Lambda_x)\|_1 \, \mathrm{d}\mu(x).$$

Fix this a. Now $1_{(a,\infty)}(\Lambda)$ is the characteristic function of a subset of $E \subseteq X \times X$. Call this set R. For each $x \in X$, we have that $R_x = \{y : (x, y) \in R\}$ is finite. Indeed, $|R_x| \leq 1/a$. We have that $\|1_{(a,\infty)}(\Lambda_x)\|_1 = |R_x|$ and similarly $\|1_{(a,\infty)}(\Lambda_x) - 1_{(a,\infty)}(\Lambda_y)\|_1 = |R_x \triangle R_y|$, where $\triangle$ indicated symmetric difference. So rewriting,

$$\int_B \sum_{y \in N_G(x)} |R_x \triangle R_y| \, \mathrm{d}\mu(x) < \epsilon \int_B |R_x| \, \mathrm{d}\mu(x)$$

By Lusin-Novikov uniformization [Ke, 18.15], since R has countable horizontal sections, we can write R as a union $R = \bigcup_n \{(g_n(x), x) : x \in X\}$ of countably many partial Borel functions g_i, and hence by taking the union of the graphs of only the first i many functions, we can find Borel sets $R_0 \subseteq R_1 \subseteq R_2 \subseteq \ldots$ such that $\bigcup_i R_i = R$, and for every i and $z \in X$ we have that $R^z_i = \{x \colon (x, z) \in R\}$ is finite. By the dominated convergence theorem, there is some R_i so that the above equation holds if we replace R by R_i. Set Π equal to this R_i. We now have

$$(*) \qquad \int_B \sum_{y \in N_G(x)} |\Pi_x \triangle \Pi_y| \, \mathrm{d}\mu(x) < \epsilon \int_B |\Pi_x| \, \mathrm{d}\mu(x)$$

Let H be the graph on X where zHz' if $(\Pi^z \cup \partial_G \Pi^z) \cap (\Pi^{z'} \cup \partial_G \Pi^{z'}) \neq \emptyset$. For each $z \in X$, Π^z and $\partial_G \Pi^z$ are finite sets. Hence H is a locally finite Borel graph. We can find a Borel $\mathbb{N}$-coloring c of H by [KST, Proposition 4.5]. For each color k, let $A_k = \{x : \exists z \in \Pi_x(c(z) = k)\}$. Then each connected component of $G \upharpoonright A_k$ corresponds to some set of the form Π^z, and so $G \upharpoonright A_k$ has finite connected components. We can use c to rewrite each side of (*) to get

$$\sum_k \int_B \sum_{y \in N_G(x)} |\{z \in \Pi_x \triangle \Pi_y : c(z) = k\}| \, \mathrm{d}\mu(x) < \epsilon \sum_k \int_B |\{z \in \Pi_x : c(z) = k\}| \, \mathrm{d}\mu(x)$$

so there is some k such that

$$\int_B \sum_{y \in N_G(x)} |\{z \in \Pi_x \triangle \Pi_y : c(z) = k\}| \, \mathrm{d}\mu(x) < \epsilon \int_B |\{z \in \Pi_x : c(z) = k\}| \, \mathrm{d}\mu(x)$$

But the right hand side of this equation is just $\int_B 1_{A_k} \, \mathrm{d}\mu = \mu(A_k)$. Further $x \in \partial A_k$ implies there is some $y \in N_G(x)$ such that $\{z \in \Pi_x \triangle \Pi_y : c(z) = k\}$ is nonempty. Hence, the function integrated on the left hand side is greater than or equal to the characteristic function of $\partial_G A_k$. Hence,

$$\mu(\partial_G A_k) < \epsilon \mu(A_k)$$

□

From this we can conclude the Connes-Feldman-Weiss theorem.

Proof of Theorem 1.4. We have already shown that every μ-hyperfinite Borel equivalence relation is μ-amenable in Proposition 2.3. Now suppose E is μ-amenable. By the Feldman-Moore theorem, we can find a Borel action of a countable group $\Gamma \curvearrowright^a X$ generating E. Let $S_0 \subseteq S_1 \subseteq \ldots \subseteq \Gamma$ be an increasing union of finite symmetric subsets of Γ whose union is Γ. Each graph $G(a, S_i)$ is hyperfinite by combining Theorem 5.2 and Theorem 4.3. Since $E = \bigcup_i E_{G(a,S_i)}$, we see that E is hyperfinite by Theorem 3.2. □

Historically, this theorem was preceded by a theorem of Ornstein and Weiss [OW] that every Borel action of an amenable countable group Γ on a standard probability space (X, μ) induces a μ-hyperfinite countable Borel equivalence relation.

Combing this with Dye's theorem, we conclude that any two aperiodic ergodic actions of an amenable group are orbit equivalent.

Corollary 5.3 (Dye [D], Ornstein-Weiss [OW]). *Suppose $\Gamma \curvearrowright^a (X, \mu)$ and $\Delta \curvearrowright^b (Y, \nu)$ are aperiodic ergodic measure-preserving actions of amenable groups on standard probability space (X, μ) and (Y, ν). Then a and b are orbit equivalent.*

Proof. E_a and E_b are μ-amenable and ν-amenable by Proposition 2.2. Hence, by Corollary 1.4, both actions induce μ-hyperfinite Borel equivalence relations E_a and E_b. Hence, both E_a and E_b are induced (modulo a nullset) by ergodic measure preserving Borel actions of $\mathbb{Z}$, which are orbit equivalent by Dye's theorem [D]. □

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