A BAIRE CATEGORY PROOF OF THE ACKERMAN-FREER-PATEL THEOREM

ANDREW S. MARKS

In this note, we give a proof of [AFP, Theorem 1.1] using the Baire category theorem. We also prove a slight generalization of [AFP, Theorem 3.19] where the underlying space is an arbitrary infinite Polish space instead of $\mathbb{R}$.

Suppose $\mathbb{A} = (A, R^A)_{R \in L}$ is a countable structure in a countable relational language L. Say $\mathbb{A}$ has **trivial definable closure** if for every finite tuple $\overline{a} \in A$, and for every $L_{\omega_1,\omega}$-formula $\phi(\overline{x}, y)$, if there is a unique $b \in \mathbb{A}$ such that $\mathbb{A} \vDash \phi(\overline{a}, b)$, then $b \in \overline{a}$. Equivalently, for all tuples $\overline{a}, \overline{b} \in A$, such that $\overline{a}$ and $\overline{b}$ are disjoint, there are infinitely many pairwise disjoint tuples $\overline{c} \in A$ such that $\mathrm{tp}^{\mathbb{A}}(\overline{a}, \overline{b}) = \mathrm{tp}^{\mathbb{A}}(\overline{a}, \overline{c})$ (see [Hod, 4.1.3]).

Lemma 0.1. *Suppose $\mathbb{A} = (A, R^{\mathbb{A}})_{R \in L}$ is a countable structure in a countable relational language L, where $\mathbb{A}$ has trivial definable closure. Then if X is an infinite Polish space, there exists a Borel L-structure $\mathbb{A}' = (X, R^{\mathbb{A}'})_{R \in L}$ on X (that is, the relations $(R^{\mathbb{A}'})_{R \in L}$ are Borel) so that for any countable dense set $D \subseteq X$, $\mathbb{A}' \restriction D$ is isomorphic to $\mathbb{A}$.*

Proof. By Morleyizing $\mathbb{A}$ (see [Hod, Section 2.6]) and expanding L, we may assume that there is a countable set T of Π_2 sentences in L such that if $\mathbb{B}$ is a countable structure, then $\mathbb{B} \vDash T$ if and only if $\mathbb{B}$ is isomorphic to $\mathbb{A}$. (After expanding the language this way and obtaining $\mathbb{A}'$, take the reduct of $\mathbb{A}'$ to the original language).

By Proposition 0.3, if X is an infinite Polish space, then there are Borel sets $\{B_s\}_{s \in \omega^{<\omega}}$ satisfying: if $s \subseteq t$ then $B_t \subseteq B_s$, for every $n \in \omega$, $\{B_s\}_{|s|=n}$ partitions X, the collection $\{B_s\}_{s \in \omega^{<\omega}}$ separates points, and every B_s contains an open subset. For example, if $X = \omega^\omega$, then let $B_s = N_s$, the basic open neighborhood determined by s. (The case $X = \omega^\omega$ suffices to prove Corollary 0.2).

Let Y be the set of injections $f\colon \omega^{<\omega} \to A$ such that if $s_0 \dots, s_n \in \omega^{<\omega}$ are pairwise incompatible, and $t_0, \dots, t_n \in \omega^{<\omega}$ are such that $s_i \subseteq t_i$ for $i \leq n$, then

$$\mathrm{tp}^{\mathbb{A}}(f(s_0), \dots, f(s_n)) = \mathrm{tp}^{\mathbb{A}}(f(t_0), \dots, f(t_n)).$$

If we equip the set of functions from $\omega^{<\omega} \to A$ with the product topology, then Y is a closed subset of this space. It is nonempty since $\mathbb{A}$ has trivial definable closure. Each $f \in Y$ yields a Borel L-structure $(X, R^f)_{R \in L}$ on X as follows: if $\overline{x} = (x_{p(0)}, \dots, x_{p(m-1)})$ is a tuple in X where $x_0, \dots, x_{n-1}$ are distinct and $p\colon m \to n$, we define

$$R^f(x_{p(0)}, \dots, x_{p(m-1)}) \leftrightarrow R^{\mathbb{A}}(f(s_{p(0)}), \dots f(s_{p(m-1)}))$$

for any sequence $s_0, \dots, s_n$ so that the sets $B_{s_0}, \dots, B_{s_n}$ are disjoint, and $x_i \in B_{s_i}$. Our definition of Y makes it clear that the truth value of $R^{\mathbb{A}}(f(s_{p(0)}), \dots f(s_{p(m-1)}))$ will be same for any such sequence $s_0, \dots, s_{n-1}$. We claim that a comeager set of $f \in Y$ have the property that the structure $\mathbb{A}' = (X, R^f)_{R \in L}$ is as desired.

Suppose $\overline{s} = s_0, \dots, s_{n-1} \in \omega^{<\omega}$ is a tuple of pairwise incompatible elements, and $\overline{a} = (a_0, \dots, a_{n-1}) \in A$ are distinct. Define the open set $U_{\overline{a}, \overline{s}} = \{f \in Y : f(s_i) = a_i\}$. Suppose $\phi \in T$ where $\phi = \forall \overline{x} \exists \overline{y} \theta(\overline{x}, \overline{y})$, where the length of $\overline{x}$ is m, and $p\colon m \to n$ is any function. Let the

Date: May 31, 2016.

length of $\overline{x}$ and $\overline{y}$ combined be $j \geq m$, and let $V_{\overline{a},\overline{s},p,\phi}$ be the set of $f \in U$ such that there exists $s_n, \ldots, s_{k-1} \in \omega^{<\omega}$ so that elements of the sequence $s_0, \ldots, s_{k-1}$ are pairwise incompatible and such that $\mathbb{A} \vDash \theta(f(s_{p^*(0)}), \ldots, f(s_{p^*(j-1)}))$ for some $p^*\colon j \to k$ extending p. We claim that $V_{\overline{a},\overline{s},p,\phi}$ is open and dense in $U_{\overline{a},\overline{s}}$.

The set $V_{\overline{a},\overline{s},p,\phi}$ is clearly open. Now $A \vDash \phi$ and so there exists a tuple $\overline{b} \in A$ such that $\theta(a_{p(0)}, \ldots, a_{p(m-1)}, \overline{b})$. Let $\overline{b}'$ enumerate all the elements of $\overline{b}$ that are disjoint from $\overline{a}$. Since there are infinitely many pairwise disjoint tuples $\overline{c} \in A$ such that $\mathrm{tp}^A(\overline{a}, \overline{b}') = \mathrm{tp}^A(\overline{a}, \overline{c})$, $V_{\overline{a},\overline{s},p,\phi}$ is dense in $U_{\overline{a},\overline{s}}$.

This suffices to prove the theorem. Suppose $f \in Y$ is generic, $D \subseteq X$ is a dense set, and $\phi = \exists \overline{x} \forall \overline{y} \theta(\overline{x}, \overline{y}) \in T$. We will show $(X, R^f)_{R \in L} \restriction D \vDash \phi$. Let $\overline{x} = (x_{p(0)}, \ldots, x_{p(m-1)})$ be a tuple in D where $x_0, \ldots, x_{n-1}$ are distinct and $p\colon n \to m$. Then there is a sequence of incompatible $s_0, \ldots, s_{n-1} \in \omega^{<\omega}$ with $x_i \in B_{s_i}$, since the B_s separate points. Since f is generic, $\theta(f(s_{p(0)}), \ldots, f(s_{p^*(j-1)}))$ is true for some $s_n, \ldots, s_{k-1} \in \omega^{<\omega}$, such that the sequence $s_0, \ldots, s_{k-1}$ is pairwise incompatible and some $p^*\colon j \to k$ extending p. Let $y_i = x_i$ for $i < n$. For $n \leq i < j$, D must contain some $y_i \in B_{s_i}$ since each B_{s_i} contains an open subset. Hence, we have shown $(X, R^f)_{R \in L} \restriction D \vDash \theta(y_{p^*(0)}, \ldots, y_{p^*(j-1)})$ and so $(X, R^f)_{R \in L} \restriction D \vDash \theta(x_{p(0)}, \ldots, x_{p(m-1)}, y_{p^*(m)}, \ldots, y_{p^*(j-1)})$ as desired. □

Recall that if L is a countable relational language, the space X_L is the set of all L-structures with universe ω. The group S_∞ of all permutations of ω acts on X_L by permuting the universe of each structure in X_L (see [K95, Section 16]).

Corollary 0.2 ([AFP, Theorem 1.1]). *Suppose $\mathbb{A} = (A, R^{\mathbb{A}})_{R \in L}$ is a countable structure in a countable relational language L. Then A has trivial definable closure if and only if there is an S_∞-invariant Borel probability measure μ on X_L that is supported on the set of structures isomorphic to $\mathbb{A}$.*

Proof. Suppose $\mathbb{A}$ has trivial definable closure. Let X be any perfect Polish space and let μ be an atomless Borel probability measure on X that assigns positive measure to every open subset of X. By Lemma 0.1, let $\mathbb{A}' = (X, R^{\mathbb{A}'})_{R \in L}$ be a Borel L-structure such that every countable dense set $D \subseteq X$ has $\mathbb{A}' \restriction D$ isomorphic to $\mathbb{A}$. Let μ^ω be the product probability measure on X^ω. Since μ is atomless and assigns positive measure to every open subset of X, μ^ω is supported on the set $Z \subseteq X^\omega$ of sequences $(x_i) \in X^\omega$ such that (x_i) is injective and dense in X. So each such (x_i) has $\mathbb{A}' \restriction \{x_i : i \in \omega\}$ isomorphic to $\mathbb{A}$.

Let $f\colon Z \to X_L$ be the function so that $f((x_i))$ is the structure on ω isomorphic to $\mathbb{A}' \restriction \{x_i : i \in \omega\}$ obtained by identifying x_i with i. Formally, $f((x_i)) = (\omega, R^{f((x_i))})_{R \in L}$ where

$$R^{f((x_i))}(n_0, \ldots, n_k) \leftrightarrow R^{\mathbb{A}'}(x_{n_0}, \ldots, x_{n_k}).$$

Then the pushforward $f_* \mu^\omega$ of μ^ω under f is supported on the set of structures isomorphic to $\mathbb{A}$. This measure is S_∞-invariant because the permutation action of S_∞ on X^ω is μ^ω-invariant.

We now prove the converse. Suppose for a contradiction that $\mathbb{A}$ has nontrivial definable closure, but there exists an S_∞-invariant Borel probability measure μ on the set of structures in X_L isomorphic to $\mathbb{A}$. Let ϕ be an $L_{\omega_1,\omega}$ formula and $\overline{a} \in A$ be parameters so that $\mathbb{A} \vDash \exists! y \notin \overline{a} \phi(\overline{a}, y)$. If $\overline{n}$ is a tuple of elements of ω and $m \notin \overline{n}$, let $A_{\overline{n},m}$ be the set of structures $\mathbb{B} \in X_L$ isomorphic to $\mathbb{A}$ so that $\overline{n}$ is lexicographically least such that $\mathbb{B} \vDash \exists! y \notin \overline{n} \phi(\overline{n}, y)$, and m is the least element not in $\overline{n}$ such that $\mathbb{B} \vDash \phi(\overline{n}, m)$. The sets $A_{\overline{n},m}$ partition the set of models isomorphic to $\mathbb{A}$. So $\mu(\bigcup A_{\overline{n},m}) = 1$. However, if $m, m' \notin \overline{n}$, then $\mu(A_{\overline{n},m}) = \mu(A_{\overline{n},m'})$ since there is an element of S_∞ that fixes $\overline{n}$ but maps m to m'. We also have that $A_{\overline{n},m}$ and $A_{\overline{n},m'}$ are disjoint. Hence, since there are countably many $m \notin \overline{n}$ we must have $\mu(A_{\overline{n},m}) = 0$ for each $\overline{n}$, since μ is a probability measure. Thus, $\mu(\bigcup A_{\overline{n},m}) = 0$ which is a contradiction. □

Proposition 0.3. *If X is an infinite Polish space, then there are Borel sets $\{B_s\}_{s\in\omega^{<\omega}}$ satisfying:*

1. *If $s \subseteq t$ then $B_t \subseteq B_s$*
2. *For every $n \in \omega$, $\{B_s\}_{|s|=n}$ partitions X*
3. *$\{B_s\}_{s\in\omega^{<\omega}}$ separates points in X*
4. *Every B_s contains an open subset.*

Proof. Since X is infinite, there exists a countably infinite collection of disjoint open subsets $(U_s)_{s\in\omega^{<\omega}}$ of X. (So $U_s \cap U_t = \emptyset$ if $s \neq t$). Let $B'_s = \bigcup\{U_t : t \supseteq s\}$. Then the B'_s satisfy (1) and (4), and for every $x \in \omega^\omega$, $\bigcap_n B'_{x\restriction n} = \emptyset$. We will find $B_s \supseteq B'_s$ satisfying (1), (2), and (3).

Let $f\colon X \setminus \bigcup_{s\in\omega^{<\omega}} U_s \to \omega^\omega$ be a Borel injection, and for every $s \in \omega^{<\omega}$, let $f_s\colon U_s \to \omega^\omega$ be a Borel injection such that $f_s(U_s) \subseteq N_s$. We may assume that the ranges of f, and the f_s are all disjoint. Define

$$B_s = f^{-1}(N_s) \cup B'_s \cup \bigcup_{t\subseteq s}\{f_t^{-1}(N_s)\}$$

□

References

[AFP] N. Ackerman, C. Freer, and R. Patel, Invariant measures concentrated on countable structures, *arXiv*: 1206.4011v3.

[Hod] W. Hodges, *Model Theory*, Encyclopedia of Mathematics and its Applications, vol. 42, Cambridge University Press, 1993.

[K95] A.S. Kechris, *Classical Descriptive Set Theory*, Springer, 1995.