Universal equivalence relations on $X^\mathbb{N}$ generated by permutation actions of countable subgroups of $S_\infty$

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Let $S_\infty$ be the group of all permutations of $\mathbb{N}$ and $X$ be a standard Borel space. Then the space $X^\mathbb{N}$ of functions from $\mathbb{N}$ to $X$ is a standard Borel space, and $S_\infty$ acts on this space by permutation where given $x \in X^\mathbb{N}$ and $g \in S_\infty$, $g \cdot x(n) = x(g^{-1}(n))$. Given any countable subgroup $G$ of $S_\infty$, we can likewise restrict this action to $G$ and consider the induced orbit equivalence relation on $X^\mathbb{N}$, which we will note $E_{X^\mathbb{N}}^G$.

Hjorth has asked [1] [2] whether given countable Borel equivalence relations $E \subseteq F$, if $E$ is universal must $F$ be universal? We are interested in this question in the setting of equivalence relations of the form $E_{X^\mathbb{N}}^G$ above. However, we will modify the question and instead ask for a stronger form of universality known as uniform universality for which we currently have more traction to prove theorems.

Suppose $G$ and $H$ are countable groups equipped with Borel actions on the standard Borel spaces $X$ and $Y$ with associated orbit equivalence relations $E_X^G$ and $E_Y^H$. Then say a homomorphism $f$ from $E_X^G$ to $E_Y^H$ is uniform if there exists a function $u : G \rightarrow H$ such that for all $g \in G$ and $x \in X$, we have $f(g \cdot x) = u(g) \cdot f(x)$. This uniformity function $u$ need not be a homomorphism from $G$ to $H$ in general, but in the case where $G$ is a free group, then $u$ may be assumed to be a homomorphism.

We say that a Borel action of a countable group $H$ on a standard Borel space $Y$ generates a uniformly universal equivalence relation $E_Y^H$ if given any countable Borel equivalence relation $E_X^G$ there exists a uniform Borel reduction from $E_X^G$ to $E_Y^H$. We say that $E_Y^H$ is uniformly weakly universal if there exists an injective uniform Borel homomorphism from every $E_G^X$ to $E_Y^H$.

For example, the usual argument that $E(\mathbb{F}_\omega, \omega^\omega)$ is universal also shows that it is uniformly universal. Of course, it is important to specify the action as well as the equivalence relation when we are discussing uniform universality.

Note for example, that any universal countable Borel equivalence relation is uniformly universal for some Borel action of a countable group that generates it. This follows from the fact that if $E$ is universal, then $F \subseteq_B E$ for any countable Borel equivalence relation $F$ [4]. Thus, if $E$ is universal and generated by a Borel action of $G$, we can embed $E(\mathbb{F}_\omega, \omega^\omega)$ into $E$, and and use this embedding to obtain a Borel action of $\mathbb{F}_\omega * G$ that generates $E$ for
which it is uniformly universal.

These notions of uniform universality and uniform weak universality are quite robust. For example, suppose $E^X_G$ is uniformly universal, $E^X_G \leq_B E^Y_H$ via $f$, and we can find a Borel partition of $X$ into continuum many $E^X_G$-invariant pieces $\{A_\alpha : \alpha \in \mathbb{R}\}$ such that each $f \restriction A_\alpha$ is equivalent to a uniform Borel homomorphism. Then results from [4] show that $E^Y_H$ is uniformly universal. Recall that two homomorphisms from $E$ to $F$ are said to be equivalent if they induce the same function on equivalence classes.

From the opposite perspective, if $E^Y_H$ is uniformly universal, then this may be witnessed in a rather strong way: every $E^X_G$ must be reducible to $E^Y_H$ with a uniformity function $u : G \to H$ that is a homomorphism. This is because $E^X_G \leq_B E(\mathbb{F}_\omega, \omega^\omega) \leq_B E^Y_H$: the usual proof that $E(\mathbb{F}_\omega, \omega^\omega)$ is uniformly universal produces a group homomorphism, and since $\mathbb{F}_\omega$ is a free group, we can also obtain a group homomorphism for the second uniform Borel reduction. Further, by [3], if these groups are recursively presented, then the resulting homomorphism $u : G \to H$ may also be chosen to be recursive.

Far more is known about uniform universality and uniform weak universality then their more general non-uniform counterparts. For example, we have the following:

**Theorem 0.1 ([3]).**

1. If $G$ is a countable group that generates a uniformly universal countable Borel equivalence relation, then $G$ must contain a nonabelian free subgroup.

2. Given any uniformly universal countable Borel equivalence relation $E^X_G$ on a standard Borel probability space $(X, \mu)$, there is an invariant Borel set $A$ of full measure such that $E^X_G \restriction A$ is not uniformly universal.

3. If $E^X_{G_0} \subseteq E^X_{G_1} \subseteq \ldots$ is an increasing sequence of non-uniformly universal countable Borel equivalence relations, then their union $E^X_{G_0 \ast G_1 \ldots}$ is not uniformly universal.

4. There exists an increasing sequence $E^X_{G_0} \subseteq E^X_{G_1} \subseteq \ldots$ of uniformly universal countable Borel equivalence relations whose union $E^X_{G_0 \ast G_1 \ldots}$ is not uniformly universal.

When the assumption of uniformity is removed, the analogues of all of the above facts are open.

It has been conjectured [3] that a countable Borel equivalence relation is universal if and only if it is uniformly universal for all Borel actions of a countable group that generate it. Certainly, all known proofs that a countable Borel equivalence relation is universal show the stronger fact that it is uniformly universal for some natural group action. So at the very least, the class of uniformly universal countable Borel equivalence relations are those which we can hope to prove universal without dramatically new techniques.
Our first theorem is a characterization of what countable subgroups of $S_\infty$ generate uniformly weakly universal countable Borel equivalence relations with their permutation actions. From [3] we know that a a group may generate a uniformly weakly universal equivalence relation iff it contains a nonabelian free subgroup. Our characterization is that the existence of such a subgroup must be witnessed in a particular manner.

**Theorem 0.2.** If the cardinality of $X$ is $\geq 2$, and $G$ is any countable subgroup of $S_\infty$, then $E_G^\infty$ is uniformly weakly universal if and only if there exists some $n \in \mathbb{N}$ and a subgroup $H \leq G$ isomorphic to $\mathbb{F}_2$ such that the map $H \to \mathbb{N}$ given by $h \mapsto h(n)$ is injective.

**Proof.** Let $E_\infty$ be the orbit equivalence relation of the left shift action of $\mathbb{F}_2$ on $2^\mathbb{N}$. In the following proof we will use $g, h$ for elements of $G$ and $\alpha, \beta, \gamma, \delta$ for elements of $\mathbb{F}_2$.

$(\Rightarrow)$: Assume that for all $n \in \mathbb{N}$ and $g_0, g_1 \in G$, that there exists some nontrivial reduced word $h$ in $g_0$ and $g_1$ such that $h(n) = n$. Now let $f$ be a uniform Borel homomorphism from $E_\infty$ to $E_G^\infty$ with uniformity function $u : \mathbb{F}_2 \to G$. Since $\mathbb{F}_2$ is free, we may assume that $u$ is a group homomorphism. We claim that $f$ is constant on a set of Lebesgue measure 1. It is enough to show that for each $n \in \mathbb{N}$, $f(x)(n)$ is constant on a set of Lebesgue measure 1.

Let $n \in \mathbb{N}$ be given. Let $\mathbb{F}_2 = \langle \alpha, \beta \rangle$ and consider words in $u(\alpha)$ and $u(\beta)$. By assumption, there must be some nonidentity $\gamma \in \mathbb{F}_2$ such that $u(\gamma)(n) = n$. We see that $f(\gamma \cdot x)(n) = u(\gamma) \cdot f(x)(n) = f(x)(u(\gamma)^{-1}(n)) = f(x)(n)$. Hence, the value assigned to $x$ by $x \mapsto f(x)(n)$ is invariant under the map $x \mapsto \gamma \cdot x$. However, since $x \mapsto \gamma \cdot x$ is an ergodic transformation, the map $x \mapsto f(x)(n)$ must therefore be constant a.e.

$(\Leftarrow)$: Let $n \in \mathbb{N}$ and $H \leq G$ be isomorphic to $\mathbb{F}_2$ as in the assumption of the Theorem. Let $u : \mathbb{F}_2 \to H$ be an isomorphism. Let the function $\hat{\cdot} : \mathbb{F}_2 \to \mathbb{N}$ be defined by $\hat{\cdot} = u(\cdot)(n)$. Then $\hat{\cdot}$ is an injection and for all $y \in X^\mathbb{N}$ and $\gamma, \delta \in \mathbb{F}_2$ we have that $(u(\gamma) \cdot y)(\hat{\delta}) = (u(\gamma) \cdot y)(u(\delta)(n)) = y(u(\gamma)^{-1}(u(\delta)(n))) = y(\hat{\gamma^{-1}\delta})$. Hence, if we restrict $y$ to the range of $\hat{\cdot}$ and act on it only with elements in the range of $u$, then this behaves like a left shift action.

We now construct an injective Borel homomorphism $f : 2^{2^\mathbb{F}_2} \to 2^\mathbb{N}$ from $E_\infty$ to $E_G^\infty$. Let $f$ be defined by $f(x)(\hat{\gamma}) = x(\gamma)$ for all $x \in 2^{2^\mathbb{F}_2}$ and $\gamma \in \mathbb{F}_2$ and $f(x)(m) = 0$ if $m$ is not in the range of $\hat{\cdot}$. It is clear that $f$ is injective. It is also clear that $f$ is a homomorphism with uniformity function $u$, since since for all $\gamma, \delta \in \mathbb{F}_2$, we have $f(\gamma \cdot x)(\hat{\delta}) = (\gamma \cdot x)(\hat{\delta}) = x(\gamma^{-1}\delta) = f(x)(\gamma^{-1}\delta) = u(\gamma) \cdot f(x)(\hat{\delta})$.

Next, we turn to universality instead of weak universality and prove that the same class of countable subgroups of $S_\infty$ as in the above theorem generate uniformly universal countable Borel equivalence relations, provided we change the cardinality of $X$ to be $\geq 3$. 


**Theorem 0.3.** If the cardinality of $X$ is $\geq 3$, and $G$ is any countable subgroup of $S_\infty$, then $E_G^{\infty \times}$ is uniformly universal if and only if there exists some $n \in \mathbb{N}$ and a subgroup $H \leq G$ isomorphic to $\mathbb{F}_2$ such that the map $H \to \mathbb{N}$ given by $h \mapsto h(n)$ is injective. Hence if $G \leq H \leq S_\infty$ are countable and $G$ generates a uniformly universal equivalence relation, then so does $H$.

**Proof.** We do not have to prove the forward direction, since this is already implied by Theorem 0.2. So suppose now that there exists a subgroup $H \leq G$ isomorphic to $\mathbb{F}_2$ such that the map $H \to \mathbb{N}$ given by $h \mapsto h(n)$ is injective. (We can obtain a subgroup isomorphic to $\mathbb{F}_2$ instead of $\mathbb{F}_2$ by using an embedding of $\mathbb{F}_3$ into $\mathbb{F}_2$). We may as well assume that $X = 3$.

By [3], the group of recursive permutations of $\mathbb{N}$ yields a universal countable Borel equivalence relation when it acts by permutation on $3^\mathbb{N}$. This proof relativizes to give the same result for the group of permutations of $\mathbb{N}$ recursive in any $z \in 2^\omega$. Fix some $z \in 2^\omega$ that codes a presentation of the group $G$. Let us replace $\mathbb{N}$ here with $\mathbb{F}_2 \times \mathbb{N}$ (there is a recursive bijection between these two sets). Let $R$ be the group of permutations of $\mathbb{F}_2 \times \mathbb{N}$ that are recursive in $z$, so that we are considering the equivalence relation on $3^{2^2 \times \mathbb{N}}$ generated by the permutation action of the group $R$. The proof from [3] yields a Borel embedding $f^*$ of $E_\infty$ into $E_R^{3^{2^2 \times \mathbb{N}}}$ with the following properties:

1. $f^*$ is uniform with the uniformity function $u^* : \mathbb{F}_2 \to R$ where for all $\gamma, \delta \in \mathbb{F}_2$ and $i \in \mathbb{N}$ we have $u^*(\gamma)((\delta, i)) = (\gamma^{-1}\delta, i)$.
2. For every infinite set $A \subseteq \mathbb{N}$ that is $\Sigma^0_2$ in $z$, there exists an $i \in A$ such $f^*(x)((\gamma, i)) \neq 0$ for every $\gamma \in \mathbb{F}_2$.
3. Let $S_k = \{(\gamma, i) : \gamma \in \mathbb{F}_2$ and $i \geq k\}$. If $h$ is a partial recursive injection $h : \mathbb{F}_2 \times \mathbb{N} \to \mathbb{F}_2 \times \mathbb{N}$ computable relative to $z$ such that there exists a $k$ such that $h \upharpoonright S_k$ is total, then for all $x, y \in 2^{\mathbb{F}_2}$, if $h^{-1} \cdot f^*(x) \upharpoonright S_k = f^*(y) \upharpoonright S_k$, then there exists a $\gamma \in \mathbb{F}_2$ such that $y = \gamma \cdot x$.

Now let $\mathbb{F}_2 = \langle \alpha, \beta \rangle$ sit canonically inside $\mathbb{F}_3 = \langle \alpha, \beta, \xi \rangle$. Let $u : \mathbb{F}_3 \to H$ be an isomorphism, and let $C = \{\gamma \xi^i : \gamma \in \mathbb{F}_2$ and $i \in \mathbb{N}\}$. Here we think of an element $\gamma \xi^i$ of $C$ as coding the pair $(\gamma, i)$. Let the function $\hat{\gamma} : C \to \mathbb{N}$ be defined by $\hat{\gamma} = u(\gamma)(n)$ for all $\gamma \in C$. This function is injective. We are now ready to define $f$, our Borel embedding of $E_\infty$ into $E_G^{\infty \times}$.

For all $x \in 2^{\mathbb{F}_2}$, $\gamma \in \mathbb{F}_2$ and $i \in \mathbb{N}$ we define $f(x)(\gamma \xi^i) = f^*(x)((\gamma, i))$. For all $m \in \mathbb{N}$ that are not in the range of $\hat{\gamma}$, we define $f(x)(m) = 0$. We claim that $f$ is a uniform reduction from $E_\infty$ to $E_G^{\infty \times}$ with uniformity function $u \upharpoonright \mathbb{F}_2$. It is clear that $f$ is homomorphic. We must show that $f$ is a cohomomorphism.

Suppose $x, y \in \mathbb{F}_2$ and $g^{-1} \cdot f(x) = f(y)$ for some $g \in G$. We must show $xE_\infty y$. There are two cases. First, suppose there exists a $k$ such
that for all $i \geq k$ and $\gamma \in \mathbb{F}_2$, we have $g(\widehat{\gamma \xi^i}) \in \text{ran}(\widehat{\cdot})$. Then using the correspondence between $(\gamma, i)$ and $\widehat{\gamma \xi^i}$, define $h : S_k \to \mathbb{F}_2 \times \mathbb{N}$ so that for all $\gamma \in \mathbb{F}_2$ and $i \geq k$ if $h((\gamma, i)) = (\delta, j)$, then $\delta \xi^j = g(\widehat{\gamma \xi^i})$ and so $h$ is computable relative to $z$. Now from property (3) of $f^*$ we have that since $h^{-1} : f^*(x) \upharpoonright S_k = f^*(y) \upharpoonright S_k$ (because $g^{-1} \cdot f(x) = f(y)$), it must be that $y = \gamma \cdot x$ for some $\gamma \in \mathbb{F}_2$. Finally, suppose that for arbitrarily large $i$, there exists a $\gamma \in \mathbb{F}_2$ such that $g(\widehat{\gamma \xi^i}) \notin \text{ran}(\widehat{\cdot})$. The set of such $i$ is $\Sigma^0_2$ relative to $z$ and hence by property (2) of $f^*$ there exists a pair $(\gamma, i)$ with $g(\widehat{\gamma \xi^i}) \notin \text{ran}(\widehat{\cdot})$ such that $f(y)(\widehat{\gamma \xi^i}) \neq 0$. However, by the definition of $f$, we see that this $\gamma$ and $i$ must have $f(x)(g(\widehat{\gamma \xi^i})) = 0$. Hence $g^{-1} \cdot f(x) \neq f(y)$.

There are a couple other classes of Borel actions of countable groups for which there is a complete classification of which are uniformly universal. The first is the left shift action of $G$ on $X^G$ when $X$ is a standard Borel space of cardinality $\geq 2$. This is uniformly universal if and only if $G$ contains a nonabelian free subgroup. The second is where $G$ acts by conjugacy on subgroups of $G$. Here also, this action generates a uniformly universal equivalence relation if and only if $G$ contains a nonabelian free subgroup [1]. Hence, in these two cases as well as Theorem 0.3, Hjorth’s question for uniform universality has a positive answer. It has been conjectured [3] that Hjorth’s question has a negative answer for equivalence relations on $2^\mathbb{N}$ generated by permutation actions of countable subgroups of $S_\infty$. The seemingly insignificant difference between $2^\mathbb{N}$ and $3^\mathbb{N}$ is a striking illustration of how subtle Hjorth’s question is.

References


