A level set algorithm

for the Mumford-Shah segmentation model

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INTRODUCTION AND MOTIVATIONS

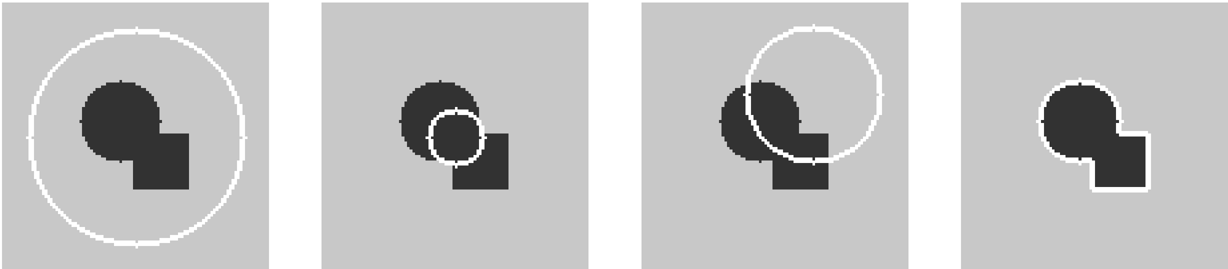
An active contour model without edges

Basic idea Fitting Term

$$\int_{inside(C)} |u_0 - c_1|^2 dxdy + \int_{outside(C)} |u_0 - c_2|^2 dxdy$$

$$\text{where} \begin{cases} c_1 = \text{average of } u_0 \text{ inside } C \\ c_2 = \text{average of } u_0 \text{ outside } C \end{cases}$$

Fit > 0 Fit > 0 Fit > 0 Fit ≈ 0



- Minimize the Fitting Term + Length(C)

The model

$$\inf_{c_1,c_2,C} F(c_1, c_2, C) = \nu \cdot \text{Length}(C)$$

$$+\mu^2 \int_{inside(C)} |u_0 - c_1|^2 dxdy + \mu^2 \int_{outside(C)} |u_0 - c_2|^2 dxdy$$

Relation with the Mumford-Shah model

C = set of curves in Ω

Find u = optimal approximation of u_0, and C:

$$\inf_{u,C} F^{MS}(u,C),$$

where

$$F^{MS}(u,C) = \mu^2 \int_\Omega |u-u_0|^2 dxdy + \int_{\Omega\setminus C} |\nabla u|^2 dxdy$$

$$+\nu \cdot \text{Length}(C),$$

u = piecewise-smooth in $\Omega \setminus C$

Our model is a particular case of M-S

$$u = \begin{cases} c_1 = \text{average}(u_0) \text{ inside } C \\ c_2 = \text{average}(u_0) \text{ outside } C. \end{cases}$$

- u is the best approximation of u_0 taking only two values c_1 and c_2, and with one edge C

- the snake C is the boundary between the sets $\{u = c_1\}$ and $\{u = c_2\}$

Level sets S. Osher-J. Sethian '87

C = curve
ϕ = level set function

such that

$$\begin{array}{lcc} \phi(x,y) = 0 & \text{on} & C \\ \phi(x,y) > 0 & \text{inside} & C \\ \phi(x,y) < 0 & \text{outside} & C \end{array}$$

$$\text{Normal} = \frac{\nabla\phi}{|\nabla\phi|}, \quad \text{Curvature} = \nabla\left(\frac{\nabla\phi}{|\nabla\phi|}\right)$$

$$\phi_t + v \cdot \nabla\phi = 0, \quad v = -\text{Normal} \cdot \text{Curvature}$$

$\longrightarrow$ Motion by Mean Curvature

The evolution: $\phi(t,x,y)$

- Given $C(0) = \{\phi(0) = 0\}$
- Obtain $C(t) = \{\phi(t) = 0\}$ by:

$$\frac{\partial\phi}{\partial t} = |\nabla\phi|\nabla\left(\frac{\nabla\phi}{|\nabla\phi|}\right), \quad \phi(0) = \phi_0$$

Allows for automatic topology changes, cusps, on a fixed rectangular grid, ...

Variational formulation and level sets

Following Zhao-Chan-Merriman-Osher '96

$C = \{(x,y) | \ \phi(x,y) = 0\}$

The Heaviside and Delta functions

$$H(z) = \begin{cases} 1, & \text{if } z \geq 0 \\ 0, & \text{if } z < 0 \end{cases}, \quad \delta(z) = H'(z)$$

- $\text{Length}(C) = \int_\Omega |\nabla H(\phi)| dxdy = \int_\Omega \delta(\phi) |\nabla \phi| dxdy$

The level set formulation of the active contour model

$$\inf_{c_1, c_2, \phi} F(c_1, c_2, \phi)$$

$$F(c_1, c_2, \phi) = \nu \int_\Omega \delta(\phi) |\nabla \phi| dxdy$$

$$+\mu^2 \int_\Omega |u_0 - c_1|^2 H(\phi) dxdy$$

$$+\mu^2 \int_\Omega |u_0 - c_2|^2 (1 - H(\phi)) dxdy$$

- $u = c_1 H(\phi) + c_2 (1 - H(\phi))$

The Euler-Lagrange equations

$$\inf_{c_1, c_2, \phi} F(c_1, c_2, \phi)$$

$$\Leftrightarrow$$

Equations for c_1 and c_2

$$c_1(\phi) = \frac{\int_\Omega u_0 \cdot H(\phi) dx dy}{\int_\Omega H(\phi) dx dy}$$

$$c_2(\phi) = \frac{\int_\Omega u_0 (1 - H(\phi)) dx dy}{\int_\Omega (1 - H(\phi)) dx dy}$$

Equation for ϕ

$$\phi_t = \delta(\phi)[\nu \nabla (\frac{\nabla \phi}{|\nabla \phi|}) - \mu^2 (u_0 - c_1)^2 + \mu^2 (u_0 - c_2)^2]$$

$$\phi(0, x, y) = \phi_0(x, y)$$

A level set algorithm

for the Mumford-Shah segmentation model

$\Omega \subset I\!R^2$, $u_0 : \Omega \to I\!R$ a given image
$\phi : \overline{\Omega} \to I\!R$ a level set function
$C = \{\phi = 0\}$
ϕ has opposite signs on each side of C
Example of ϕ: signed distance function to C

Two functions: u^+, u^-, s.t.
$u(x,y) = u^+(x,y)$ if $\phi(x,y) \geq 0$,
$u(x,y) = u^-(x,y)$ if $\phi(x,y) < 0$.

The M-S functional in level set formulation:

$$F(u^+, u^-, \phi) = \mu^2 \Big(\int_{\phi \geq 0} |u^+ - u_0|^2 dxdy$$

$$+ \int_{\phi < 0} |u^- - u_0|^2 dxdy \Big)$$

$$+ \Big(\int_{\phi > 0} |\nabla u^+|^2 dxdy + \int_{\phi < 0} |\nabla u^-|^2 dxdy \Big)$$

$$+ \nu \cdot \text{Length}\{\phi = 0\}.$$

Using H and δ
$u = u^+ H(\phi) + u^-(1 - H(\phi))$, for all (x, y),

we obtain the following minimization problem:

$$\inf_{u^+, u^-, \phi} F(u^+, u^-, \phi),$$

where

$$F(u^+, u^-, \phi) =$$

$$\mu^2 \left(\int_\Omega |u^+ - u_0|^2 H(\phi) dxdy + \int_\Omega |u^- - u_0|^2 (1 - H(\phi)) dxdy \right)$$

$$+ \left(\int_\Omega |\nabla u^+|^2 (1 - H(-\phi)) dxdy + \int_\Omega |\nabla u^-|^2 (1 - H(\phi)) dxdy \right)$$

$$+ \nu \int_\Omega \delta(\phi) |\nabla \phi|.$$

The Euler-Lagrange equations

Fix $\phi = 0$
Consider u^+ and u^- as C^1 functions, with continuous derivatives up to the boundary
$\{\phi = 0\}$).

Min $F(u^+, u^-, \phi)$ with respect to u^+, u^-:

$$\mu^2(u^+ - u_0) = \triangle u^+ \text{ on } \phi > 0, \; \frac{\partial u^+}{\partial \vec{n}} = 0 \text{ on } \{\phi = 0\},$$

and

$$\mu^2(u^- - u_0) = \triangle u^- \text{ on } \phi < 0, \; \frac{\partial u^-}{\partial \vec{n}} = 0 \text{ on } \{\phi = 0\}.$$

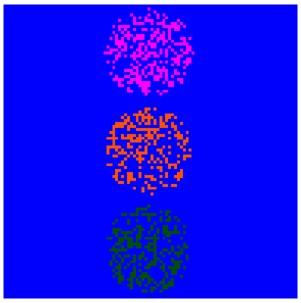
Fix u^+ and u^-.
Min $F(u^+, u^-, \phi)$ with respect to ϕ:

$$\phi_t = \delta(\phi)(\nu \nabla(\frac{\nabla \phi}{|\nabla \phi|}) - \mu^2 |u^+ - u_0|^2$$

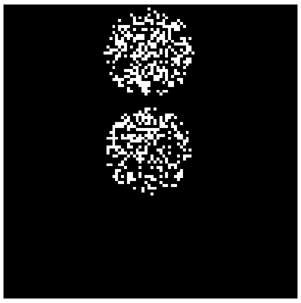
$$-|\nabla u^+|^2 + \mu^2 |u^- - u_0|^2 + |\nabla u^-|^2) = 0.$$

Active Contours without Edges

for Color Images

Color 

Intensity 

1 

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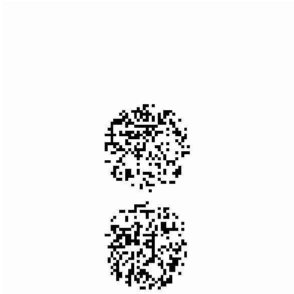
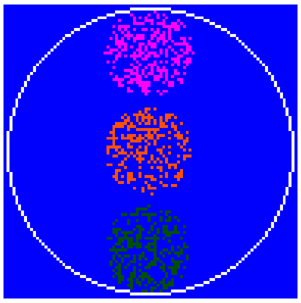
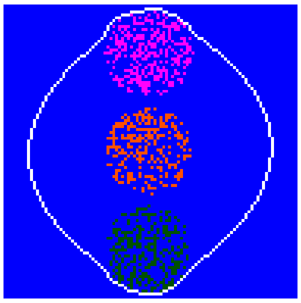
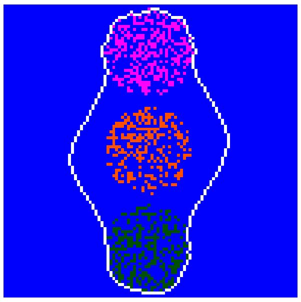
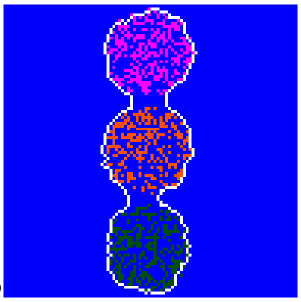
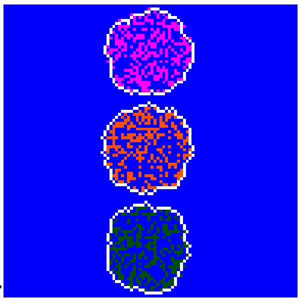
3 

B. Sandberg 0 

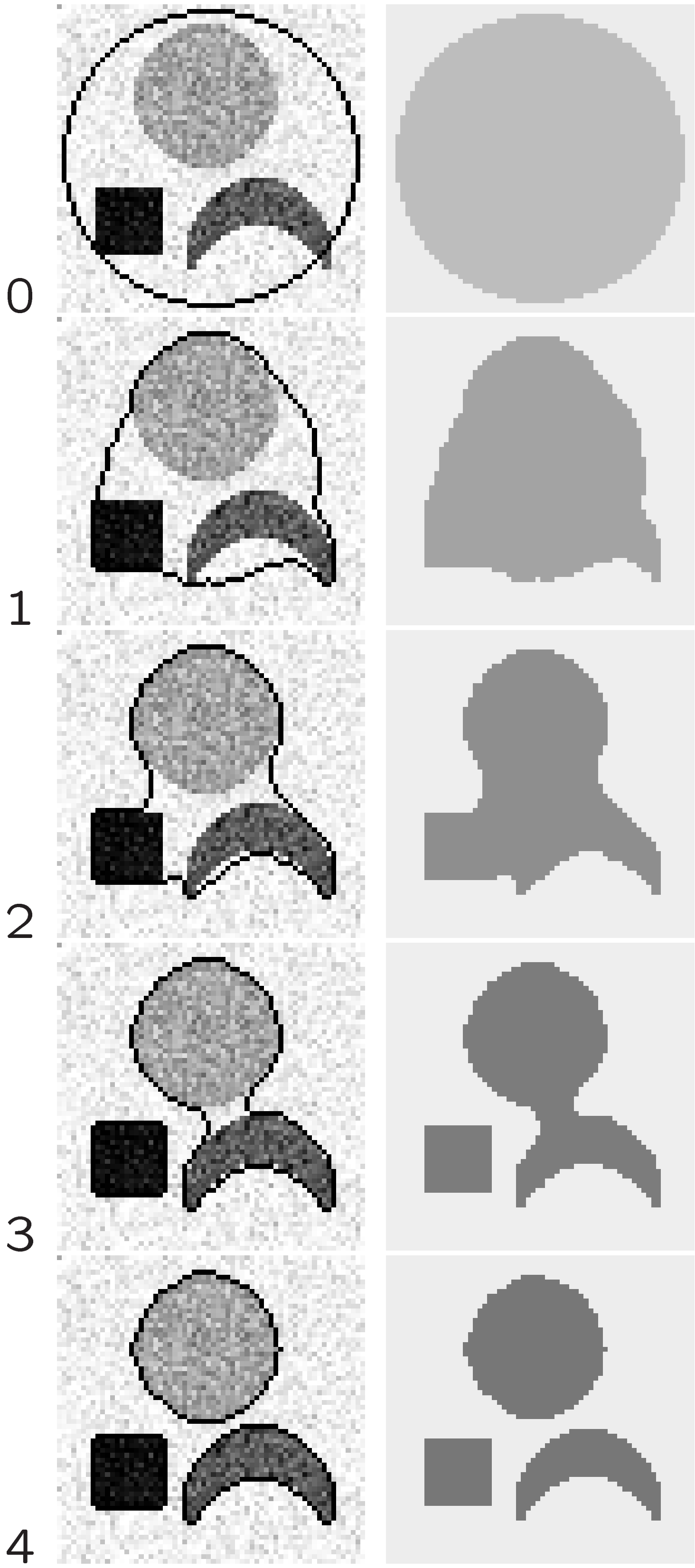
1 

2 

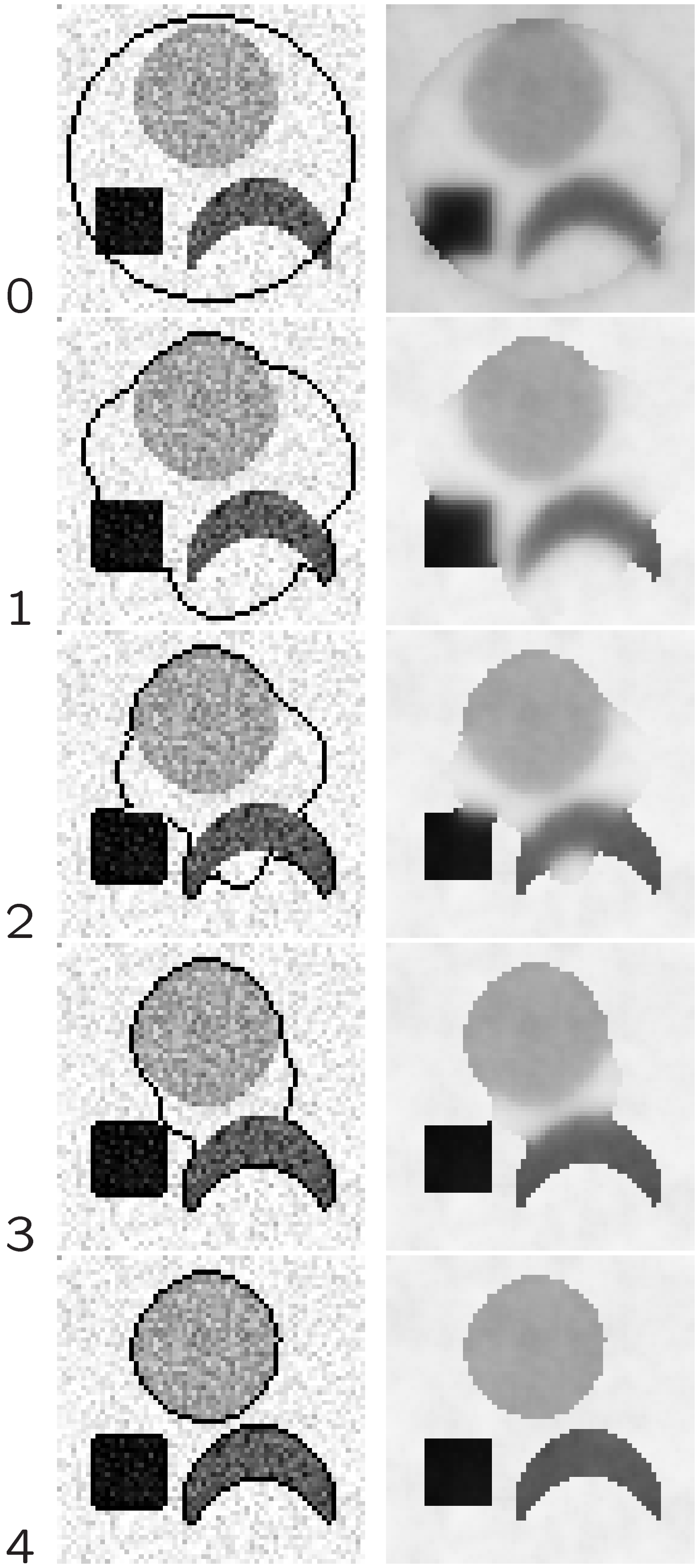
3 

4 

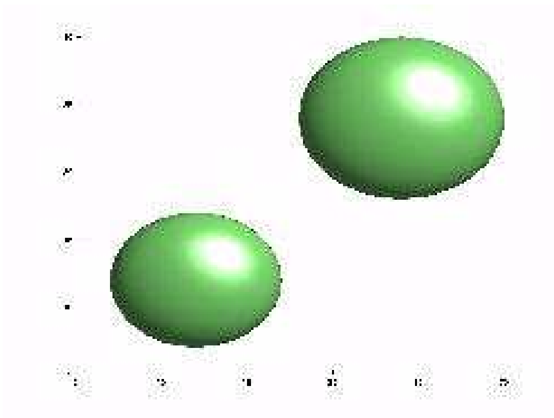
Active Contours c^{+}, c^{-}, ϕ

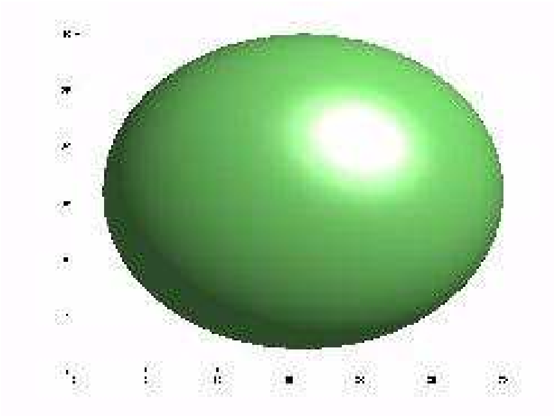


Mumford-Shah by Level Sets u^+, u^-, ϕ



Active Contours for Volumetric Images





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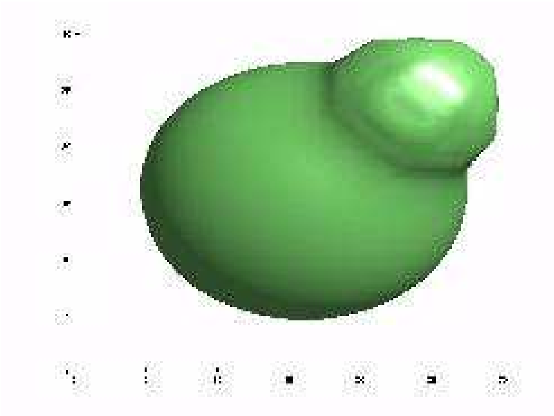

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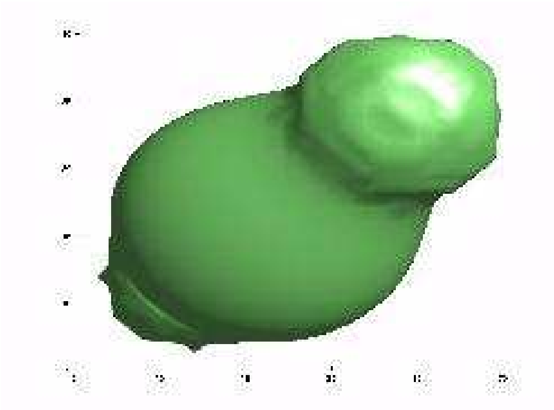

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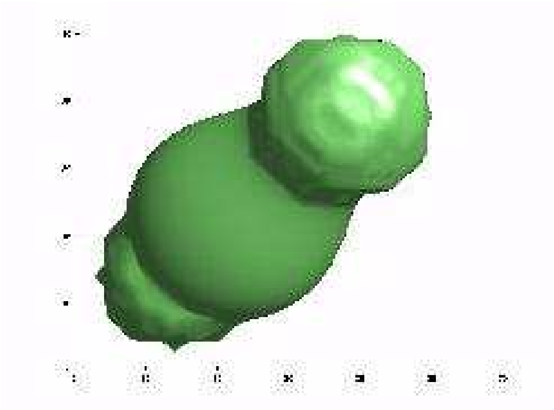

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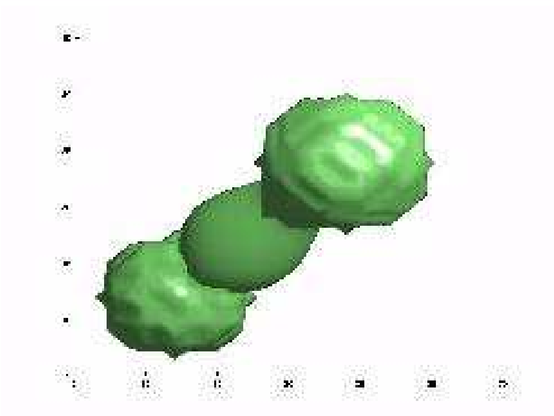

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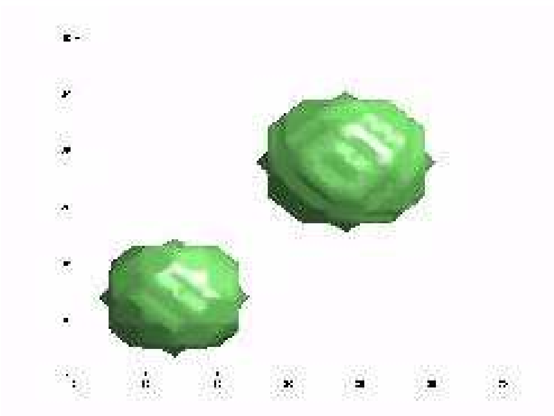

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