

TRIGONOMETRY PROBLEMS

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These problems are exercises in trigonometry. Incidentally I'm making the point that though trigonometry is high school material, a trigonometry problem can be interesting and challenging. In fact trigonometry apparently is often taught as a completely mechanical series of routines, without engaging students in any nonroutine mathematical thinking. On the whole this group of problems is fairly straightforward and manipulative compared with Putnam practice problems as a group. No strokes of genius are required. Some of the first few problems may be completely familiar to you. But if not, they could be well worthwhile. Later on the problems get livelier. Some are geometry problems rather than trigonometry problems. Incidentally, some of these are not presented in the usual style of a competition problem. Competition problems usually aren't divided into parts, and hardly ever have anything like a hint. Also they never ask you to prove for yourself some more or less well known result, as many of these questions do. But that's OK because these problems are intended as training problems.

- (1) Give a geometric proof for the formulas for the sine and cosine of the sum of two angles.
- (2) Derive formulas for $\sin 2A$, $\cos 2A$, $\tan 2A$, $\sin 3A$, $\cos 3A$, $\tan 3A$, $\cot 3A$. Find formulas expressing $\cos nA$ as a polynomial $T_n(\cos A)$. First try $n = 2, 3, 4, 5$. T_n is called the n th Chebyshev polynomial. See if you can find a general expression for the coefficients in the n th Chebyshev polynomial.
- (3) Derive half angle formulas for \sin , \cos , \tan .
- (4) Without looking it up, find geometric proofs for the cosine law and the law of sines. (There's no point in doing this if you remember the proofs offhand. But not everybody does.)
- (5) Show that $\sin a + \sin 3a + \sin 5a + \dots + \sin(2n-1)a = \sin^2 na / \sin a$
- (6) Show that $\cos a + \cos 3a + \cos 5a + \dots + \cos(2n-1)a = \sin 2na / 2 \sin a$
- (7) $\sin \pi/n + \sin 2\pi/n + \dots + \sin n\pi/n = \cot \pi/2n$. Can you draw a picture to illustrate this ?
- (8) $\cos \pi/n + \cos 2\pi/n + \dots + \cos n\pi/n = -1$.
- (9) Show that $\sin 15 \text{ deg} = (\sqrt{6} - \sqrt{2})/4$, $\sin 75 \text{ deg} = (\sqrt{6} + \sqrt{2})/4$, $\sin 18 \text{ deg} = (-1 + \sqrt{5})/4$, $\sin 36 \text{ deg} = \sqrt{10 - 2\sqrt{5}}/4$, $\sin 54 \text{ deg} = (1 + \sqrt{5})/4$, $\sin 72 \text{ deg} = \sqrt{10 + 2\sqrt{5}}/4$.
- (10) Show that the three angle bisectors of a triangle meet in a point.
- (11) Show that the three perpendicular bisectors of a triangle meet in a point. (Each edge has a midpoint, and the line perpendicular to the edge through the midpoint is called the perpendicular bisector.)
- (12) A median of a triangle is a line from one vertex to the midpoint of the other two vertices. Show that the three medians of a triangle meet in a point.
- (13) Show that the three altitudes of a triangle meet in a point.

- (14) Consider a triangle with side lengths a, b, c and angles A, B, C . Show that

$$\sin A + \sin B + \sin C = 4 \cos A/2 \cos B/2 \cos C/2$$

$$\cos A + \cos B + \cos C = 4 \sin A/2 \sin B/2 \sin C/2$$

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C$$

$$\tan(B/2) \tan(C/2) + \tan(C/2) \tan(A/2) + \tan(A/2) \tan(B/2) = 1$$

- (15) It would be a good idea to do at least the last part of the previous problem before trying this. Let r be the radius of the inscribed circle of the triangle. (Do you know how to show that there is an inscribed circle, and how to construct its center?) Show that

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

where $s = (a + b + c)/2$. Then show that the area of the triangle is given by Heron's formula: $A = \sqrt{s(s-a)(s-b)(s-c)}$.

- (16) Given a triangle, there are 4 circles tangent to its three sides, if the sides are extended to infinite lines. One is the inscribed circle. The other three are the escribed circles. Each is tangent to one of the edges of the triangle and to two of the extended sides. Give a geometric construction for the escribed circles. Can you find a formula for the radius of an escribed circle?
- (17) In three dimensional Euclidean space, consider a parallelepiped with vertex O , the origin, and three edges OA, OB, OC . The Gram determinant

$$\begin{vmatrix} OA \cdot OA & OA \cdot OB & OA \cdot OC \\ OB \cdot OA & OB \cdot OB & OB \cdot OC \\ OC \cdot OA & OC \cdot OB & OC \cdot OC \end{vmatrix}$$

is the square of the volume of the determinant. (Why?) You can use the Gram determinant to obtain an alternative proof of Heron's formula: Given a triangle ABC in R^2 , place a copy of ABC in the plane $z = 1$ in R^3 . What is the relation between the area of ABC and the volume of the tetrahedron $OABC$?