SUMMATION PROBLEMS: FALL 2006 PUTNAM TRAINING

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(1) Find:

$$\sum_{n=1}^{N} \frac{1}{n(n+1)} = ?$$

(2) Find:

$$\sum_{n=1}^{N} \frac{1}{n(n+1)(n+2)} = ?$$

(3) Find:

$$\sum_{n=1} \infty \frac{a}{n(n+1)(n+2)(n+3)} = ?$$

(4) Show:

$$\sum_{k=1}^{N} k^2 = N(N+!)(2N+1)/3$$

(5) Show:

$$\sum_{k=1}^{N} k^3 = (\frac{N(N+1)}{2})^2$$

(6) Suppose $p(x)$ is a degree m polynomial. Then there is a degree $m+1$ polynomial $q(x)$ such that

$$\sum_{k=1}^{n} p(k) = q(n) \quad \text{forall n} \in \mathbf{N}$$

(7)

$$\sum_{k=1}^{n} \frac{k}{k^4+k^2+1} = ?$$

(8)

$$\prod_{n=2}^{\infty} \frac{n^3-1}{n^3+1} = ?$$

(9) In 1593 Viete proved that:

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty} \cos(\frac{x}{2^n})$$

Now prove it yourself. Actually, Archimedes knew something logically equivalent to Viete's formula, but he didn't use anything like modern notation. He might have objected to the way it's written here on the grounds that the statement is actually fairly complicated, and only looks simple because the notation (for instance the infinite product sign) hides the true complexity of the statement. Like other Greek mathematicians he used a lot of ordinary long winded prose, so when a statement about limits comes up, he makes it explicit exactly what is meant by a limit. Nowadays we use symbols, for instance lim and $\int$, that hide the fairly complicated definitions of limit and integral. (And many a student takes AP calculus, and uses the notations lim and $\int$ for a whole year, without ever having a clear idea what they mean.) Using his knowledge of a result equivalent to Viete's, Archimedes showed that $3\frac{10}{71} < \pi < 3\frac{10}{70}$, but he could have found upper and lower fractions approximating π to any desired degree of accuracy.

(10)
$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\frac{\sqrt{2+\sqrt{2}}}{2}\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots$$
(A fun looking formula!)

(11)
$$\frac{3}{\pi} = \frac{\sqrt{2+\sqrt{3}}}{2}\frac{\sqrt{2+\sqrt{2+\sqrt{3}}}}{2}\frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}{2}\cdots$$

(12)
$$\sum_{n=1}^{\infty} \cot^{-1}(n^2+n+1) = ?$$

(13)
$$1/\theta - \cot\theta = \sum_{k=1}^{\infty} 2^{-k}\tan\frac{\theta}{2^k}$$
In particular, you could choose $\theta = \pi/2$. You can get a formula for $2/\pi$ involving nested radicals. It is not particularly attractive, important or useful.

(14)
$$\sum_{k=0}^{n} \cos ka = \frac{\sin(n+1)a/2}{\sin a/2}\cos na/2$$

(15)
$$1/2 + \sum_{k=1}^{n} \cos kt = \frac{\sin(n+1/2)t}{2\sin t/2}$$
(unless $t = 2p\pi$ for some $p \in \mathbf{Z}$. In that case the sum equals $n+1/2$.) Note: Many of these summation problems are not particularly important in the general scheme of things. But this one is. It gives the formula for the "Dirichlet kernel" $D_n(t)$ which is crucial in the theory of Fourier series.

(16)
$$\sum_{k=1}^{n} \sin ka = \frac{\sin(n+1)a/2}{\sin a/2}\sin na/2$$

(17) Note: this is rather unlike most of the other problems in this group. It is not meant to be proved by induction on n.
$$\prod_{k=1}^{n-1} \sin k\pi/n = n2^{1-n}$$

(18)
$$\sum_{k=1}^{n} \cos(2k-1)a = \frac{1}{2}\frac{\sin 2na}{\sin a}$$
(19)
$$\sum_{k=1}^{n} \sin(2k-1)a = \frac{\sin^2 na}{\sin a}$$
(20)
$$\sum_{n=0}^{\infty} \frac{x^{2^n}}{1-x^{2^{n+1}}} = ?$$
(For what values of x does this sum converge? You should find the sum of the series for any x for which the series converges.

(21) WLP 2001 B3.

(22) WLP 1985 B2.

(23) For any k we can define
$$F_m(t) = \frac{1}{m+1}\sum_{n=0}^{m} D_n(t)$$
where the "Dirichlet" kernel was defined above. Show that
$$F_m(t) = \frac{1}{2(m+1)}\left[\frac{\sin(n+1)t/2}{\sin t/2}\right]^2$$
except for $t = 2p\pi$, $p \in \mathbf{Z}$. In that case $F_m(t) = \frac{m+1}{2}$. $F_m(t)$is the mth Fejer kernel and is also very important in Fourier series.