Recurrence relations
Luminita Vese
November 14, 2006

Definition: A recurrence relation for a sequence $\{a_n\}$ is a formula which expresses the nth term of the sequence in terms of one or more of the previous terms of the sequence.

• Typically the recurrence relation is known and one wants to determine explicitly the sequences which satisfy the given recurrence relation. Sometimes one wants to find its limit, convergence properties, etc.

• Many natural functions are easily expressed as recurrences:
$a_n = a_{n-1} + 1$, $a_1 = 1 \Rightarrow a_n = n$ (polynomial)
$a_n = 2a_{n-1}$, $a_0 = 1 \Rightarrow a_n = 2^n$ (exponential)
$a_n = na_{n-1}$, $a_1 = 1 \Rightarrow a_n = n!$ (factorial)

• It is often easy to find a recurrence as the solution of a counting problem.

• Mathematical induction provides a useful tool to solve recurrences: guess a solution and prove it by induction (as we will see).

Example: Find the sequence satisfying $u_n = 2u_{n-1} + 1$, $n \geq 1$, $u_0 = 0$.
Solution:
- To guess the solution: compute u_n for small values of n for insight. Thus $u_0 = 0$, $u_1 = 1$, $u_2 = 3$, $u_3 = 7$, $u_4 = 15$, $u_5 = 31$, $u_6 = 63$, $u_7 = 127$, ...
- Prove $u_n = 2^n - 1$ by induction.

• Linear recurrences, with finite history, constant coefficients can always be solved mechanically.

Examples:
- Fibonacci sequence $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$.

To find the characteristic polynomial, we rewrite $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$, or $F_{n+2} - F_{n+1} - F_n = 0$. The characteristic equation is $x^2 - x - 1 = 0$, with roots $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Then there are constants A and B such that $F_n = A\alpha^n + B\beta^n$, and A, B are found using F_0 and F_1.

In general, if a root α has multiplicity m, then its term will be $P(n)\alpha^n$, where $P(n) = A_0 + nA_1 + ...n^{m-1}A_{m-1}$ is a polynomial in n of degree $m - 1$, instead of the constant A.

- Suppose $u_0 = 0$, $u_1 = 1$, and $u_n = u_{n-1} + 2u_{n-2}$, $n \geq 2$. What is the characteristic equation for this recursion ? What are its roots ? Show that $u_n = \frac{1}{3}\Big(2^n - (-1)^n\Big)$ for all $n \geq 0$.

- Suppose $u_0 = 0$, $u_1 = 1$, $u_2 = 2$, and $u_n = 3u_{n-2} + 2u_{n-3}$ for $n \geq 3$. Show that $u_n = \frac{1}{9}\Big((3n - 4)(-1)^n + 2^{n+2}\Big)$ for all $n \geq 0$.

• Plug the recurrence back into itself until you see a pattern.

Problems: (from easy to more difficult)

1. (easy) Let Z be the set of integers. Prove that if $f : Z \to Z$ satisfies $f(f(n)) = f(f(n+2)+2) = n$ for all n, and $f(0) = 1$, then $f(n) = 1 - n$. Hint: use induction.

2. (easy but not a nice solution) Prove that the sequence $a_0 = 2$, 3, 6, 14, 40, 152, 784, ... with general term $a_n = (n+4)a_{n-1} - 4na_{n-2} + (4n-8)a_{n-3}$ is the sum of two well-known sequences.

Hint: Just try subtracting off various simple sequences until you recognize the result. Then prove it by induction.

3. An event is a hit or a miss. The first event is a hit, the second is a miss. Thereafter the probability of a hit equals the proportion of hits in the previous trials (so, for example, the probability of a hit in the third trial is 1/2). What is the probability of exactly 50 hits in the first 100 trials ?

4. You have a set of n biased coins. The mth coin has probability $\frac{1}{2m+1}$ of landing heads ($m = 1, 2, ..., n$) and the results for each coin are independent. What is the probability that if each coin is tossed once, you get an odd number of heads?

5. The polynomials $p_n(x)$ are defined as follows: $p_0(x) = 1$; $p'_{n+1}(x) = (n+1)p_n(x+1)$, $p_{n+1}(0) = 0$ for $n \geq 0$. Factorize $p_{100}(1)$ into distinct primes.

6. Let S be the set of points (x, y) in the plane such that the sequence a_n defined by $a_0 = x$, $a_{n+1} = (a_n^2 + y^2)/2$ converges. What is the area of S ?

7. x is a real. Define $a_{i,0} = x/2^i$, $a_{i,j+1} = a_{i,j}^2 + 2a_{i,j}$. What is $\lim_{n\to\infty} a_{n,n}$?

8. (moderately hard) Let $r(n)$ be the number of $n \times n$ matrices $A = (a_{ij})$ such that:
(1) each $a_{ij} = -1, 0$, or 1; and
(2) if we take any n elements a_{ij}, no two in the same row or column, then their sum is the same.

Find rational numbers a, b, c, d, u, v, w such that $r(n) = au^n + bv^n + cw^n + d$.

9. Define the sequences x_i and y_i as follows. Let $(x_1, y_1) = (0.8, 0.6)$ and let $(x_{n+1}, y_{n+1}) = (x_n \cos y_n - y_n \sin y_n, x_n \sin y_n + y_n \cos y_n)$ for $n \geq 1$. Find $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$.

10. For any real a define $f_a(x) = [ax]$. Let n be a positive integer. Show that there exists an a such that for $1 \leq k \leq n$, $f_a^k(n^2) = n^2 - k = f_{a^k}(n^2)$, where f_a^k denotes the $k-$fold composition of f_a.

11. Let N be a fixed positive integer. For real α, define the sequence x_k by: $x_0 = 0$, $x_1 = 1$, $x_{k+2} = (\alpha x_{k+1} - (N-k)x_k)/(k+1)$. Find the largest α such that $x_{N+1} = 0$ and the resulting x_k.

12. $u_1 = 1$, $u_2 = 2$, $u_3 = 24$, $u_n = (6u_{n-1}^2 u_{n-3} - 8u_{n-1}u_{n-2}^2)/(u_{n-2}u_{n-3})$. Show that u_n is always a multiple of n.

Other problems (not difficult)

Show that u_n converges and find its limit:

$u_n = \log(1 + u_{n-1})$, $u_0 > 0$. (Hint: use an inequality)

$u_{n+1} = u_n + u_n^2$, $-1 < u_0 < 0$.

$u_n = \sqrt{u_{n-1}u_{n-2}}$, with $u_0 = 1$, $u_1 = 2$. (Hint: transform it into a linear recurrence with constant coefficients).