PROBLEM SET: GEOMETRY

Problem 1. Let $ABCD$ be a rectangle inscribed in a circle of radius 1. Let the length of the side BC be equal to b, and the length of the side AB be equal to h. Let BTC be the isosceles triangle ($|BT| = |TC|$) which is also inscribed in this circle, and such that BC is exactly the intersection of the rectangle and the triangle. Find the value of h such that the rectangle and the triangle have the same area.

Problem 2. Prove that any convex pentagon whose vertices (no three of which are collinear) have integer coordinates must have area $\geq 5/2$.

(*Hint:* Use the following

Pick's theorem. Given an integer lattice polygon with i internal integer lattice points and b boundary points, the area of the polygon is $S = i + \frac{b}{2} - 1$.

Here a lattice polygon is a polygon whose vertices are points with integer coordinates on the plane,)

Problem 3. A 2×3 rectangle has vertices at $(0,0)$,$(2,0)$, $(0,3)$ and $(2,3)$. It rotates 90 degrees clockwise about the point $(2,0)$. It then rotates 90 degrees clockwise about the point $(5,0)$, then 90 degrees clockwise about the point $(7,0)$ and, finally, 90 degrees clockwise about the point $(10,0)$. (The side originally on the x-axis is now back on the x-axis.) Find the area of the region above the x-axis and below the curve traced out by the point whose initial position is $(1,1)$.

Problem 4. Let C_1 and C_2 be circles whose centers are 10 units apart, and whose radii are 1 and 3. Find, with proof, the locus of all points M for which there exists points X on C_1 and Y on C_2 such that M is the midpoint of the line segment XY.

Problem 5. A rectangle, $HOMF$, has sides $HO = 11$ and $OM = 5$. A triangle ABC has H as the intersection of the altitutes, O the center of the circumscribed circle, M the midpoint of BC, and F the foot of the altitute from A. What is the length of BC?

(*Hint:* A very straightforward method to solve this problem is by introducing the system of coordinates on the plane

Another way to solve this problem is to use the following property of a triangle: *In an arbitrary triangle the following three "centers":*

- *the point G of intersection of medians;*
- *the point H of intersections of altitudes;*
- *the center O of the circumscribed circle;*

always lie on the same line, called the Euler line. Moreover, $|GH| = 2|OG|$. You can try to prove this statements as an excercise in elementary geometry).

Problem 6. A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

Problem 7. Let H be the unit hemisphere $\{(x,y,z) \ : \ x^2+y^2+z^2=1 \ , \ z \geq 0\}$, C be the unit circle $\{(x,y,0) \ : \ x^2+y^2=1\}$, and P the regular pentagon inscribed in C. Determine the surface area of that portion of H lying over the planar region inside P, and write your answer in the form $A\sin\alpha + B\sin\beta$, where A, B, α, β are real numbers.