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Problèmes variationnels et EDP pour l'analyse d'images et l'évolution de courbes

par

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Résumé

Ce travail, composé de trois parties, est consacré à l'étude d'un problème variationnel issu de l'analyse d'images et à quelques équations aux dérivées partielles elliptiques dégénérées du second ordre, liées à la reconstruction d'images bruitées et floues et à l'évolution de courbes.

Dans la **Première Partie**, nous étudions un problème de débruitage et de déconvolution par des méthodes variationnelles. En analyse d'images, un problème important est la reconstruction d'une image originale u, à partir d'une image initiale observée u_0. Ici, nous considérons que le modèle de dégradation reliant u à u_0 est linéaire, de la forme suivante:

$$u_0 = Ku + \eta,$$

où K est un opérateur de dégradation (en général une convolution) et η représente le bruit aléatoire.

Le modèle de reconstruction que nous étudions ici est posé comme un problème variationnel. On cherche u parmi les minima de l'énergie

$$F_\alpha(u) = \frac{1}{2}\int_\Omega |Ku - u_0|^2 dxdy + \alpha \int_\Omega \phi(|Du|)dxdy, \tag{0.0.1}$$

où $\Omega \subset \mathbb{R}^2$ est un ouvert borné (un rectangle en général) et les images u_0 et u sont des fonctions de Ω dans $\mathbb{R}$ qui associent à chaque pixel $(x, y) \in \Omega$ son niveau de gris (ou l'intensité lumineuse) $u_0(x, y)$ et $u(x, y)$. $K : L^2(\Omega) \to L^2(\Omega)$ est un opérateur linéaire et continu (en général, K est l'identité I où une convolution). Le premier terme dans l'énergie F_α est un terme d'attache aux données et le second est un terme de régularisation (le paramètre $\alpha > 0$ permet de régler l'influence de chaque terme dans l'énergie). Le choix de la fonction $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ est guidé par le principe suivant, classique en analyse d'images:

(P) *L'image reconstruite u doit être formée par des zones homogènes et séparées par des bords francs.*

Donc le modèle doit diffuser la solution dans les régions à faible variation de niveaux de gris mais dans le même temps doit préserver les contours de ces régions, c'est-à-dire respecter les fortes discontinuités.

En associant au problème de minimisation l'équation d'Euler, qui s'écrit formellement:

$$-\alpha \mathrm{div}\Big(\frac{\phi'(|Du|)}{|Du|}Du\Big) + K^*Ku = K^*u_0, \tag{0.0.2}$$

ou son équivalent sous forme non conservative, nous allons déterminer les hypothèses nécessaires sur ϕ, pour satisfaire le principe (P). ϕ doit être "quadratique" au voisinage de l'origine, avec une croissance au plus linéaire à l'infini. De plus, pour que le problème soit mathématiquement bien-posé, ϕ doit être convexe et croissante à l'infini. Par exemple, le choix de la fonction des hypersurfaces minimales $\phi(z) = \sqrt{1 + z^2}$ convient. Les critères de choix de la fonction ϕ et les exemples que nous utiliserons dans ce travail sont présentés dans le premier chapitre de la partie I.

Nous avons fait l'étude théorique du problème de minimisation associé à F_α. La difficulté est due au fait que ϕ est linéaire à l'infini et qu'on doit travailler sur l'espace non réflexif $L^1(\Omega)$ ou sur un espace construit à partir de L^1: l'espace des fonctions à variations bornées $BV(\Omega)$, bien adapté pour modéliser la reconstruction d'images. Dans le deuxième chapitre, nous présentons donc l'existence et l'unicité de la solution du problème de minimisation, sur l'espace $BV(\Omega)$, en utilisant la notion de fonction convexe de mesures et des résultats de relaxation. Nous donnons une caractérisation du sous-différentiel de la fonctionnelle, ainsi que des conditions d'optimalité de la solution. A la fin de ce chapitre, nous étudions dans le même cadre le problème d'évolution associé à la minimisation de la fonctionnelle.

Dans le troisième chapitre, on étudie deux méthodes d'approximation pour le problème continu (0.0.1). Pour la première approximation, on introduit une suite de fonctionnelles plus régulières $F_{\alpha,\varepsilon}$, ayant des minima u^ε plus réguliers et, en utilisant la notion de $\Gamma-$convergence, on montre la convergence de u^ε vers u, la solution du problème sur $BV(\Omega)$. Pour la deuxième méthode, quand la solution est supposée plus régulière (par exemple dans $H^1(\Omega)$), on associe au problème des variables dites duales et on démontre un résultat de régularité, ainsi que la convergence d'un algorithme, qui permet d'approximer l'équation d'Euler qui est non linéaire, par une suite de problèmes linéaires.

En fin, dans le chapitre 4, nous présentons deux méthodes de discrétisation du problème variationnel et de l'équation d'Euler associée, par des différences finies. La première est une discrétisation classique de l'équation sous forme conservative, tandis que dans la seconde méthode, adaptée à des images constantes par morceaux, nous approximons directement la fonctionnelle. Nous présentons des résultats numériques en dimension 1, pour comparer les deux algorithmes et les différents potentiels choisis.

La **Deuxieme Partie** de cette thèse est consacrée à l'étude de quelques équations non linéaires, elliptiques ou paraboliques dégénérées de second ordre, dans le cadre de la théorie des solutions de viscosité, avec des applications en analyse d'images et à la convexification de fonctions via l'évolution de courbes.

Le chapitre 5 est une introduction de la théorie des solutions de viscosité. Nous présentons les définitions et quelques résultats généraux, utilisés dans la suite.

Dans le chapitre 6, nous considérons quelques équations quasilinéaires elliptiques, et nous donnons comme exemple l'équation d'Euler de la première partie, écrite sous forme développée. A la fin de ce chapitre, nous présentons une discrétisation de l'équation, dans le cadre général d'une équation quasilinéaire, basée sur un schéma aux différences finies déjà utilisé pour l'opérateur de courbure.

Le chapitre 7 est consacré à une méthode de convexification de fonctions, par des techniques de la "mean curvature motion". Etant donnée une fonction initiale u_0 à convexifier, on cherche en suite $u(t,\cdot)$ (avec $u(0)=u_0$) comme solution de l'équation

$$\frac{\partial u}{\partial t} = \sqrt{1+|D_x u|^2}\, g(\lambda_{min}(D_x^2 u)),$$

où la fonction $g : I\!R \to I\!R$ est $g(z)=0$ si $z \geq 0$ et $g(z)=z$ si $z<0$ et où $\lambda_{min}(D_x^2 u)$ est la plus petite valeur propre de la matrice hessienne de u. Cette équation produit un déplacement du graphe de la fonction initiale, dans la direction de la normale, avec une vitesse dépendant de la plus petite valeur propre de D^2u.

Quand $t \to \infty$, $u(t, \cdot)$ converge vers l'enveloppe convexe de u_0. Nous présentons une discrétisation par des différences finies du problème et des résultats numériques en une et deux dimensions.

Finalement, dans la **Troisième Partie** nous présenterons des résulats numériques pour la reconstruction d'images, en utilisant les deux algorithmes du Chapitre 4 et celui du Chapitre 6. Nous faisons une étude comparative des résultats.

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Part I

Image reconstruction by variational methods

This part of the thesis is devoted to the study by variational methods of a denoising-deblurring problem arising from image reconstruction, in the case of an additive noise. The problem is, given an observed and degraded image u_0, to find an original or reconstructed image u, which must be "closed" to u_0, but more "regular".

The reconstruction model is posed as a minimization problem, in which the energy that we minimize depends on the solution u and on its gradient Du, by a regularizing term of the form $\int \phi(|Du|)$.

In Chapter 1, we precise the degradation model, together with the reconstruction model, which will be study in this part, and we propose some necessary assumptions on the potential ϕ, in order to apply this model to denoise and deblur signals and images. For instance, ϕ may be a function with at most a linear growth at infinity.

Chapter 2 is devoted to the theoretical study of the problem, which is not always mathematically well-posed. Of course that, in order to ensure the existence and the uniqueness of a solution u, we need to suppose that the potential ϕ is convex, but the difficulty becomes from the fact that we must work on the space L^1, or BV, the space of functions with bounded variation, which is adapted to model images. In this context, we give in the general case an existence and uniqueness result in BV, and we show optimality conditions on the solution, by calculating the subdifferential of the functional. We also study the evolution problem associated to the Euler equation.

In Chapter 3, we present some algorithms to approach the solution of the continuous problem, by using the $\Gamma - convergence$ and the so-called dual variables, and we add a regularity result. We show how we can associate to the image u an auxiliary variable (or dual), and how the regularizing term in the energy can be represented by the infimum of quadratic functions.

Finally, Chapter 4 is devoted to the numerical study of the minimization problem, of the associated Euler equation and evolution problem, in order to apply all these results to denoise and deblur signals and images. In the end of this chapter, we present several experimental results in dimension one, and we give our opinion and conclusion about the choice of the potential ϕ and of the algorithm, in order to obtain good results for different cases. The two-dimensional reconstruction will be treated in Part III.

Chapter 1

Introduction

It is well known that during the formation, transmission or recording process, images are degraded by noise and blur. Then, an important problem in image analysis is the reconstruction of an original image u, describing a real scene, from an observed image u_0. The transformation (or degradation) connecting u to u_0 is in general the result of two phenomena. The first phenomenon is deterministic and is related to the mode of image acquisition (for example, the computation of integral projections in tomography), to possible defects of the imaging system (blur created by a wrong lens adjustment, by a moving,...) or other phenomena, like atmospheric turbulence. The second phenomenon is random: the noise, inherent degradation in any signal transmission. In this thesis, we deal only with additive noise. Then, the relation connecting u to u_0, considering the linear degradation model, which takes into account for both blur and noise, can be written as:

$$u_0 = Ku + \eta, \tag{1.0.1}$$

where K is a linear operator, representing the blur and η is a random noise.

The reconstruction problem of u can be identified, in that way, with an inverse problem: find u, from (1.0.1). In general, this problem is ill-posed in the sense of Hadamard: the information provided by u_0 and the model (1.0.1) is not sufficient to ensure the existence, uniqueness and stability of a solution u.

It is therefore necessary to regularize the problem by adding an a-priori constraint on the solution. The most classical and frequent approach in image reconstruction is a stochastic approach, based, in the framework of Bayesian estimation, on the use of the Maximum A Posteriori (MAP) estimation. Supposing that u is a Markov field (which constitutes an a-priori constraint), then the MAP criterion identifies with a minimization problem in which the energy will depend on the image u and on the gradient. We are not going into the details of this approach, we refer the reader to the original article of S. Geman and D. Geman [32], or for a clear and synthetic presentation, to a work of P. Charbonnier [16].

Our purpose being to study the problem of image reconstruction via the calculus of variations and partial differential equations, we do not develop here a new model. We will study for a continuous image the model described by S. Geman and D. Geman for a numerical image (or a slight modification of this).

The outline of this chapter is as follows: in Section 1.1 we will make a description of the degradation model considered here, and of the corresponding reconstruction problem; we give the necessary assumptions in order to apply this problem to signal and image recovery. In Section 1.2 we present related models, in the context of variational methods and PDE's for image analysis.

1.1 Description of the model. Assumptions

In continuous variables the observed image u_0 and the reconstructed image u can be represented by functions of $\Omega \subset \mathbb{R}^2 \to \mathbb{R}$ which associate, to the pixel $(x,y) \in \mathbb{R}^2$, its grey level $u_0(x,y)$ or $u(x,y)$; Ω is the support of the image (a rectangle in general). The grey levels being in finite number, we can assume that the observation $u_0(x,y)$ verifies $0 \le u_0(x,y) \le 1, \forall (x,y) \in \Omega$.

The stochastic model proposed by S. Geman and D. Geman for image reconstruction leads, as we pointed out in the introduction, to search a solution among the minima of the energy

$$F_\alpha(u) = \frac{1}{2}\int_\Omega |Ku - u_0|^2 dxdy + \alpha \int_\Omega \phi(|Du|)dxdy. \tag{1.1.1}$$

K is a linear operator of $L^2(\Omega) \to L^2(\Omega)$ (generally, K is the identity I or a convolution) and the first integral represents an attached term on the data. The function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ will be defined and symbolizes the regularizing term (hence, a constraint on the solution). In [32], S. Geman and D. Geman have added a regularizing term of the form

$$\int_\Omega \Big(\phi\big(|\frac{\partial u}{\partial x}|\big) + \phi\big(|\frac{\partial u}{\partial y}|\big)\Big)dxdy, \tag{1.1.2}$$

corresponding (for numerical images) to a regularization on lines and columns. This term, unlike ours, is not rotationally invariant but, since it is more easy to approximate this term, this could be useful for images with parallels edges to the axes. The number $\alpha \in \mathbb{R}^+$ is a parameter which allows to balance the influence of each integral in the energy $F_\alpha(u)$.

If $\alpha = 0$, $F_\alpha(u)$ is

$$F_0(u) = \frac{1}{2}\int_\Omega |Ku - u_0|^2 dxdy,$$

and so, the energy is reduced only to the attached term on the data, and the problem

$$\inf_u F_0(u) \tag{1.1.3}$$

corresponds to the least-squares method associated to the equation (1.0.1). Formally, every solution of (1.1.3) verifies the equation

$$K^*Ku = K^*u_0, \tag{1.1.4}$$

where we have denoted by K^* the adjoint operator of K. Generally, (1.1.4) is an ill-posed problem: K^*K is not always invertible, or the problem is often unstable.

To overcome this difficulty, we either look for a solution in a smaller set (where we have some compactness), or we add to the attached term on the data, a regularizing term (for instance, by taking $\alpha > 0$ in (1.1.1)). This method, which is due to Tichonov [59], is the one we use here and the additional term is represented in (1.1.1) by $\int_\Omega \phi(|Du|)dxdy$. Now, it remains to find some appropriate conditions on the function ϕ, in order to satisfy the following principle of image analysis:

> (P) *The reconstructed image must be formed by homogeneous regions, separated by sharp edges.*

The model must, therefore, diffuse within the regions where the variations of grey levels are weak and otherwise, it must preserve the boundaries of these regions, that is to say: it must respect the strong variations of grey levels.

So, assuming that the integrals in $F_\alpha(u)$ have made sense, then any function realizing the minimum of F_α, must verify formally (for instance, in the sense of distribution), the Euler equation: $F_\alpha^{'}(u) = 0$, or

$$-\alpha \mathrm{div}\Big(\frac{\phi^{'}(|Du|)}{|Du|}Du\Big) + K^*Ku = K^*u_0. \tag{1.1.5}$$

Now, we will try to obtain some sufficient assumptions on ϕ, in order to respect, as much as possible, the principle (P).

For the moment, we look at the equation (1.1.5), written in conservative form. To diffuse the image in the homogeneous regions, that is to say: where $|Du| \ll \varepsilon$, with $\varepsilon > 0$ as a threshold parameter, we can assume, for instance, that there exists $L_1 \in]0, +\infty[$ such that

$$\lim_{z \to 0} \frac{\phi^{'}(z)}{z} = L_1 > 0.$$

In this way, if u is a minimum of F_α, in a neighborhood of a point $x \in \Omega$, where $|Du(x)| \ll \varepsilon$, (1.1.5) is locally a diffusion equation (like the Laplacian) in all directions. On the other hand, to preserve the boundaries of these regions, that is to say: where $|Du| \gg \varepsilon$, we can assume that

$$\lim_{z \to \infty} \frac{\phi^{'}(z)}{z} = 0.$$

Then, in a neighborhood of a point $x \in \Omega$ such that $|Du(x)| \gg \varepsilon$, the diffusion will be weak.

We can assume, in addition, that the function

$$z \mapsto \frac{\phi^{'}(z)}{z} \text{ is continuous and decreasing on } [0, +\infty[, \tag{1.1.6}$$

in order to have some monotonicity and continuous dependence on $|Du|$.

Now, writing the equation (1.1.5) in a nonconservative form (or expanded form), we will obtain more assumptions on ϕ, in order to respect the principle (P). To do this, let for each point (x, y) where $|Du(x, y)| \neq 0$, the vectors $T(x, y) = \frac{Du(x,y)}{|Du(x,y)|}$ in the gradient direction, and $\xi(x, y)$ in the orthogonal direction to $T(x, y)$. With the usual notations

$u_x, u_y, u_{xx}, \ldots$ for the first and second partial derivatives of u, and by formally developing the divergence operator, the equation (1.1.5) can be written as:

$$-\alpha\Big(\frac{\phi^{'}(|Du|)}{|Du|}\Big)u_{\xi\xi} - \alpha\phi^{''}(|Du|)u_{TT} + K^*Ku = K^*u_0, \tag{1.1.7}$$

where we have denoted by $u_{\xi\xi}$ and u_{TT} the second derivatives of u in the directions $\xi(x,y)$ and $T(x,y)$ respectively:

$$u_{\xi\xi} = \frac{1}{\mid Du \mid^2}(u_x^2u_{yy} + u_y^2u_{xx} - 2u_xu_yu_{xy}),$$
$$u_{TT} = \frac{1}{\mid Du \mid^2}(u_x^2u_{xx} + u_y^2u_{yy} + 2u_xu_yu_{xy}).$$

If u is regular (at least continuous), we can interpret the principle (P) in the following manner: locally, we represent a contour $\mathcal{C}$ separating two homogeneous regions of the image, by a level curve of u: $\mathcal{C} = \{\ (x,y);\ u(x,y) = c\ \}$. In this case, the vector $T(x,y)$ is normal to $\mathcal{C}$ at $(x,y) \in \mathcal{C}$, as in Figure 1.1.

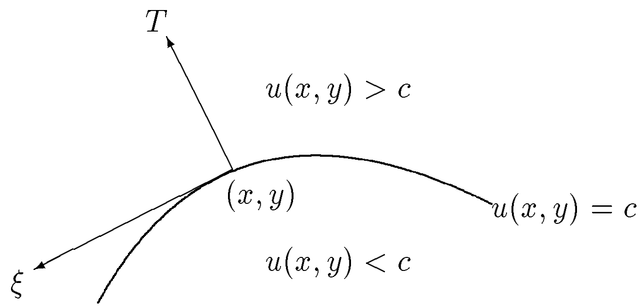


Figure 1.1: A contour $\mathcal{C}$ separating two homogeneous regions.

In the interior of the homogeneous regions: $\Big\{(x,y), u(x,y) < c\Big\} \cup \Big\{(x,y), u(x,y) > c\Big\}$, where the variations of u are weak, we desire to encourage smoothing. If we assume that $\phi^{''}$ exists, with

$$\phi^{'}(0) = 0 \text{ and } \phi^{''}(0) > 0, \tag{1.1.8}$$

we obtain that in a neighborhood of $z = 0$, (1.1.7) is formally equivalent to

$$-\alpha\phi^{''}(0)(u_{\xi\xi} + u_{TT}) + K^*Ku = K^*u_0, \tag{1.1.9}$$

and since for any orthogonal directions T and ξ we always have

$$u_{\xi\xi} + u_{TT} = u_{xx} + u_{yy} = \triangle u,$$

then (1.1.9) can be written as

$$-\alpha\phi^{''}(0) \triangle u + K^*Ku = K^*u_0. \tag{1.1.10}$$

Therefore, at the points where the image has a weak gradient, u locally satisfies (1.1.10), which is a uniformly elliptic equation having strong regularizing properties on the solution, in all directions.

On the contrary, in a neighborhood of a contour $\mathcal{C}$, the image presents a strong gradient. If we wish to better preserve this contour (or edge), it is preferable to diffuse only in the parallel direction to $\mathcal{C}$, i.e. in the ξ-direction. For this, it will be sufficient, in (1.1.7), to annihilate, for strong gradients, the coefficient of u_{TT} and to assume that the coefficient of $u_{\xi\xi}$ does not vanish, that is:

$$\lim_{z\to+\infty} \phi''(z) = 0, \tag{1.1.11}$$

$$\lim_{z\to+\infty} \frac{\phi'(z)}{z} = m > 0, \tag{1.1.12}$$

which allows to reduce (1.1.7) in a neighborhood of $+\infty$ to an equation of the form:

$$-\alpha m u_{\xi\xi} + K^*Ku = K^*u_0,$$

which can be interpreted as a regularizing equation in the ξ-direction.

But, (1.1.11) and (1.1.12) are not compatible, and one must make a compromise between these two hypothesis: for example, by assuming that $\phi''(z)$ and $\frac{\phi'(z)}{z}$ both converge to zero as $z \to \infty$, but with different speeds. More precisely, we will assume that:

$$\lim_{z\to+\infty} \phi''(z) = \lim_{z\to+\infty} \frac{\phi'(z)}{z} = 0, \tag{1.1.13}$$

$$\lim_{z\to+\infty} \frac{\phi''(z)}{\frac{\phi'(z)}{z}} = 0, \tag{1.1.14}$$

that is to say: ϕ'' converges faster to 0 than $\frac{\phi'(z)}{z}$, which makes preponderant the coefficient of $u_{\xi\xi}$ in (1.1.7) (this last condition is related or equivalent "at infinity" to (1.1.6), in the case when $\phi' \geq 0$).

The preceding assumptions on ϕ are rather of qualitative type and represent a-priori the properties which we desire to obtain on the solution. But, these are not sufficient to prove that the model is mathematically well-posed. Then, in order to use the direct method of calculus of variations to ensure the existence and the uniqueness of a solution to the minimization problem, we assume that ϕ is a convex (sometimes, strictly convex) and nondecreasing function on $\mathbb{R}^+$, such that:

$$\lim_{z\to+\infty} \phi(z) = +\infty.$$

This growth at infinity does not be too strong, because this does not penalize strong gradients (or formation of edges). Hence, we assume at most a *linear* growth at infinity, that is:

$$\begin{cases} \text{There exist constants } c > 0 \text{ and } b \geq 0, \text{ such that} \\ \phi(z) \leq cz + b, \forall z \in \mathbb{R}^+. \end{cases} \tag{1.1.15}$$

This inequality is related to (1.1.13). But the convexity of ϕ implies that we must also assume the following inequality:

$$cz - b \leq \phi(z), \forall z \in I\!\!R^+, \tag{1.1.16}$$

and then a natural space on which we will be able to seek a solution and where the functional F_α is finite and well-defined, will be

$$V = \{u \in L^2(\Omega),\ Du \in (L^1(\Omega))^2\},$$

or the more general space $BV(\Omega)$, the space of functions with bounded variation. In Chapter 2 we present the study of the minimization problem on the space $BV(\Omega)$ and in Chapter 3 we will give a regularity result on the space V.

Of course, there are many functions ϕ verifying the above assumptions, in order to ensure the existence and the uniqueness of the problem (1.1.1), and the principle (P) as much as possible.

The function $\phi : I\!\!R^+ \to I\!\!R^+$ must be convex, increasing, "quadratic" in a neighborhood of the origin and with a "linear" growth at infinity.

For this type, our choice will be one of the two following functions denoted ϕ_1 and ϕ_2, defined by:

$$\phi_1(z) = \begin{cases} \dfrac{1}{2}z^2 & \text{if } z \leq 1, \\ \\ z - \dfrac{1}{2} & \text{if } z > 1, \end{cases} \quad \text{and } \phi_2(z) = \sqrt{1+z^2} - 1.$$

We assume, for simplicity, that the threshold parameter ε is equal to one and we note that ϕ_2 is called the function of minimal surfaces.

We will see now what are the differences between these two functions:

1) ϕ_2 is a strictly convex function, while ϕ_1 is not.

2) $\phi_2 \in C^2(I\!\!R^+)$, while $\phi_1 \in C^2(I\!\!R^+ - \{1\})$.

3) It is easy to write the equation (1.1.7) for these two functions: for ϕ_1, if $|Du| < 1$, it becomes:

$$K^*Ku - \alpha \triangle u = K^*u_0, \tag{1.1.17}$$

and if $|Du| > 1$, we get:

$$K^*Ku - \frac{\alpha}{|Du|}u_{\xi\xi} - 0 \cdot u_{TT} = K^*u_0, \tag{1.1.18}$$

where

$$\frac{u_{\xi\xi}(x)}{|Du(x)|} = \mathrm{div}\Big(\frac{Du(x)}{|Du(x)|}\Big)$$

represents the curvature of the level curve of u passing by x.

Now, for ϕ_2, the equation is:

$$K^*Ku - \frac{\alpha}{\sqrt{1+|Du|^2}}u_{\xi\xi} - \frac{\alpha}{\sqrt{1+|Du|^2}^3}u_{TT} = K^*u_0. \tag{1.1.19}$$

In this way, we can see that, with the function ϕ_1, the contours will be better preserved than with ϕ_2, since, if $|Du| > 1$, the coefficient of u_{TT} in (1.1.18) is zero, while in (1.1.19) is not. Moreover, in the homogeneous regions, where $|Du| < 1$, by the equation (1.1.17), the diffusion is isotrope, since (in (1.1.19)):

$$\frac{\alpha}{\sqrt{1+|Du|^2}} \neq \frac{\alpha}{\sqrt{1+|Du|^2}^3}$$

(of course, the differences between these two coefficients tends to zero, when $|Du| \to 0$). By (1.1.17), this diffusion will be more important than by (1.1.19), since we have, for $0 < |Du| < 1$:

$$\frac{\alpha}{\sqrt{1+|Du|^2}} < \alpha, \ \frac{\alpha}{\sqrt{1+|Du|^2}^3} < \alpha$$

(with equality only if $|Du| = 0$).

Several authors, like Rudin-Osher [49], Chambolle-Lions [14], Chan-Mullet [15], have considered the function $\phi(z) = z$ and then $\int_\Omega \phi(|Du|)dx$ becomes the total variation of u. With this function, the behavior of the equation (1.1.5) at infinity is the same, as with ϕ_1 (or ϕ_2), but the difference is in a neighborhood of the origin, since we have:

$$\lim_{z \to 0} \frac{\phi'(z)}{z} = +\infty.$$

Maybe the diffusion in the homogeneous regions can be stronger than with ϕ_1 or ϕ_2, from this behavior but, in order to discretize the associated Euler equation, we must define $\frac{\phi'(z)}{z}$ at the origin. Then, it is necessary to regularize this function. For instance, we can take $\epsilon > 0$ small enough and to replace $\phi(z)$ by one of the two following functions, defined by: $\phi_\epsilon(z) = \sqrt{\epsilon^2 + z^2}$, as in [15], or by

$$\phi_\epsilon(z) = \begin{cases} \dfrac{1}{2\epsilon} z^2 & \text{if } z \leq \epsilon, \\ \\ z - \dfrac{\epsilon}{2} & \text{if } z > \epsilon, \end{cases}$$

as in [14], and then to make $\epsilon \to 0$. But, as we can see, these two regularized functions are like ϕ_2 and ϕ_1 defined before, if we introduce a parameter. Therefore, in the numerical results, we will not explicitly use this function $\phi(z) = z$ as a potential in the energy.

Remark 1.1.1

In this thesis, we will consider, in addition, two other functions, denoted by ϕ_4 and ϕ_5, which still satisfy the conditions (1.1.8),(1.1.13) and (1.1.14), or (1.1.6) (in order to obtain the principle (P) on the solution), but which are not convex. In this case, the minimization problem (1.1.1) will be, of-course, mathematically ill-posed, but the corresponding numerical approximations and results, developed in this work, are very satisfactory.

As we have showed before, it is clear that, to reconstruct edges, we must consider functions ϕ with *at most* a linear growth at infinity. So, the two convex functions ϕ_1

and ϕ_2 have, obviously, a linear growth at infinity. But, a better and weaker growth condition, which does not penalize the formation of edges in the energy, can be the following: ϕ growths sublinearly at infinity (that is to say: we do not assume (1.1.16)). Unfortunately, this condition introduces, in general, an hypothesis of non convexity on ϕ at infinity (for $z \geq z_0 > 0$). For instance, we can consider one of the two following functions: $\phi_4, \phi_5 : I\!R^+ \to I\!R^+$, defined by

$$\phi_4(z) = \frac{1}{2}\log(1+z^2) \quad \text{and} \quad \phi_5(z) = \frac{z^2}{2(1+z^2)}.$$

We observe that ϕ_5 has an horizontal asymptote at infinity and, moreover, due to this property, the formation of edges has no contribution in the energy, while the function ϕ_4 has a behavior "between" ϕ_1 (or ϕ_2) and ϕ_5.

As in the convex case (that is, for ϕ_1 and ϕ_2), we write here the equivalent of the equation (1.1.7) with ϕ_4 and ϕ_5.

For ϕ_4, we have:

$$K^*Ku - \alpha\frac{1}{1+|Du|^2}u_{\xi\xi} - \alpha\frac{1-|Du|^2}{(1+|Du|^2)^2}u_{TT} = K^*u_0, \tag{1.1.20}$$

and for ϕ_5:

$$K^*Ku - \alpha\frac{1}{(1+|Du|^2)^2}u_{\xi\xi} - \alpha\frac{1-3|Du|^2}{(1+|Du|^2)^3}u_{TT} = K^*u_0. \tag{1.1.21}$$

Since ϕ_4 and ϕ_5 are non-convex, for $z \geq z_0$ (with $z_0 = 1$ for ϕ_4 and $z_0 = 1/\sqrt{3}$ for ϕ_5), we can say that, for large gradients, where $|Du| \geq z_0$, the term

$$-\alpha\phi''(|Du|)u_{TT}$$

represents, in each equation (1.1.20) and (1.1.21), an anti-diffusion term, which will reinforce the edges. In this way, the edges will be better preserved, but, in the homogeneous regions, and when the noise is important, the diffusion will be weaker.

We conclude this section with a global comparison of all these functions (we consider, as before, the threshold parameter ε equal to one).

If we look at the equation (1.1.5), which depends only on $\frac{\phi'(z)}{z}$ ($z \geq 0$), we see that for each function ϕ_i, $i = 1,2,4,5$, $z \mapsto \frac{\phi_i'(z)}{z} : [0,+\infty[\to]0,+\infty[$ is a decreasing function, such that:

$$\lim_{z\to 0}\frac{\phi'(z)}{z} = 1, \ \lim_{z\to\infty}\frac{\phi'(z)}{z} = 0.$$

About the diffusion in the homogeneous regions (where $|Du| \ll 1$), we can say, from the equation (1.1.7), that among the four functions, the diffusion will be stronger with the function ϕ_1. This can be also seen by looking at the equation written in the following form:

$$K^*Ku - \alpha \bigtriangleup u - \alpha\mathrm{div}[(\frac{\phi'(|Du|)}{|Du|} - 1)Du] = K^*u_0.$$

We observe that the coefficient $\frac{\phi'(|Du|)}{|Du|}-1$, for $|Du|<1$, is equal to zero for ϕ_1 and strictly negative for the other functions ϕ_2, ϕ_4 and ϕ_5.

Now, to preserve the contours of the homogeneous regions, the diffusion must to be "weak" where $|Du| \gg 1$, and only in the orthogonal direction to the gradient, that is, in the $\xi-$direction. We have already seen from the equation (1.1.7), since $\phi_i''(|Du|)<0$ if $|Du|>1$ for ϕ_4 and ϕ_5, then by using one of these two functions, the contours will be better preserved.

We wish to add a last potential, denoted ϕ_3 and defined by:

$$\phi_3 : I\!R^+ \to I\!R^+, \quad \phi_3(z) = \begin{cases} \dfrac{1}{2}z^2 & \text{if } z \leq 1, \\ \\ \frac{z^3}{1+z^2} & \text{if } z > 1. \end{cases}$$

The function ϕ_3 is linear at infinity and convex.

In Chapter 4 (for one-dimensional reconstruction) and Part III (for two-dimensional reconstruction), we will compare several numerical results by using these five functions and we will give our opinion and conclusion about the choice of ϕ in the algorithms, in order to obtain good results, for different types of signals and images.

1.2 Related models

We study here, in Part I, a problem which becomes more and more classical, and there exists a lot of papers devoted to the same problem. We do not develop a new model, we only wish to complete the theoretical study of the problem in the framework of functions with bounded variation, in the general case, to give some characterizations for the solution and convergent algorithms. Finally, we wish to test and compare the potentials defined before, in order to denoise and deblur signals and images.

We first mention, in the discrete case, the works of S. Geman, D. Geman, G. Reynolds, C. Yang ([32], [33], [34]), whose considered a regularizing term of the form:

$$\int_\Omega \Big(\phi\big(|\frac{\partial u}{\partial x}|\big) + \phi\big(|\frac{\partial u}{\partial y}|\big)\Big) dx dy,$$

corresponding, for numerical images, to a regularization on lines and columns. This term, unlike ours, is not rotationally invariant. We also mention the interesting works of L. Blanc-Féraud, M. Barlaud, G. Aubert and P. Charbonnier ([17], [18]), whose considered non-convex potentials ϕ and made interesting algorithms to compute the solution.

Now, in the continuous case, we refer the reader to works of S. Osher, L. Rudin, E. Fatemi, T. Chan, G. Golub, P. Mulet ([49], [50], [15]), whose considered the potential $\phi(z) = z$. From a theoretical point of view, in the context of functions with bounded variation, we refer to works of R. Acart, C. R. Vogel (for example: [1], where the authors have considered the cases $\phi(z) = z$ and $\phi(z) = \sqrt{1+z^2}$, the function of minimal surfaces), and to a very interesting and complete paper of A. Chambolle, P.L. Lions ([14], in the case: $\phi(z) = z$).

Finally, we wish to mention some very closed papers to this problem, of C. Schnörr and al ([51], [52], [53]), whose considered a variational problem of the same type, with regularity and numerical results.

Chapter 2

The variational problem on the space BV

In this chapter, we study the minimization problem introduced before, in the convex case, in the framework of functions with bounded variation. We recall here the functional F_α that we want to minimize:

$$F_\alpha(u) = \int_\Omega |Ku - u_0|^2 dx + \alpha \int_\Omega \phi(|Du|)dx \tag{2.0.1}$$

(in fact, we will minimize the lower semicontinuous relaxation of F_α in $BV(\Omega)$).

In Section 2.1, we give, in addition to the conditions established in the previous chapter, the sufficient assumptions on K and ϕ in order to ensure the existence and the uniqueness of a solution to the minimization problem in $BV(\Omega)$, the space of functions with bounded variation. We will first recall some general definitions and properties for $BV(\Omega)-$functions and some relaxation results for convex functionals of measures.

The Section 2.2 is devoted to the existence and the uniqueness of the minimum. In Section 2.3, we will give a characterization of the subdifferential of the functional and then we will deduce optimality conditions for the solutions. Finally, in Section 2.4, using the theory of maximal monotone operators, we study the evolution equation associated to the minimization problem.

2.1 Notations, assumptions and preliminary results

Let Ω be an open, bounded and connected subset of $\mathbb{R}^N$, with Lipschitz boundary Γ; we shall use standard notations for the Sobolev and Lebesgue spaces $W^{1,p}(\Omega)$ and $L^p(\Omega)$. For the theoretical study of the problem, we can take $\alpha = 1$, for simplicity, and the functional F_α will be denoted by F.

To ensure the existence and the uniqueness of the minima of (2.0.1), we make the following assumptions:

$H1$. $\phi : \mathbb{R} \to \mathbb{R}^+$ is an even and convex function, nondecreasing on $\mathbb{R}^+$, such that:
 (i) $\phi(0) = 0$ (without loss of generality).
 (ii) There exist $c > 0$ and $b \geq 0$ such that: $cz - b \leq \phi(z) \leq cz + b$, $\forall z \in \mathbb{R}^+$.

$H2$. $K : L^p(\Omega) \to L^2(\Omega)$ is a linear and continuous operator, where $p = \frac{N}{N-1}$ if $N \geq 2$ and $p = 2$ if $N = 1$;

$H3$. $K\chi_\Omega \neq 0$;

$H4$. K is injective or ϕ is strictly convex.

Remark 2.1.1

Since ϕ is convex and finite on $I\!R$, then it is continuous. Moreover, its asymptote (or recession) function ϕ^∞ exists (see, for instance, Goffman-Serrin [37]) and it is finite (from $H1(ii)$):

$$\phi^\infty(z) := \lim_{t\to\infty} \frac{\phi(tz)}{t} \in [0; +\infty).$$

In fact, $c = \lim_{t\to\infty} \frac{\phi(t)}{t}$ and $\phi^\infty(z) = cz \cdot \mathrm{sign} z$.

Remark 2.1.2

Thanks to $H1(ii)$, the functional $j(u) := \int_\Omega \phi(|Du|)dx$ is finite and well-defined on the space $W^{1,1}(\Omega)$. But as it is well known, $W^{1,1}(\Omega)$ is a non reflexive Banach space and moreover, the minimization problem (2.0.1) can has no solution on this space. For these reasons, we must work with functions with bounded variation and use the notions of convex function of measures and relaxed functionals on measures, to get the existence of a minimum. Moreover, the space of $BV-$functions is the proper class for many basic image processing tasks, because it allows discontinuities along curves or edges, while $W^{1,1}$-functions may not.

Example: Suppose $E \subset \Omega$ has C^2 boundary and consider the characteristic function χ_E, defined by:

$$\chi_E(x) = \begin{cases} 1 \text{ if } x \in \Omega, \\ 0 \text{ if } x \in \Omega \backslash E. \end{cases}$$

Then $\chi_E \in BV(\Omega)$, since $TV(\chi_E) := \int_\Omega |D\chi_E| = \mathcal{H}^{N-1}(\partial E) < \infty$, but $\chi_E \notin W^{1,1}(\Omega)$, according, for instance, to Evans-Gariepy [30], Theorem 2 (characterization of Sobolev functions), Section 4.9.2. Therefore, the boundary of E, ∂E, could represent an edge in a image. We note that $\mathcal{H}^{N-1}(\partial E)$ is called the *perimeter* of E in Ω ([35]).

Remark 2.1.3

Examples of linear and continuous operators K from $L^p(\Omega)$ into $L^2(\Omega)$ include the identity operator (K=I) if $N = 1, 2$ and convolutions with a positive kernel. In image analysis, for $K = k * u$, the kernel k must satisfy $k(x) \geq 0$, $k(x) \to 0$ rapidly as $|x| \to \infty$, and $\int_{I\!R^N} k(x) = 1$. Generally, k is the heat kernel or a function which satisfies in addition the following properties: $k(x) = k(|x|)$, $k(|x|) = 0$ if $|x| \geq 1$ and $k \in C^\infty$ (see, for instance, Osher-Rudin [44]). For these particular cases, k belongs to $L^2(\Omega)$, and then, for $u \in L^p(\Omega)$, $Ku := k * u$ is well-defined, linear and continuous from $L^p(\Omega)$ into $L^2(\Omega)$, even if $N > 2$. The assumption $H3$ says that K does not annihilate constant functions. This will guarantee the $BV-$coercivity of the functional and it is always true for the convolution operator.

Now we introduce the notations and preliminary results for the space $BV(\Omega)$ and we recall the notion of lower semicontinuity on this space.

We denote by $\mathcal{L}_N$ (or sometimes by dx) the Lebesgue $N-$dimensional measure in $I\!R^N$ and by $\mathcal{H}^\alpha$ the $\alpha-$dimensional Hausdorff measure. We also set $|E| = \mathcal{L}_N(E)$, the Lebesgue measure of a measurable set $E \subset I\!R^N$. We will use the notation $\mathcal{B}(\Omega)$ for the family of the Borel subsets of Ω. If $x, y \in I\!R^N$, then $x \cdot y$ will denote their scalar product.

Given a vector-valued measure $\mu : \mathcal{B}(\Omega) \to I\!R^M$, we adopt the notation $|\mu|$ for its total variation. We recall that

$$|\mu|(A) = \sup\Big\{ \sum_{j=1}^{M} \int_\Omega v_j d\mu_j : v = (v_1, ..., v_M) \in C_0(A; I\!R^M), \|v\|_\infty \leq 1 \Big\},$$

where $C_0(A; I\!R^M)$ denotes the closure, in the sup norm, of continuous functions with compact support in A. We denote by $\mathcal{M}(\Omega)$ the set of all signed measures on Ω with bounded total variation.

The usual $weak*$ topology on $\mathcal{M}(\Omega)$ is defined as the weakest topology on $\mathcal{M}(\Omega)$ for which the maps $\mu \to \int_\Omega \psi d\mu$ are continuous for every continuous function ψ vanishing on $\partial\Omega$.

We say that $u \in L^1(\Omega)$ is a function of bounded variation ($u \in BV(\Omega)$) if its distributional derivative $Du = (D_1 u, ..., D_N u)$ belongs to $\mathcal{M}(\Omega)$.

The space $BV(\Omega)$ endowed with the norm

$$\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + |Du|(\Omega),$$

is a Banach space.

The product topology of the strong topology of $L^1(\Omega)$ for u and of the $weak*$ topology of measures for Du will be called the $weak*$ topology of BV, and will be denoted by $BV - w*$. We recall that every bounded sequence in $BV(\Omega)$ admits a subsequence converging in $BV - w*$. This sequence is also relatively compact in $L^p(\Omega)$ for $1 \leq p < \frac{N}{N-1}$ and $N \geq 1$, and relatively weakly compact in $L^p(\Omega)$ for $p = \frac{N}{N-1}$ and $N \geq 2$ (Giusti [35], Acart-Vogel [1]).

We have also an extension to $BV-$functions of the Poincaré-Wirtinger inequality ([10], [1]): for $u \in BV(\Omega)$, let

$$\bar{u} := \frac{1}{|\Omega|} \int_\Omega u(x) dx.$$

Then there exists $M > 0$ such that

$$\|u - \bar{u}\|_{L^p(\Omega)} \leq M |Du|(\Omega),$$

where $p = \infty$ for $N = 1$ and $p = \frac{N}{N-1}$ for $N > 1$. But, for $N = 1$, we can take $p = 2$ instead of $p = \infty$. We deduce that if $u \in BV(\Omega)$, then $u \in L^p(\Omega)$ ($BV(\Omega)$ is continuously embedded in $L^p(\Omega)$).

For any function $u \in L^1(\Omega)$ we denote by S_u the complement of the Lebesgue set of u, i.e. $x \notin S_u$ if and only if there exists $\tilde{u}(x) \in I\!R$ such that

$$\lim_{\rho \to 0^+} \rho^{-N} \int_{B_\rho(x)} |u(y) - \tilde{u}(x)| dy = 0.$$

The limit $\tilde{u}(x)$ denotes the *approximate limit* of u at x and $\tilde{u}$ is a Borel function equal to u almost everywhere. The set S_u is Lebesgue negligible.

If $u \in BV(\Omega)$, then u is differentiable almost everywhere on $\Omega \backslash S_u$ and ∇u coincides with the Radon-Nikodym derivative of Du with respect to $\mathcal{L}_N$. Moreover, the Hausdorff dimension of S_u is at most $(N-1)$ and for $\mathcal{H}^{N-1}$-almost every $x \in S_u$ it is possible to find $u^+(x), u^-(x) \in I\!R$ unique, with $u^+(x) > u^-(x)$ and $\nu \in S^{n-1}$ such that

$$\lim_{\rho \to 0^+} \rho^{-N} \int_{B_\rho^\nu(x)} |u(y) - u^+(x)| dy = \lim_{\rho \to 0^+} \rho^{-N} \int_{B_\rho^{-\nu}(x)} |u(y) - u^-(x)| dy = 0,$$

where $B_\rho^\nu(x) = \{y \in B_\rho(x) : (y-x) \cdot \nu > 0\}$ and $B_\rho^{-\nu}(x) = \{y \in B_\rho(x) : (y-x) \cdot \nu < 0\}$ (we suppose that the normal ν "points towards the larger value" of u; we have denoted by $B_\rho(x)$ the ball centered in x of radius ρ).

We have the Lebesgue decomposition

$$Du = \nabla u \cdot \mathcal{L}_N + D_s u,$$

where $\nabla u \in (L^1(\Omega))^N$ is the Radon-Nikodym derivative of Du and $D_s u$ is singular, with respect to $\mathcal{L}_N$. We also have the decomposition for $D_s u$:

$$D_s u = C_u + J_u,$$

where

$$J_u = (u^+ - u^-)\nu \cdot \mathcal{H}^{N-1}_{|S_u}$$

is the *Hausdorff part* or *jump part* and C_u the *Cantor part* of Du. We recall that the measure C_u is singular with respect to $\mathcal{L}_N$ and it is "diffuse", i.e. $C_u(S) = 0$ for every set S of Hausdorff dimension $N-1$. Hence, we have for every $B \in \mathcal{B}(\Omega)$, that

$$D_s u(B \backslash S_u) = C_u(B \backslash S_u) \quad \text{and} \quad D_s u(B \cap S_u) = J_u(B \cap S_u).$$

Finally, we can write Du and its total variation on Ω, $|Du|(\Omega)$, as

$$\begin{gathered} Du = \nabla u \cdot \mathcal{L}_N + C_u + (u^+ - u^-)\nu \cdot \mathcal{H}^{N-1}_{|S_u}, \\ |Du|(\Omega) = \textstyle\int_\Omega |\nabla u| dx + \int_{\Omega \backslash S_u} |Cu| + \int_{S_u} (u^+ - u^-) d\mathcal{H}^{N-1}; \end{gathered}$$

(we recall that $u^+ > u^-$).

It is then possible to define the convex function of measures $\phi(|\cdot|)$ on $\mathcal{M}(\Omega)$, which is for Du:

$$\phi(|Du|) = \phi(|\nabla u|) \cdot \mathcal{L}_N + \phi^\infty(1)|D_s u|,$$

and the functional

$$J(u) = \phi(|Du|)(\Omega) = \int_\Omega \phi(|\nabla u|) dx + \phi^\infty(1) \int_\Omega |D_s u|$$

(see Goffman-Serrin [37], who show that the functional $\phi(|\cdot|)(\Omega)$ is weakly* lower semi-continuous on $\mathcal{M}(\Omega)$, or Demengel-Temam [28]). It is also simple to see that $J(\cdot)$ is convex on $BV(\Omega)$ (for this, we use that ϕ is convex and increasing on $I\!R_+$).

By the decomposition of $D_s u$, the properties of C_u, J_u and the definition of the constant c, the functional J can be written as:

$$J(u) = \int_\Omega \phi(|\nabla u|)dx + c\int_{\Omega\setminus S_u} |C_u| + c\int_{S_u} (u^+ - u^-)d\mathcal{H}^{N-1}.$$

Now, the functional $J : BV(\Omega) \to [0, +\infty)$ is lower semicontinuous with respect to the $BV - w*$ topology and less than or equal to j, where j is defined by:

$$j(u) = \begin{cases} \int_\Omega \phi(|\nabla u|)dx & \text{if } \ u \in W^{1,1}(\Omega), \\ +\infty & \text{if } \ u \in BV(\Omega)\setminus W^{1,1}(\Omega). \end{cases}$$

We note that the functional j is not lower semicontinuous on $BV(\Omega)$ (or $L^p(\Omega), L^1(\Omega)$) with respect to the $BV - w*$ (or L^p, L^1 topologies). Moreover, for each $u \in BV(\Omega)$, there exists (see Demengel-Temam [28], page 692) a sequence $\{u_n\}_{n\geq 1} \in C^\infty(\Omega)\cap W^{1,1}(\Omega)$ such that $u_n \to u$ as $n \to \infty$ in $BV - w^*$ and

$$J(u) = \phi(|Du|)(\Omega) = \lim_{n\to\infty} \int_\Omega \phi(|\nabla u_n|)dx = \lim_{n\to\infty} j(u_n).$$

In this way, we get that J is the relaxation of j on $BV - w^*$, that is:

$$J(u) = \bar{j}(u) := \inf\{\liminf_{n\to\infty} j(u_n) : u_n \in BV(\Omega), u_n \to u \text{ in } BV - w^*\},$$

($\bar{j}$ is the greatest $BV - w^*$ lower semicontinuous functional less than or equal to j).

For more general results of relaxation for functionals defined on measures, we refer the reader to the works by Bouchitté, Braides, Buttazzo, Dal Maso and al ([8], [7], [9]).

It is then natural to consider, instead of $j(u)$, $J(u)$ for the second term of $F(u)$ in (2.0.1) and now we will denote the functional on $BV(\Omega)$ by $\hat{F}$ (this will be equal to F in $W^{1,1}(\Omega)$):

$$\hat{F}(u) = \int_\Omega |Ku - u_0|^2 dx + \int_\Omega \phi(|\nabla u|)dx + c\int_\Omega |D_s u|.$$

Remark 2.1.4

According to a result of Bouchitté-Dal Maso [9], we have that J is the lower semicontinuous envelope of j with respect to the L^p topology, with $p = \frac{N}{N-1}$. Then, if $u_0 \in L^2(\Omega)$ and since K is linear and continuous from $L^p(\Omega)$ into $L^2(\Omega)$, the functional

$$\hat{F}(u) = \int_\Omega |Ku - u_0|^2 dx + J(u)$$

is the lower semicontinuous envelope of F on $L^p(\Omega)$. In fact, if $u \in BV(\Omega)$, then there exists a sequence $u_n \in W^{1,1}(\Omega)$ such that $u_n \to u$ (as $n \to \infty$) in $L^p(\Omega)$, and:

$$\hat{F}(u) = \lim_{n\to\infty} \hat{F}(u_n) = \lim_{n\to\infty} F(u_n).$$

2.2 The minimization problem

In this section we study the existence and the uniqueness of a solution for the minimization problem

$$\inf_u \left\{ \hat{F}(u) = \int_\Omega (Ku - u_0)^2 dx + \int_\Omega \phi(|\nabla u|) dx + c \int_\Omega |D_s u| \right\}, \tag{2.2.1}$$

for $u \in BV(\Omega)$ (we recall that $BV(\Omega) \subset L^p(\Omega)$, with $p = 2$ if $N = 1$ and $p = \frac{N}{N-1}$ if $N \geq 2$), $Du = \nabla u \cdot \mathcal{L}_N + D_s u$, $Ku \in L^2(\Omega)$ and $u_0 \in L^2(\Omega)$.

To do this, we essentially follow Acart-Vogel [1] to show that

$$\phi(|Du|)(\Omega) + \|Ku - u_0\|_{L^2(\Omega)}$$

is coercive on $BV(\Omega)$ and Chambolle-Lions [14], for passing to the limit on the minimizing sequences.

Proposition 2.2.1 *Let $u_0 \in L^2(\Omega)$. Under the assumptions $H1 - H4$, there exists a unique solution $u \in BV(\Omega)$ of (2.2.1), satisfying $Ku \in L^2(\Omega)$.*

Proof.

Step 1: Existence

In this first part of the proof, the existence, we will denote by M a strictly positive constant, but which may be different from line to line.

Let $\{u_n\}_{n\geq 1}$ be a minimizing sequence for (2.2.1). Then $u_n \in BV(\Omega)$ thanks to $H1(ii)$ and we have:

$$|Du_n|(\Omega) = \int_\Omega |\nabla u_n| dx + |D_s u_n|(\Omega) \leq M, \ \forall n \geq 1,$$

where $Du_n = \nabla u_n dx + D_s u_n$ is the Lebesgue decomposition of Du_n.

Now we will prove that $|\int_\Omega u_n| \leq M$, $\forall n \geq 1$.

Let

$$w_n = \frac{\int_\Omega u_n}{|\Omega|} \chi_\Omega, \text{ and } v_n = u_n - w_n.$$

Then $\int_\Omega v_n = 0$ and $Dv_n = Du_n$. Hence, $|Dv_n|(\Omega) \leq M$. Using the Poincaré-Wirtinger inequality, we obtain that:

$$\|v_n\|_{L^p(\Omega)} \leq M.$$

We also have:

$$\begin{aligned} M &\geq \|Ku_n - u_0\|_2^2 = \|Kv_n + Kw_n - u_0\|_2^2 \\ &\geq (\|Kv_n - u_0\|_2 - \|Kw_n\|_2)^2 \\ &\geq \|Kw_n\|_2(\|Kw_n\|_2 - 2\|Kv_n - u_0\|_2) \\ &\geq \|Kw_n\|_2[\|Kw_n\|_2 - 2(\|K\| \cdot \|v_n\|_p + \|u_0\|_2)]. \end{aligned}$$

Let $x_n = \|Kw_n\|_2$ and $a_n = \|K\| \cdot \|v_n\|_p + \|u_0\|_2$. Then

$$x_n(x_n - 2a_n) \leq M, \text{ with } 0 \leq a_n \leq \|K\| \cdot M + \|u_0\|_{L^2(\Omega)} = M', \forall n \geq 1.$$

Hence, we obtain:

$$0 \leq x_n \leq a_n + \sqrt{a_n^2 + M} \leq M'',$$

which implies

$$\|Kw_n\|_2 = |\int_\Omega u_n dx| \cdot \frac{\|K\chi_\Omega\|_2}{|\Omega|} \leq M'', \forall n \geq 1,$$

and thanks to $H3$, we obtain that $|\int_\Omega u_n dx|$ is uniformly bounded.

Again, by the Poincaré-Wirtinger inequality, we have:

$$\|u_n - \frac{\int_\Omega u_n}{|\Omega|}\|_{L^p(\Omega)} \leq C|Du_n|(\Omega) \leq C \cdot M,$$

with $p = 2$ if $N = 1$ and $p = \frac{N}{N-1}$ if $N \geq 2$. Finally, we obtain:

$$\|u_n\|_{L^p(\Omega)} = \|u_n - \frac{\int_\Omega u_n}{|\Omega|} + \frac{\int_\Omega u_n}{|\Omega|}\|_{L^p(\Omega)} \leq \|u_n - \frac{\int_\Omega u_n}{|\Omega|}\|_{L^p(\Omega)} + |\int_\Omega u_n| \leq M'''.$$

Hence, u_n is bounded in $L^p(\Omega)$ and, in particular, in $L^1(\Omega)$. Then, u_n is also bounded in $BV(\Omega)$, and there is a subsequence u_n and $u \in BV(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $L^p(\Omega)$ and in $BV - w^*$, $Du_n \rightharpoonup Du$ weakly* in $\mathcal{M}(\Omega)$. Moreover, Ku_n converges weakly to Ku in $L^2(\Omega)$, from $H2$.

Finally, we have (from the above relaxations results in $BV - w^*$):

$$\int_\Omega (Ku - u_0)^2 dx \leq \liminf_{n\to\infty} \int_\Omega (Ku_n - u_0)^2 dx,$$

and

$$\int_\Omega \phi(|Du|) \leq \liminf_{n\to\infty} \int_\Omega \phi(|Du_n|),$$

that is to say:

$$\hat{F}(u) \leq \liminf_{n\to\infty} \hat{F}(u_n)$$

and u is a minimum of $\hat{F}$.

Step 2: Uniqueness

Let $u, v \in BV(\Omega)$ be two solutions of the minimization problem (2.2.1). Then we deduce that $Ku = Kv$, since, on the contrary, if $Ku \neq Kv$, then we must have

$$\hat{F}(\frac{1}{2}u + \frac{1}{2}v) < \frac{1}{2}\hat{F}(u) + \frac{1}{2}\hat{F}(v) = \inf \hat{F},$$

because $\hat{F}$ is the sum of two convex functions with independent variables, Ku and Du, the first being strictly convex. But this inequality is impossible for u and v minima of $\hat{F}$. Hence, $u = v$ if K is injective. But, if we only have that ϕ is strictly convex, then $Du = Dv$, which implies that $u = v + C$ and $K \cdot C = 0$. From $(H3)$, we get that $C = 0$ and then $u = v$. ■

2.3 Characterization of the solutions

In this section we will characterize the solution of the minimization problem by calculating the subdifferential of $\hat{F}(u)$. We use some of the technics of R. Temam for the problem of minimal surfaces ([28]).

We assume $H1 - H4$ and that $u_0 \in L^2(\Omega)$.

We first extend $\hat{F}$ to $L^p(\Omega)$: for $u \in L^p(\Omega) \backslash BV(\Omega)$, $\hat{F}(u) = +\infty$ and for $u \in BV(\Omega)$, with $Du = \nabla u dx + C_u + J_u$, $\hat{F}(u)$ is given by

$$\hat{F}(u) = \int_\Omega (Ku - u_0)^2 dx + \int_\Omega \phi(|\nabla u|) dx + c \int_{\Omega \backslash S_u} |C_u| + c \int_{S_u} |J_u|.$$

The definition of the subdifferential $\partial \hat{F}$ at u is the following (see Ekeland-Temam [29]): let $u \in L^p(\Omega)$ and $\xi \in L^{p'}(\Omega)$. Then

$$\xi \in \partial \hat{F}(u) \quad \text{iff} \quad \{\ \hat{F}(u) \in I\!R \text{ and } \hat{F}(u) - \int_\Omega \xi u \leq \hat{F}(v) - \int_\Omega \xi v, \forall v \in L^p(\Omega)\ \}$$

and we have that $\hat{F}(u) = \inf_{v \in L^p(\Omega)} \hat{F}(v)$ if and only if $0 \in \partial \hat{F}(u)$ and for this reason it is natural to provide a characterization of $\partial \hat{F}$.

The subdifferential of $\hat{F}$

Let $u \in BV(\Omega) \subset L^p(\Omega)$ and $\xi \in L^{p'}(\Omega)$. We say that $\xi \in \partial \hat{F}(u)$ if u achieves the minimum on $BV(\Omega)$ of the following variational problem:

$$(\mathcal{P}_1) \qquad \inf_{v \in BV(\Omega)} \Big\{ \hat{F}(v) - \int_\Omega \xi v dx \Big\}.$$

By Remark 2.1.4 (see Section 2.1), we can replace in $(\mathcal{P}_1)$ the infimum on $BV(\Omega)$ by the infimum on $W^{1,1}(\Omega)$ ($W^{1,1}(\Omega) \subset BV(\Omega) \subset L^p(\Omega)$). Using the following property: if $v \in W^{1,1}(\Omega)$, then $D_s v = 0$ (see Evans-Gariepy [30]), the problem becomes

$$(\mathcal{P}_2) \qquad \inf_{v \in W^{1,1}(\Omega)} \Big\{ \int_\Omega (Kv - u_0)^2 dx + \int_\Omega \phi(|\nabla v|) dx - \int_\Omega \xi v dx \Big\}.$$

Now, the problems $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ have the same infimum, which belongs to $I\!R$, because we have assumed that $\xi \in \partial \hat{F}(u)$, that is: u is a solution of $(\mathcal{P}_1)$.

Now, we will write $(\mathcal{P}_2^*)$, the dual of $(\mathcal{P}_2)$, in the sense of Ekeland-Temam [29].

Let $\mathcal{F} : W^{1,1}(\Omega) \to I\!R$, $\mathcal{G} : L^2(\Omega) \times L^1(\Omega)^N \to I\!R$, $\mathcal{G}_1 : L^2(\Omega) \to I\!R$ and $\mathcal{G}_2 : L^1(\Omega)^N \to I\!R$, such that

$$\mathcal{F}(v) = -\textstyle\int_\Omega v \xi dx = - < v, \xi >_{L^p \times L^{p'}},$$
$$\mathcal{G}_1(w_0) = \textstyle\int_\Omega (w_0 - u_0)^2 dx, \ \mathcal{G}_2(\bar{w}) = \int_\Omega \phi(|\bar{w}|) dx,$$
$$\mathcal{G}(w) = \mathcal{G}_1(w_0) + \mathcal{G}_2(\bar{w}),$$

with $w = (w_0, \bar{w}) = (w_0, w_1, ..., w_N) \in L^2(\Omega) \times L^1(\Omega)^N$.

Then, $(\mathcal{P}_2^*)$ is given by:

$$(\mathcal{P}_2^*) \qquad \sup_{p^* \in L^2(\Omega)\times L^\infty(\Omega)^N} \Big\{ -\mathcal{F}^*(\Lambda^* p^*) - \mathcal{G}^*(-p^*) \Big\},$$

where the operator $\Lambda : W^{1,1}(\Omega) \to L^2(\Omega) \times L^1(\Omega)^N$ is defined by:

$$\Lambda u = (Ku, D_1 u, D_2 u, ..., D_N u)$$

and Λ^* is the adjoint.

We compute $\mathcal{F}^*$ and $\mathcal{G}^*$ using the definition of the Legendre transform (or polar): if $V = W^{1,1}(\Omega)$ with V^* the dual, then

$$\mathcal{F}^*(\Lambda^* p^*) = \sup_{v \in W^{1,1}(\Omega)} < \Lambda^* p^* + \xi, v >_{V \times V^*} = \begin{cases} 0 \text{ if } \Lambda^* v^* + \xi = 0 \ \text{ on } V, \\ +\infty \text{ elsewhere .} \end{cases}$$

It is simple to see that

$$\mathcal{G}^*(p^*) = \mathcal{G}_1^*(p_0^*) + \mathcal{G}_2^*(\bar{p}^*),$$

where $p^* = (p_0^*, \bar{p}^*) = (p_0^*, p_1^*, ..., p_N^*)$.

We have that:

$$\mathcal{G}_1^*(p_0^*) = \int_\Omega \Big(\frac{(p_0^*)^2}{4} + p_0^* u_0 \Big) dx.$$

Since $\phi : I\!R \to I\!R^+$ is convex, l.s.c. and even, we also have [29]:

$$\mathcal{G}_2^*(\bar{p}^*) = \int_\Omega \phi^*(|\bar{p}^*|) dx,$$

if $|\bar{p}^*(\cdot)| \in Dom(\phi^*)$.

In this way, we can also write $(\mathcal{P}_2^*)$ in the following form:

$$(\mathcal{P}_2^*) \qquad \sup_{p^* \in \mathcal{K}} \Big\{ -\int_\Omega \Big(\frac{(p_0^*)^2}{4} - p_0^* u_0 \Big) dx - \int_\Omega \phi^*(|\bar{p}^*|) dx \Big\},$$

where

$$\mathcal{K} = \{ p^* \in L^2(\Omega) \times L^\infty(\Omega)^N : |\bar{p}^*(x)| \in Dom(\phi^*), \Lambda^* p^* + \xi = 0 \text{ in } \mathcal{D}'(\Omega) \}.$$

We can simply see from $H1(ii)$ that, if $m \in I\!R$, then $m \in Dom(\phi^*)$ if and only if $|m| \leq c = \phi^\infty(1)$ (see also Demengel-Temam [28]).

From $\Lambda^* p^* + \xi = 0$ in $\mathcal{D}'(\Omega)$, we get:

$$\begin{aligned} &< \Lambda^* p^*, w > + < \xi, w > = < p^*, \Lambda w > + < w, \xi > \\ &= < p_0^*, Kw > + < \bar{p}^*, Dw > + < w, \xi > = 0, \ \forall w \in \mathcal{D}(\Omega). \end{aligned}$$

Then, we have:

$$K^* p_0^* - \text{div} \bar{p}^* + \xi = 0 \ \text{ in } \mathcal{D}'(\Omega).$$

For p^* satisfying this relation, we obtain that $\text{div}\bar{p}^* \in L^{p'}(\Omega)$ and then, we can define (by a theorem of Lions-Magenes [10]) the trace of $\bar{p}^* \cdot \nu$ on $\Gamma = \partial\Omega$, where ν represents the unit normal to Γ and integrating by parts, we get, for $v \in W^{1,1}(\Omega)$:

$$\begin{aligned}\textstyle\int_\Gamma \bar{p}^* \cdot \nu v d\Gamma &= \textstyle\int_\Omega \sum_{i=1}^N (D_i \bar{p}_i^* v) dx + \int_\Omega \sum_{i=1}^N (\bar{p}_i^* D_i v) dx \\ &= < K^* p_0^*, v > + < v, \xi > - < p_0^*, Kv > - < v, \xi > = 0.\end{aligned}$$

In this way, we deduce for $p^* \in \mathcal{K}$, that $\bar{p}^* \cdot \nu = 0 \ d\Gamma -$ a.e. on Γ.

Finally, we rewrite $\mathcal{K}$ in the following way:

$$\mathcal{K} = \{p^* \in L^2(\Omega) \times L^\infty(\Omega)^N : |\bar{p}^*(x)| \leq c, K^* p_0^* - \text{div}\bar{p}^* + \xi = 0 \text{ in } \mathcal{D}'(\Omega), \bar{p}^* \cdot \nu = 0 \text{ on } \Gamma\}.$$

Then $\inf(\mathcal{P}_2) = \sup(\mathcal{P}_2^*) \in I\!R$ and $(\mathcal{P}_2^*)$ has a solution $M \in \mathcal{K}$ [29]. This solution is unique if ϕ^* is strictly convex, which is equivalent to saying that $\phi \in C^1(I\!R)$, according to a result of Rockafellar [48].

Now let us write that u is a solution of $(\mathcal{P}_1)$, that M is a solution of $(\mathcal{P}_2^*)$ and that $\inf(\mathcal{P}_1) = \inf(\mathcal{P}_2) = \sup(\mathcal{P}_2^*)$ (the extremality relations):

$$\begin{aligned}\int_\Omega (Ku - u_0)^2 dx + \int_\Omega \phi(|\nabla u|) dx + c\int_{\Omega\setminus S_u} |C_u| + c\int_{S_u} (u^+ - u^-) d\mathcal{H}^{N-1} - \int_\Omega \xi u dx = \\ - \int_\Omega \Big(\frac{M_0^2}{4} - M_0 u_0\Big) dx - \int_\Omega \phi^*(|\bar{M}|) dx,\end{aligned}$$

where $M \in L^2(\Omega) \times L^\infty(\Omega)^N, |\bar{M}(x)| \leq c$, $K^* M_0 - \text{div}\bar{M} + \xi = 0$ in $\mathcal{D}'(\Omega)$ and $\bar{M} \cdot \nu = 0$ $d\Gamma-$a.e. on Γ.

Following Demengel-Temam [28], we can associate to u and $\bar{M}$ a bounded unsigned measure denoted $Du \cdot \bar{M}$ which is defined, as a distribution on Ω, by:

$$< Du \cdot \bar{M}, \psi > = - \int_\Omega u(\text{div}\bar{M})\psi dx - \int_\Omega \bar{M} \cdot (\nabla\psi) u dx, \ \forall \psi \in C_0^\infty(\Omega).$$

By the generalized Green's formula

$$\int_\Omega Du \cdot \bar{M} = - \int_\Omega u \cdot \text{div}\bar{M} + \int_\Gamma u(\bar{M} \cdot \nu) d\Gamma,$$

we get, since $\bar{M} \cdot \nu = 0$, $d\Gamma -$ a.e.

$$\begin{aligned}&\int_\Omega (Ku - u_0)^2 dx + \int_\Omega \phi(|\nabla u|) dx + c\int_{\Omega\setminus S_u} |C_u| + c\int_{S_u} (u^+ - u^-) d\mathcal{H}^{N-1} \\ &+ \int_\Omega M_0 K u + \int_\Omega \Big(\frac{M_0^2}{4} - M_0 u_0\Big) dx + \int_\Omega Du \cdot \bar{M} + \int_\Omega \phi^*(|\bar{M}|) dx = 0.\end{aligned}$$

Using the decomposition $Du = \nabla u dx + C_u + (u^+ - u^-)\nu d\mathcal{H}^{N-1}|_{S_u}$, where $C_u(S_u) = 0$, we finally have:

$$\begin{aligned}&\textstyle\int_\Omega (Ku - u_0)^2 dx + \int_\Omega M_0 K u dx + \int_\Omega \Big(\frac{M_0^2}{4} - M_0 u_0\Big) dx \\ &+ \textstyle\int_\Omega \Big(\phi(|\nabla u|) + \nabla u \cdot \bar{M} + \phi^*(|\bar{M}|)\Big) dx \\ &+ \textstyle\int_{\Omega\setminus S_u} (c|C_u| + \bar{M} C_u) + \int_{S_u} (u^+ - u^-)(c + \bar{M} \cdot \nu) d\mathcal{H}^{N-1} = 0.\end{aligned}$$

Now, we have the following:

1^o. $(Ku-u_0)^2-(-M_0)u_0+\left(\frac{(-M_0)^2}{4}+(-M_0)u_0\right)\geq 0$, by the definition of $\mathcal{G}_1^*$ and for $dx-$a.e. $x\in\Omega$;

2^o. $\phi(|\nabla u(x)|)+\bar{M}(x)\cdot\nabla u(x)+\phi^*(|\bar{M}(x)|)\geq\phi(|\nabla u(x)|)-|\bar{M}(x)|\cdot|\nabla u(x)|+\phi^*(|\bar{M}(x)|)\geq 0$, by the definition of ϕ^* and for $dx-$a.e. $x\in\Omega$, such that ∇u is defined.

3^o. $C_u<<|C_u|$ and there exists $h\in L^1(|C_u|)^N$ such that $|h|=1$ and $C_u=h|C_u|$ (the Radon-Nikodym theorem). Hence, we obtain that $c|C_u|+\bar{M}\cdot C_u=(C+\bar{M}\cdot h)|C_u|\geq 0$, because $|\bar{M}|\leq c$.

4^o. When u^+ and u^- are defined, we have: $u^+-u^-\geq 0$ and also $c+\bar{M}\cdot\nu\geq 0$.

We can now give a characterization of $\xi\in\partial\hat{F}(u)$:

Proposition 2.3.1 *Let $\xi\in L^{p'}(\Omega)$ and $u\in BV(\Omega)$, with $Du=\nabla u dx+C_u+J_u$ the decomposition of Du. Then $\xi\in\partial\hat{F}(u)$ if and only if there exists $M:\Omega\to I\!R^{N+1}$ with $M(x)=(M_0(x),\bar{M}(x))\in I\!R\times I\!R^N$, $|\bar{M}(\cdot)|\leq c$ and $M_0\in L^2(\Omega)$, such that*

$$\phi(|\nabla u|)+\nabla u\cdot\bar{M}+\phi^*(|\bar{M}|)=0\ dx-a.e.\quad x\in\Omega. \tag{2.3.1}$$

If

$$\mathcal{H}=\{x\in\Omega\backslash S_u: c+\bar{M}(x)\cdot h(x)=0, C_u=h|C_u|, h\in L^1(|C_u|)^N, |h|=1\},$$

then

$$supp(|C_u|)\subset\mathcal{H}, \tag{2.3.2}$$
$$c+\bar{M}\cdot\nu=0,\ |\bar{M}|=c\quad d\mathcal{H}^{N-1}-a.e.\ x\in S_u, \tag{2.3.3}$$
$$K^*M_0-div\bar{M}+\xi=0\quad in\ \ \mathcal{D}'(\Omega), \tag{2.3.4}$$
$$-M_0=2(Ku-u_0)\quad dx-a.e.\ in\ \Omega, \tag{2.3.5}$$
$$\bar{M}\cdot\nu=0\ d\Gamma-a.e.\ on\ \Gamma. \tag{2.3.6}$$

If in addition ϕ is differentiable, then we can compute $\bar{M}$ as:

$$\bar{M}(x)=-\frac{\phi'(|\nabla u(x)|)}{|\nabla u(x)|}\nabla u(x),\ dx-a.e.\ x\in\Omega,\ if\ |\nabla u(x)|\neq 0 \tag{2.3.7}$$

and $\bar{M}(x)=(0,...,0)$ if $|\nabla u(x)|=0$.

Finally, we have a characterization for a solution u of $\{\inf_{v\in L^p(\Omega)}\hat{F}(v)\}$ by taking $\xi=0$ and writing that $0\in\partial\hat{F}(u)$.

Proof.

The direct implication has just been proved. Conversely, if such an M exists, it is easy to check that M is a solution of $(\mathcal{P}_2^*)$ and u is a solution of $(\mathcal{P}_1)$, which amounts to saying that $\xi\in\partial\hat{F}(u)$.

Now, if ϕ is differentiable, we only show how we obtain the expression of M, the other results follow from before.

Let $x \in \Omega$ such that (2.3.1) is true at x and we denote by $\bar{M}_i(x)$ (and by $\nabla_i u(x)$), $i = 1, ..., N$, the components of $\bar{M}(x)$ ($\nabla u(x)$ respectively). We have the follows:

$$\phi^*(|-\bar{M}(x)|) = -\bar{M}(x) \cdot \nabla u(x) - \phi(|\nabla u(x)|) = \sup_{T \in I\!R^N} \{-\bar{M} \cdot T - \phi(|T|)\}.$$

If $|\nabla u(x)| \neq 0$, then for $T = \nabla u(x)$ we have that

$$\nabla_T(-\bar{M} \cdot T - \phi(|T|)) = (0, ..., 0),$$

which implies that

$$\bar{M}_i(x) = -\frac{\phi'(|\nabla u(x)|)}{|\nabla u(x)|} \nabla_i u(x).$$

We also deduce from 2^o and (2.3.1) that:

$$\phi(|\nabla u(x)|) - |\bar{M}(x)| \cdot |\nabla u(x)| + \phi^*(|\bar{M}(x)|) = 0.$$

Then

$$\phi^*(|\bar{M}(x)|) = |\bar{M}(x)| \cdot |\nabla u(x)| - \phi(|\nabla u(x)|) = \sup_{t \in I\!R} \{|\bar{M}(x)| \cdot t - \phi(t)\},$$

which implies that $|\bar{M}(x)| = \phi'(t)$ for $t = |\nabla u(x)|$. Hence, if $|\nabla u(x)| = 0$, then $\bar{M}(x) = (0, ..., 0)$, by using that $\phi'(0) = 0$. ■

Remark 2.3.2

Unfortunately, this functional is not lower semicontinuous on $SBV(\Omega)$, the space of special functions with bounded variation, introduced by De Giorgi, Ambrosio [26], and defined by:

$$SBV(\Omega) = \Big\{u \in BV(\Omega) : Du = \nabla u dx + J_u \Leftrightarrow C_u = 0\Big\}.$$

This fact is proved in Bouchitté-Braides-Buttazzo [7], and then we cannot say a-priori that the solution belongs to $SBV(\Omega)$, as in image segmentation, where the Mumford and Shah functional, defined by (see [43], [25])

$$MS(u) = \frac{1}{2} \int_\Omega |u - u_0|^2 dx + \int_\Omega |\nabla u|^2 dx + \int_{S_u} d\mathcal{H}^{N-1}, \tag{2.3.8}$$

is convex *and* lower semicontinuous on $SBV(\Omega)$. Moreover, the space $SBV(\Omega)$ is, maybe, more adapted than $BV(\Omega)$ to model the reconstructed images, which must satisfy the principle (P). In order to show that the solution to the minimization problem belongs to $SBV(\Omega)$, it is necessary for instance to prove that $\mathcal{H} = \emptyset$, which is equivalent to having: $|\bar{M}(x)| < c$ for all $x \notin S_u$, but for the moment we have no answer to this question.

Remark 2.3.3

The dual problem $(\mathcal{P}_2^*)$ with the solution $M = (M_0, \bar{M})$ can offer a new method to compute numerically the solution u, at least for the case $K = I$. Indeed, if we solve the following constrained minimization problem with the solution $\bar{M}$:

$$\inf_{\bar{M}:|\bar{M}(x)|\leq c} \Big\{ \int_\Omega \Big(\frac{(\mathrm{div}\bar{M})^2}{4} - (\mathrm{div}\bar{M})u_0 \Big) dx + \int_\Omega \phi^*(|\bar{M}|)dx \Big\},$$

then we can compute u by the relations: $M_0 = \mathrm{div}\bar{M}$, $2(u - u_0) = -M_0$, from Proposition 2.3.1. We have remarked that ϕ^* is strictly convex, if and only if $\phi \in C^1(I\!R)$, and this is true for the potentials ϕ_1, ϕ_2 and ϕ_3 defined in Chapter I. For instance, for ϕ_1 and $m \geq 0$, we have $\phi_1^*(m) = \frac{m^2}{2}$ (for $m \in [0,1]$), which is obviously a $C^2([0,1])$-function. On the contrary, for ϕ_2, we have (for $m \geq 0$) $\phi_2^*(m) = -\sqrt{1-m^2} + 1$, which is defined and regular only in $m \in [0,1)$ (it is not necessary for the problem to consider the case $m < 0$). For these two functions, the constant c is equal to one. This problem could be solved by a quasi-Newton method with constraints, but we will not consider and treat this possible approach here.

2.4 The evolution problem

In this section, we study the evolution equation associated to the stationary problem $0 \in \partial\hat{F}(u)$, in the particular cases of one and two dimensions, i.e. $N = 1$ and $N = 2$. In this way, $BV(\Omega)$ is continuously embedded in the Hilbert space $L^2(\Omega)$, this being necessary in order to apply general results on maximal monotone operators and evolution equations on Hilbert spaces ([11]). The function ϕ will satisfy the same assumptions as in the previous sections of this chapter. For the moment, we only assume that $K : L^2(\Omega) \to L^2(\Omega)$ is linear and continuous.

We can associate the following evolution problem to the minimization of the functional F (written in (2.0.1), finite only in $W^{1,1}(\Omega)$), for instance, if $\phi \in C^1(I\!R)$:

$$(Ev_1) \qquad \begin{cases} 0 = \dfrac{\partial u}{\partial t} + \mathcal{A}u & \text{in } \;]0,\infty[\times\Omega, \\ u(0,x) = u_0(x) & \text{for } \; x \in \Omega, \\ \dfrac{\phi'(|Du|)}{|Du|} Du \cdot \nu = 0 & \text{on } \; \partial\Omega, \end{cases}$$

where $u : [0,\infty) \times \Omega \to I\!R$ is the unknown function and $\mathcal{A}$ is the operator:

$$\mathcal{A}u = 2K^*(Ku - u_0) - \mathrm{div}\Big(\frac{\phi'(|\nabla u|)}{|\nabla u|}\nabla u\Big).$$

To study such an evolution problem, we wish to apply the nonlinear semigroup theory and the notion of maximal monotone operator ([11]).

But, unfortunately, the operator $\mathcal{A}$ is the subdifferential of the functional F, recalled here: $F : L^2(\Omega) \to I\!R \bigcup \{+\infty\}$,

$$F(u) = \begin{cases} \int_\Omega (Ku - u_0)^2 dx + \int_\Omega \phi(|\nabla u|)dx & \text{if } u \in W^{1,1}(\Omega) \\ +\infty \quad \text{if } u \in L^2(\Omega)\backslash W^{1,1}(\Omega), \end{cases}$$

which is not lower semicontinuous on $L^2(\Omega)$ and then $\mathcal{A}$ is not maximal monotone. To overcome this difficulty, as in the previous sections, we will consider the relaxed functional $\hat{F}$ of F on $L^2(\Omega)$:

$$F(u) = \begin{cases} \int_\Omega (Ku-u_0)^2dx + \int_\Omega \phi(|\nabla u|)dx + c|D_s u|(\Omega) \text{ if} \begin{cases} u \in BV(\Omega) \\ \text{and } Du = \nabla u dx + D_s u \end{cases} \\ +\infty \quad \text{if } u \in L^2(\Omega)\backslash BV(\Omega) \end{cases}$$

and the following evolution problem on $L^2(\Omega)$:

$$(Ev_2) \qquad \begin{cases} 0 \in \dfrac{\partial u}{\partial t} + \partial \hat{F}(u) & \text{on }]0,\infty[\times\Omega \\ u(0,x) = u_0(x) & \text{for } x \in \Omega. \end{cases}$$

It is easy to establish the following theorem, from the above relaxation results and a general result of an evolution equation governed by a maximal monotone operator.

Theorem 2.4.1 *Let $\Omega \subset \mathbb{R}^N$ be open, bounded and connected subset of $\mathbb{R}^N$ $(N=1,2)$ with Lipschitz boundary $\Gamma = \partial\Omega$. Let $u_0 \in Dom(\partial\hat{F})$. Then there exists a unique function $u(t) : [0,+\infty[\to L^2(\Omega)$ such that:*

$$(2.4.1) \qquad u(t) \in Dom(\partial\hat{F}) \ \ \forall t > 0, \quad \frac{\partial u}{\partial t} \in L^\infty((0,+\infty); L^2(\Omega)),$$

$$(2.4.2) \qquad -\frac{\partial u}{\partial t} \in \partial\hat{F}(u(t)), \quad a.e.\ t \in]0,+\infty[, \quad u(0) = u_0.$$

If $\hat{u}$ is a solution of $(2.4.1)-(2.4.2)$, with $\hat{u}_0$ instead of u_0, then:

$$(2.4.3) \qquad \|u(t) - \hat{u}(t)\|_{L^2(\Omega)} \leq \|u_0 - \hat{u}_0\|_{L^2(\Omega)}, \quad \forall t \geq 0.$$

Let $Du(\cdot,t) = \nabla u(\cdot,t)dx + D_s u(\cdot,t)$ be the Lebesgue decomposition of $Du(\cdot,t)$. Then, for almost every $t > 0$, there exists $M(t,\cdot) \in L^2(\Omega) \times L^\infty(\Omega)^N$, $M = (M_0, \bar{M}) = (M_0, M_1, ..., M_N)$ satisfying $(2.3.1), (2.3.2), (2.3.3), (2.3.5), (2.3.6)$ and, instead of $(2.3.4)$:

$$-\frac{du}{dt} + K^* M_0 - div\bar{M} = 0 \ \ in\ \mathcal{D}'(\Omega).$$

If, in addition, ϕ is differentiable, then $M(t,x)$ is given by $(2.3.7)$.

Proof. The functional $\hat{F}$ is clearly convex, proper and lower semicontinuous in $L^2(\Omega)$, from Remark 2.1.4. Then $\partial\hat{F}$ is maximal monotone and (2.4.1),(2.4.2) follow from the nonlinear semigroup theory (Thm. 3.1, [11]). The other conditions follow immediately from (2.4.2) and the characterization of $\partial\hat{F}$. ∎

Remark 2.4.2

For each $t > 0$, the map $u_0 \to u(t)$ is a contraction of $Dom(\partial\hat{F})$ in $Dom(\partial\hat{F})$. We denote by $S(t)$ its unique extension by continuity to a continuous nonexpansive semigroup on $\overline{Dom(\partial\hat{F})} = Dom\hat{F} = BV(\Omega)$ (see for instance [11], [62]). If $u_0 \in BV(\Omega)$, then $u(t) = S(t)u_0$ is called the generalized solution of (Ev_2). Moreover, $S(t)u_0 \in Dom(\partial\hat{F})$ for all $t > 0$, that is to say: the operator $\partial\hat{F}$ has a regularizing effect.

Behavior of the solutions as $t \to +\infty$

Now let ϕ and K satisfying all the assumptions $H1 - H4$ and $u_0 \in Dom(\partial\hat{F})$. Then the problem (Ev_2) has a unique solution $u(t) : [0; +\infty[\to L^2(\Omega)$, which satisfies (2.4.1)-(2.4.3) and we also know that $\hat{F} : L^2(\Omega) \to I\!R \cup \{+\infty\}$ has a unique minimum $\bar{u}$ on $L^2(\Omega)$, and of-course that $\bar{u} \in BV(\Omega)$.

We shall now prove, as in [39], that $u(t)$ converges strongly in $L^1(\Omega)$ and weakly in $L^2(\Omega)$ to $\bar{u}$ as $t \to \infty$.

First, we give a result of Bruck [12] which proves the weak convergence in $L^2(\Omega)$ to $\bar{u}$.

Proposition 2.4.3 *(Bruck [12]). Let H be a Hilbert space and $\mathcal{A}$ the subdifferential ∂F of a proper lower semicontinuous function $F : H \to]-\infty, +\infty]$ which assumes a minimum in H.*

If $u : [0, \infty[\to H$ is absolutely continuous and satisfies

$$u(t) \in Dom(\mathcal{A}) \quad \forall t \geq 0,$$
$$0 \in \frac{\partial u}{\partial t} + \mathcal{A}u \quad a.e.$$
$$\|\frac{\partial u}{\partial t}\|_H \in L^\infty(0, \infty),$$

then $u(t)$ has a weak limit $\bar{u}$ in H as $t \to \infty$ and $\bar{u}$ belongs to $\mathcal{A}^{-1}(0)$.

Theorem 2.4.4 *Let $u_0 \in Dom(\partial\hat{F})$. Then the solution u of (Ev_2) converges as $t \to \infty$ to the minimum $\bar{u}$ of $\hat{F}$ in the following sense:*

$$u(t) \to \bar{u} \text{ strongly in } L^1(\Omega) \text{ and weakly in } L^2(\Omega).$$

Proof.

The existence of a weak limit $\bar{u}$ in $L^2(\Omega)$ is a consequence of the existence result from Section 2.2, Theorem 2.4.1 and Proposition 2.4.3. It remains to show the strong convergence in $L^1(\Omega)$.

We prove that $\hat{F}(u(t))$ is uniformly bounded and therefore $u(t)$ will be uniformly bounded in $BV(\Omega)$.

From

$$-\frac{\partial u}{\partial t} \in \partial\hat{F}(u(t)),$$

by the definition of the subdifferential, we have

$$\hat{F}(u(t)) + \int_\Omega \frac{\partial u}{\partial t} u(t)dx \leq \hat{F}(v) + \int_\Omega \frac{\partial u}{\partial t} v dx, \ \forall v \in L^2(\Omega).$$

Let $v = \bar{u}$. Then

$$\hat{F}(u(t)) \leq \hat{F}(\bar{u}) + \int_\Omega \frac{\partial u}{\partial t}(\bar{u} - u(t))dx \leq \hat{F}(\bar{u}) + \|\frac{\partial u}{\partial t}\|_{L^2(\Omega)} \cdot \|\bar{u} - u(t)\|_{L^2(\Omega)}.$$

Now, $\hat{F}(\bar{u}) < \infty$ because $\bar{u} \in Dom(\hat{F}) = BV(\Omega)$, $\|\frac{\partial u}{\partial t}\|_{L^2(\Omega)}$ is uniformly bounded from (2.4.1) and also $\|\bar{u} - u(t)\|_{L^2(\Omega)}$, from the weak convergence.

Finally, as in Section 2.2, there is a subsequence $u(t_n)$ which converges in $L^1(\Omega)$ to a limit, which must to be $\bar{u}$. Moreover, all the sequence $u(t)$ converges strongly in $L^1(\Omega)$ to $\bar{u}$ (for instance, by contradiction). ∎

Remark 2.4.5

Unfortunately, we have obtained the strong convergence of $u(t)$ to $\bar{u}$ only in $L^1(\Omega)$, and not in $L^2(\Omega)$, since $Dom(\hat{F}) = BV(\Omega)$ is only continuously embedded in $L^2(\Omega)$. Generally, we could obtain the strong convergence in $L^2(\Omega)$ under the following assumption on $\hat{F}$: for each $C \geq 0$, the set $\{u \in L^2(\Omega) : \hat{F}(u) + \|u\|^2_{L^2(\Omega)} \leq C\}$ is strongly compact (see [11]), which is not true in our case.

Chapter 3

Approximations in continuous variables

This chapter is devoted to the study of convergence results in continuous variables for some algorithms, in order to solve the minimization problem, or the associated Euler equation.

In Section 3.1 we show, in the general $N-$ dimensional case, how we can approximate the solution to the minimization problem on the space $BV(\Omega)$, by a sequence of more regular functions (for instance, $H^1(\Omega)$-functions, if $N = 1, 2$), using the notion of $\Gamma-$ *convergence*.

In Section 3.2, we will show how we can associate to the image (the solution), assumed to be now more regular, an auxiliary variable (or dual) and how the regularizing term in the energy can be represented by the infimum of quadratic functions.

In Section 3.3, we present, in addition to the existence and uniqueness result from Chapter 2, Section 2.2, a regularity result, by using the dual variables.

Finally, in Section 3.4, we describe an algorithm based on dual variables, to approach the continuous solution, together with a convergence result in $H^1(\Omega)$.

3.1 Approximation by $\Gamma-$ *convergence*

In order to solve numerically the minimization problem (2.2.1), we first need to regularize it and to work on a more regular space than $BV(\Omega)$, because we do not know how to approximate directly in the energy the term

$$\int_\Omega |D_s u| = \int_{\Omega \setminus S_u} |C_u| + \int_{S_u} |u^+ - u^-| d\mathcal{H}^{N-1},$$

for $u \in BV(\Omega)$, and which is more complicated than the term $\int_{S_u} d\mathcal{H}^{N-1}$, which appears in the Mumford and Shah functional from image segmentation (see (2.3.8), and [25]; in this case, the solution u belongs to $SBV(\Omega)$, i.e. $C_u = 0$ and, moreover, the jump of u on S_u, $[u] := u^+ - u^-$, not appears in the functional).

Therefore, it is necessary to approach in some sense the functional $\hat{F}$ by a sequence $(F_\varepsilon)_{\varepsilon>0}$ of quadratic functionals, finite, lower semicontinuous and well-defined on a subspace of $W^{1,p}(\Omega)$ (we recall that $p=2$ if $N=1$ and $p=\frac{N}{N-1}$ if $N\geq 2$), and where the functions have the singular part of the gradient equal to zero.

There are many possibilities to construct the sequence $(F_\varepsilon)_{\varepsilon>0}$, and we consider here two cases. The most classical approximation and regularization is obtained by defining $\phi_{1\varepsilon}:I\!R^+\to I\!R^+$, $\phi_{1\varepsilon}(z)=\phi(z)+\varepsilon z^2$.

We can also approach and regularize the function ϕ, which now is assumed to be continuously differentiable on $]0,+\infty[$, in the following manner: let $\phi_{2\varepsilon}:I\!R^+\to I\!R^+$ being $(\varepsilon>0)$

$$\phi_{2\varepsilon}(z)=\begin{cases}\frac{\phi'(\varepsilon)}{2\varepsilon}z^2+\phi(\varepsilon)-\frac{\varepsilon\phi'(\varepsilon)}{2}, \text{ if } z\leq\varepsilon,\\ \phi(z), \text{ if } \varepsilon\leq z\leq\frac{1}{\varepsilon},\\ \frac{\varepsilon\phi'(\frac{1}{\varepsilon})}{2}z^2+\phi(\frac{1}{\varepsilon})-\frac{\phi'(\frac{1}{\varepsilon})}{2\varepsilon}, \text{ if } z\geq\frac{1}{\varepsilon}.\end{cases}\tag{3.1.1}$$

By the following assumption (as in (1.1.6))

$$]0,\infty[\ni z\mapsto\frac{\phi'(z)}{z} \text{ is continuously decreasing,}\tag{3.1.2}$$

we have that $\phi_{2\varepsilon}(z)\geq\phi(z)$, for all $z\geq 0$ (this is of-course true for the previous approximation $\phi_{1\varepsilon}$ of ϕ).

Now, choosing one of this two sequences $(\phi_{i\varepsilon})_{\varepsilon>0}$, we define the sequence $(F_{i\varepsilon})_{\varepsilon>0}$, $i=1,2$, by

$$F_{i\varepsilon}(u)=\begin{cases}\int_\Omega|Ku-u_0|^2dx+\int_\Omega\phi_{i\varepsilon}(|\nabla u|)dx, \text{ if } u\in W^{1,1}(\Omega),\ \nabla u\in L^2(\Omega),\\ +\infty, \text{ elsewhere.}\end{cases}$$

We also define

$$\bar{F}(u)=\begin{cases}F(u), \text{ if } u\in W^{1,1}(\Omega),\ \nabla u\in L^2(\Omega),\\ +\infty, \text{ elsewhere}\end{cases}$$

($\bar{F}$ is the restriction of F from Chapter 2 Section 2.1, to functions $u\in W^{1,1}(\Omega)$, with $\nabla u\in L^2(\Omega)$).

Sometimes, we will use the notation $(F_\varepsilon)_{\varepsilon>0}$ instead of $(F_{i\varepsilon})_{\varepsilon>0}$, F_ε being one of these two approximations.

From now on, we will suppose the assumptions $H1-H4$ from Chapter 2, Section 2.1. For the results concerning the sequence $(F_{2\varepsilon})_{\varepsilon>0}$, we need in addition to assume that $\phi\in C^1(0,+\infty)$ and (3.1.2).

Proposition 3.1.1 *For every $\varepsilon>0$, the functional F_ε has a unique minimum $u^\varepsilon\in W^{1,1}(\Omega)$, with $\nabla u^\varepsilon\in L^2(\Omega)$.*

Proof. Let u_n be a minimizing sequence for F_ε. Then $u_n\in W^{1,1}(\Omega)$, with $\nabla u_n\in L^2(\Omega)$, and there exists a constant $M>0$ such that:

$$\int_\Omega|Ku_n-u_0|^2dx\leq M,\ \|\nabla u_n\|_{L^2(\Omega)}\leq M,$$

from the construction of ϕ_ε. Then we prove, as in the existence result from Chapter 2, Section 2.2, that

$$\|u_n\|_{L^p(\Omega)} \leq M.$$

Then there is $u \in W^{1,1}(\Omega)$, with $\nabla u \in L^2(\Omega)$, and a subsequence of u_n, still called u_n, such that:

$$u_n \rightharpoonup u \text{ weakly in } L^p(\Omega), \ \nabla u_n \rightharpoonup \nabla u \text{ weakly in } L^2(\Omega).$$

Since ϕ_ε is convex and continuous, and $K : L^p(\Omega) \to L^2(\Omega)$ is linear and continuous, we obtain that

$$F_\varepsilon(u) \leq \liminf_{n\to\infty} F_\varepsilon(u_n),$$

that is to say: u is a minimum of F_ε, and this will be called u^ε. The uniqueness is deduced as in Chapter 2, Section 2.2. ∎

Now, to show that $(u^\varepsilon)_{\varepsilon>0}$ converges to the unique minimum of $\hat{F}$, we will use the notion of $\Gamma - convergence$ and its relation with the pointwise convergence, presented by Dal Maso in [24]).

Let X be a topological space. The set of all open neighborhoods of x in X will be denoted by $\mathcal{N}(x)$. Let (F_h) be a sequence of functions from X into $\overline{I\!R}$.

Definition 3.1.2

The $\Gamma -$ *lower limit* and the $\Gamma -$ *upper limit* of the sequence (F_h) are the functions from X into $\overline{I\!R}$ defined by

$$(\Gamma - \liminf_{h\to\infty} F_h)(x) = \sup_{U\in\mathcal{N}(x)} \liminf_{h\to\infty} \inf_{y\in U} F_h(y),$$
$$(\Gamma - \limsup_{h\to\infty} F_h)(x) = \sup_{U\in\mathcal{N}(x)} \limsup_{h\to\infty} \inf_{y\in U} F_h(y).$$

If there exists a function $F : X \to \overline{I\!R}$ such that $\Gamma - \liminf_{h\to\infty} F_h = \Gamma - \limsup_{h\to\infty} F_h = F$, then we write $F = \Gamma - \lim_{h\to\infty} F_h$ and we say that the sequence (F_h) $\Gamma -$ *converges* to F (in X) or that F is the $\Gamma -$ *limit* of (F_h) (in X).

We will also use the following two results from [24]:

Proposition 3.1.3 *If (F_h) is a decreasing sequence converging to F pointwise, then (F_h) $\Gamma -$ converges to the lower semicontinuous envelope of F in X, denoted by sc^-F.*

Corollary 3.1.4 *Suppose that (F_h) is equi-coercive and $\Gamma -$ converges to a function F, with a unique minimum point x_0 in X. Let (x_h) be a sequence in X such that x_h is a minimum for F_h in X for every $h \in I\!N$. Then (x_h) converges to x_0 in X and $(F_h(x_h))$ converges to $F(x_0)$.*

Proposition 3.1.5 *The sequence $(u^\varepsilon)_{\varepsilon>0}$ from Proposition 3.1.1 converges in $L^1(\Omega)$ to the unique minimum u of $\hat{F}$ and $F_\varepsilon(u^\varepsilon)$ converges to $\hat{F}(u)$.*

Proof. In our case, for $X = L^1(\Omega)$, we have that $F_\varepsilon(u) \searrow \bar{F}(u)$ as $\varepsilon \searrow 0$, for every $u \in W^{1,1}(\Omega)$, with $\nabla u \in L^2(\Omega)$. Of course, the sequence (F_ε) is equi-coercive in $L^1(\Omega)$, since $F_\varepsilon \geq \hat{F}$ for all $\varepsilon > 0$ and $\hat{F}$ is coercive in $L^1(\Omega)$. To apply the above results, we need to check that $\hat{F} = sc^-\bar{F}$ in $L^1(\Omega)$. We consider two steps.

1) *$\hat{F}$ is lower semicontinuous in $L^1(\Omega)$ with respect to the L^1-topology*

It is easy to verify this: let $u, u_n \in L^1(\Omega)$, such that $u_n \to u$ in $L^1(\Omega)$, as $n \to \infty$ and $\liminf_{n\to\infty} \hat{F}(u_n) < +\infty$. Then, as $\hat{F}(u_n)$ is bounded (or for a subsequence), we deduce that $u_n \in BV(\Omega)$ with $\|u_n\|_{BV(\Omega)}$ uniformly bounded. Then $u \in BV(\Omega)$, $u_n \rightharpoonup u$ in $L^p(\Omega)$ and $u_n \rightharpoonup u$ in $BV - w^*$, as $n \to \infty$. Finally, we have:

$$\hat{F}(u) \leq \liminf_{n\to\infty} \hat{F}(u_n),$$

that is: 1).

2) *$\hat{F}$ is the lower semicontinuous envelope of $\bar{F}$ in $L^1(\Omega)$, with respect to the L^1-topology*

From 1), it suffices to show that, for $u \in BV(\Omega)$, there exists a sequence $u_n \in W^{1,1}(\Omega)$, with $\nabla u_n \in L^2(\Omega)$, such that

$$u_n \to u \text{ in } L^1(\Omega) \text{ as } n \to \infty \text{ and } \hat{F}(u) = \liminf_{n\to\infty} \bar{F}(u_n).$$

Let $u \in BV(\Omega)$. From Remark 2.1.4, we have that $\hat{F}$ is the lower semicontinuous envelope in $L^p(\Omega)$, of its restriction $\hat{F}|_{W^{1,1}(\Omega)} = F$. Then there is a sequence $u_n \in W^{1,1}(\Omega)$, such that $u_n \to u$ in $L^p(\Omega)$ and

$$\hat{F}(u) = \lim_{n\to\infty} \hat{F}(u_n) = \lim_{n\to\infty} F(u_n).$$

Now, for each $u_n \in W^{1,1}(\Omega)$, there is a sequence $(u_n^k)_{k\in\mathbb{N}} \in W^{1,1}(\Omega) \cap C^\infty(\overline{\Omega})$, such that $u_n^k \to u_n$ in $W^{1,1}(\Omega)$, as $k \to \infty$ (see Evans-Gariepy [30]). In particular, we have that $\nabla u_n^k \in L^2(\Omega)$ and $u_n^k \to u_n$ in $L^p(\Omega)$, as $k \to \infty$, by the Sobolev embedding. Then, since the map $u \mapsto \int_\Omega \phi(|Du|)dx$ is a convex and continuous function from $W^{1,1}(\Omega)$ into $\mathbb{R}$ (see, for instance, Demengel-Temam [28]), and K is linear and continuous from $L^p(\Omega)$ into $L^2(\Omega)$, we deduce in addition that:

$$\hat{F}(u_n) = F(u_n) = \lim_{k\to\infty} \bar{F}(u_n^k).$$

Then, by a double approximation argument, we deduce that, for $u \in BV(\Omega)$, there exists $u^n \in W^{1,1}(\Omega)$, with $\nabla u^n \in L^2(\Omega)$, such that $u^n \to u$ in $L^p(\Omega)$ (and, in particular, in $L^1(\Omega)$), and:

$$\hat{F}(u) = \lim_{n\to\infty} \bar{F}(u^n),$$

and then 2) is proved.

In this way, we get that the sequence $(u^\varepsilon)_{\varepsilon>0}$ converges in $L^1(\Omega)$ to u, the unique minimum of $\hat{F}$ (equation (2.2.1)). ■

Now, to compute numerically u^ε, with $\varepsilon > 0$ small enough, we can use the Euler equation with the solution u^ε, the minimum of F_ε, which is now Gâteaux-differentiable at each point. But, unfortunately, the problem is still nonlinear. To overcome this difficulty, we will construct a sequence of functions $(u_n^\varepsilon)_{n\in\mathbb{N}}$, which will converges to u^ε, and u_n^ε will be the solution of a linear equation. In the following sections of this chapter, written in collaboration with Gilles Aubert (see [5]), we describe an algorithm in continuous variables to approach u^ε, solution of the Euler equation, by using the so-called dual variables. We will also present a regularity result of the minimization problem.

Remark 3.1.6

In this section, we have presented the results in the general $N-$ dimensional case. But, in practice, for signal and image reconstruction, we will consider $N=1$ or $N=2$. Then $p=2$ and the regularizing sequence u^ε will belong to $H^1(\Omega)$.

3.2 Auxiliary variable. Half-quadratic reduction

We will show in this section how we can associate to the image u an auxiliary variable (or dual) and how the regularizing term in the energy (1.1.1) can be represented by the infimum of quadratic functions. We recall that the energy $F_\alpha(u)$ is

$$F_\alpha(u) = \int_\Omega |Ku-u_0|^2 dxdy + \alpha \int_\Omega \phi(|Du|)dxdy, \tag{3.2.1}$$

the regularizing term being

$$L_\phi(u) = \int_\Omega \phi(|Du|)dxdy. \tag{3.2.2}$$

To develop this idea, we use the Fenchel-Legendre transform (Rockafellar [48] or Ekeland-Temam [29]). We recall that if $l(\xi)$ is a convex function of $I\!R^N$ into $I\!R$, then its Fenchel-Legendre transform (or polar) is the convex function $l^*(\xi^*)$ defined by

$$l^*(\xi^*) = \sup_{\xi \in I\!R^N} (\xi \cdot \xi^* - l(\xi));$$

($\xi \cdot \xi^*$ is the usual scalar product). This definition can be extended, without difficulty, on infinite dimensional spaces. Let Ω be an open set of $I\!R^N$, l a convex continuous function of $I\!R^N \to I\!R$, and for $u \in L^\gamma(\Omega)^N$, let the functional

$$L(u) = \int_\Omega l(u(x))dx.$$

Then the polar of L denoted L^* is defined on $L^{\gamma'}(\Omega)^N$, the dual space of $L^\gamma(\Omega)^N$, where $\frac{1}{\gamma} + \frac{1}{\gamma'} = 1$, by

$$L^*(u^*) = \sup_{u \in L^{\gamma'}(\Omega)^N} \Big\{ \int_\Omega u(x)u^*(x) - \int_\Omega l(u(x))dx \Big\}.$$

If, in addition, l is non negative or if l verifies an inequality of the type $l(\xi) \geq a(x) - b \cdot |\xi|^\eta_{I\!R^N}$, with $a(x) \in L^1(\Omega), b \geq 0$ and $\eta \in [1,\infty)$, and if there exists $u_0 \in L^\infty(\Omega)^N$ such that $L(u_0) < \infty$, then we can prove (see Ekeland-Temam [29], chap. IX), that $L^*(u^*)$ is written as:

$$L^*(u^*) = \int_\Omega l^*(u^*(x))dx.$$

Of course, we can reiterate the process and define

$$L^{**}(u) = \int_\Omega l^{**}(u(x))dx.$$

Since l is convex, we have $l(\xi) = l^{**}(\xi)$ and then $L^{**}(u) = L(u)$.

We use this notion of polarity in our problem with $N = 2, \gamma = \gamma' = 2$. Let, for $\xi, \xi^* \in I\!\!R^2$:

$$l(\xi) = \frac{|\xi|^2}{2} - \phi(|\xi|), \tag{3.2.3}$$

$$\psi(\xi^*) = l^*(\xi^*) - \frac{|\xi^*|^2}{2}, \tag{3.2.4}$$

as well as the functionals defined on $L^2(\Omega)^2$ by:

$$\begin{aligned} \Phi(u) &= \int_\Omega \phi(|u(x,y)|)dxdy \qquad \Big(= \int_\Omega \big(\frac{|u(x,y)|^2}{2} - l(u(x,y)) \big) dxdy \Big), && (3.2.5) \\ \Psi(b) &= \int_\Omega \psi(b(x,y))dxdy. && (3.2.6) \end{aligned}$$

The following theorem proves that Φ and Ψ are dual in a certain sense.

Theorem 3.2.1 *If ϕ (extended by parity on $I\!\!R$) verifies*
(H1) there exist constants $C > 0$ and $B \geq 0$,
$$C|z| - B \leq \phi(z) \leq C|z| + B, \forall z \in I\!\!R$$
(H5) the function $z \to \frac{z^2}{2} - \phi(z)$ is convex on $I\!\!R$,
then

$$\Phi(u) = \inf_{b \in L^2(\Omega)^2} \int_\Omega \big(\frac{|u-b|^2}{2} + \psi(b) \big) dxdy, \tag{3.2.7}$$

$$\Psi(b) = \sup_{u \in L^2(\Omega)^2} \int_\Omega \big(- \frac{|u-b|^2}{2} + \phi(|u|) \big) dxdy. \tag{3.2.8}$$

Proof. We prove (3.2.7). Let $\rho(u)$ be the value of the infimum in (3.2.7):

$$\rho(u) = \int_\Omega \frac{|u|^2}{2} dxdy + \inf_b \int_\Omega \big(\frac{|b|^2}{2} - b \cdot u + \psi(b) \big) dxdy.$$

This can be also written, with (3.2.4), as:

$$\begin{aligned} \rho(u) &= \int_\Omega \frac{|u|^2}{2} dxdy + \inf_b \int_\Omega \big(\frac{|b|^2}{2} - b \cdot u + l^*(b) - \frac{|b|^2}{2} \big) dxdy \\ &= \int_\Omega \frac{|u|^2}{2} dxdy - \sup_b \int_\Omega \big(b \cdot u - l^*(b) \big) dxdy. \end{aligned}$$

Then, (by Ekeland-Temam [29]):

$$\rho(u) = \int_\Omega \frac{|u|^2}{2} dxdy - \int_\Omega l^{**}(u) dxdy.$$

From (H5) we have that $l^{**}(\xi) = l(\xi)$, $\forall \xi \in I\!\!R^2$, hence

$$\rho(u) = \int_\Omega \frac{|u|^2}{2} dxdy - \int_\Omega l(u) dxdy = \Phi(u).$$

(3.2.8) can be proved in the same manner. ∎

In Section 3.3, we want to seek a solution u on the space $V = \Big\{u \in L^2(\Omega), Du \in L^1(\Omega)^2\Big\}$. But in order to use the duality, we will assume for the moment that $u \in H^1(\Omega)$.

By using the relation (3.2.7), $F_\alpha(u)$ is written, for $u \in H^1(\Omega)$, as

$$F_\alpha(u) = \int_\Omega (Ku - u_0)^2 dxdy + \alpha \inf_{b \in L^2(\Omega)^2} \int_\Omega \Big(\frac{|Du - b|^2}{2} + \psi(b)\Big) dxdy$$

and then:

$$\inf_{u \in H^1(\Omega)} F_\alpha(u) = \inf_{u \in H^1(\Omega)} \inf_{b \in L^2(\Omega)^2} \Big[\int_\Omega (Ku - u_0)^2 dxdy + \alpha \int_\Omega \Big(\frac{|Du - b|^2}{2} + \psi(b)\Big) dxdy\Big].$$

Because we can always invert the infinima, we get:

$$\inf_{u \in H^1(\Omega)} F_\alpha(u) = \inf_{b \in L^2(\Omega)^2} \Big[\alpha \int_\Omega \psi(b) dxdy + \inf_{u \in H^1(\Omega)} \int_\Omega \Big((Ku - u_0)^2 + \alpha \frac{|Du - b|^2}{2}\Big) dxdy\Big]$$

and the method is now clear: we fix $b \in L^2(\Omega)^2$, and we solve the problem:

$$(\mathcal{P}_b) \qquad \inf_{u \in H^1(\Omega)} \Big[\int_\Omega \Big((Ku - u_0)^2 + \alpha \frac{|Du - b|^2}{2}\Big) dxdy\Big].$$

If K satisfies the previous assumptions (H2), (H3) and (H4), then $(\mathcal{P}_b)$ has an unique solution, which is, formally, solution of the Euler equation:

$$(3.2.9) \qquad \begin{cases} -\frac{\alpha}{2} \triangle u_b + K^* K u = K^* u_0 - \frac{\alpha}{2} \mathrm{div} b, & \text{in } \mathcal{D}'(\Omega), \\ \\ \frac{\partial u_b}{\partial \eta} = 0, & \text{on } \partial\Omega. \end{cases}$$

We have then, for all $v \in H^1(\Omega)$ and for all b:

$$(3.2.10) \int_\Omega \Big((u_0 - Ku_b)^2 + \frac{\alpha}{2} |Du_b - b|^2\Big) dxdy \leq \int_\Omega \Big((u_0 - Kv)^2 + \alpha \frac{|Dv - b|^2}{2}\Big) dxdy.$$

By adding $\int_\Omega \psi(b) dxdy$ on each both side of (3.2.10), and by passing to the infimum in b, we get, for all $v \in H^1(\Omega)$,

$$(3.2.11) \qquad \inf_{b \in L^2(\Omega)^2} \Big[\int_\Omega \Big((u_0 - Ku_b)^2 + \frac{\alpha}{2} |Du_b - b|^2 + \alpha\psi(b)\Big) dxdy \leq \int_\Omega \Big((u_0 - Kv)^2 + \alpha\phi(|Dv|)\Big) dxdy.$$

Denoting by

$$T(b) = \int_\Omega \Big((u_0 - Ku_b)^2 + \frac{\alpha}{2} |Du_b - b|^2 + \alpha\psi(b)\Big) dxdy,$$

it is then sufficient, in order to prove that our algorithm allows us to solve the initial reconstruction problem in, to obtain the existence of $b_0 \in L^2(\Omega)^2$, such that: $b_0 \in L^2(\Omega)^2$, such that:

$$(3.2.12) \qquad \begin{aligned} T(b_0) &= \inf_{b \in L^2(\Omega)^2} T(b), \text{ with} \\ T(b_0) &= \int_\Omega \Big((u_0 - Ku_{b_0})^2 + \alpha\phi(|Du_{b_0}|)\Big) dxdy = F_\alpha(u_{b_0}). \end{aligned}$$

We will then deduce, with (3.2.11), that

$$F_\alpha(u_{b_0}) \leq F_\alpha(v), \qquad \forall v \in H^1(\Omega). \tag{3.2.13}$$

At this stage we must precisely formulate the mathematical assumptions in order to ensure the existence and uniqueness of a solution in V. There is a difficulty which is due to the fact that we work with sets constructed from the *non reflexive* Banach space $L^1(\Omega)$.

3.3 A regularity result

For simplifying, we will suppose that $K = I$ on $L^2(\Omega)$ (which corresponds to a denoising problem). We also suppose that the weighting parameter α is equal to 1, which does not modify the theoretical study of the problem (its presence and adjustment are fundamental in the applications). The studied functional is therefore

$$F(u) = \int_\Omega (u - u_0)^2 dxdy + \int_\Omega \phi(|Du|)dxdy. \tag{3.3.1}$$

The basic assumptions which we will suppose verified along this section are as follows:

$$u_0 \in L^\infty(\Omega) \text{ and } 0 \leq u_0(x,y) \leq 1, \text{ a.e. } (x,y) \in \Omega \tag{3.3.2}$$

(3.3.3) $\phi : I\!R \to I\!R$ is even, of class C^2, non decreasing on $I\!R^+$ and there exist constants

$$C > 0, B \geq 0, \text{ such that } C|z| - B \leq \phi(z) \leq C|z| + B, \forall z \in I\!R,$$

$$0 < \phi''(t) < 1, \forall t \in I\!R. \tag{3.3.4}$$

Remark 3.3.1

From (3.3.4), the functions $\phi(t)$ and $\dfrac{t^2}{2} - \phi(t)$ are strictly convex (i.e. the hypothesis (H5) which is strengthened).

Thanks to (3.3.4), with the notations of the previous section, $F(u)$ can be written, for $u \in H^1(\Omega)$, as

$$F(u) = \inf_{b \in L^2(\Omega)^2} \int_\Omega (u - u_0)^2 + \int_\Omega \Big(\frac{|b - Du|^2}{2} + \psi(b)\Big)dxdy.$$

Proposition 3.3.2 *For fixed b in $L^2(\Omega)^2$ and for u_0 satisfying (3.3.2), the problem*

$$\inf_{u \in H^1(\Omega)} \int_\Omega \Big((u - u_0)^2 + \frac{|b - Du|^2}{2}\Big)dxdy \tag{3.3.5}$$

has an unique solution $u_b \in H^1(\Omega)$ verifying the Euler equation

$$- \triangle u_b + 2u_b = 2u_0 - div b \qquad in \mathcal{D}'(\Omega). \tag{3.3.6}$$

Proof. The functional

$$F_b(u) = \int_\Omega \Big((u-u_0)^2 + \frac{|b-Du|^2}{2}\Big)dxdy$$

being continuous, strictly convex and coercive on $H^1(\Omega)$, then, by the classical theory of calculus of variations, there exists an unique $u_b \in H^1(\Omega)$ such that:

$$F_b(u_b) \le F_b(u), \qquad \forall u \in H^1(\Omega), \tag{3.3.7}$$

which is equivalent, writing that u_b is a stationary point of F_b, to

$$2\int_\Omega (u_b - u_0)v dxdy + \int_\Omega (Du_b - b)\cdot Dv dxdy = 0, \qquad \forall v \in H^1(\Omega).$$

Then we obtain (3.3.6), choosing $v \in \mathcal{D}(\Omega)$. ■

Remark 3.3.3

For the moment, we will not include in (3.3.6) the usual condition on the boundary $\partial\Omega$ of Ω: $\dfrac{\partial u}{\partial \eta}(x) = 0$, the H^1-regularity of u_b being not sufficient to define the value on the boundary of the normal derivative.

Hence, we have for all $u \in H^1(\Omega)$, and fixed b,

$$\int_\Omega (u_b-u_0)^2 dxdy + \int_\Omega \frac{|b-Du_b|^2}{2} dxdy \le \int_\Omega (u-u_0)^2 dxdy + \int_\Omega \frac{|b-Du|^2}{2} dxdy, \tag{3.3.8}$$

and by adding $\psi(b)$ on each both side of (3.3.8) and taking the infimum in b, for all $u \in H^1(\Omega)$, we obtain

$$\inf_{b\in L^2(\Omega)^2} \int_\Omega \Big((u_b-u_0)^2 + \frac{|b-Du_b|^2}{2} + \psi(b)\Big)dxdy \le \int_\Omega ((u-u_0)^2 + \phi(|Du|))dxdy = F(u). \tag{3.3.9}$$

We recall that

$$T(b) = \int_\Omega \Big((u_b-u_0)^2 + \frac{1}{2}|b-Du_b|^2 + \psi(b)\Big)dxdy.$$

Now, we must prove that the problem $\inf_{b\in L^2(\Omega)^2} T(b)$ has a solution b_0, which will involve the existence of a function u_0, solution of the initial problem

$$F(u_0) \le F(u), \qquad \forall u \in V.$$

First of all, we will state some properties of the dual function ψ.

Lemma 3.3.4 *If ϕ verifies* (3.3.3) *and* (3.3.4), *then the function ψ defined by (3.2.4) has the following properties:*

$$\xi^* \to \psi(\xi^*) \textit{ is strictly convex,} \tag{3.3.10}$$

$$\textit{there exist constants } C' > 0 \textit{ and } B' \ge 0 \textit{ that} \tag{3.3.11}$$
$$C'|\xi^*| - B' \le \psi(\xi^*) \le C'|\xi^*| + B', \qquad \forall \xi^* \in I\!R^2.$$

Proof. We recall the definition of $\psi(\xi^*)$. If $l(\xi)$ denotes the strictly convex function (from (3.3.4))

$$l(\xi) = \frac{|\xi|^2}{2} - \phi(|\xi|),$$

then $\psi(\xi^*)$ is defined by

$$\psi(\xi^*) = l^*(\xi^*) - \frac{|\xi^*|^2}{2};$$

(l^* denotes the Fenchel-Legendre transform of l).

We prove (3.3.11):

$$\begin{aligned}
\psi(\xi^*) &= \sup_{\xi} \Big(\xi^* \cdot \xi - l(\xi)\Big) - \frac{|\xi^*|^2}{2} \\
&= \sup_{t\geq 0} \sup_{|\xi|=t} \Big(\xi^* \cdot \xi - \frac{|\xi|^2}{2} + \phi(|\ \xi|)\Big) - \frac{|\xi^*|^2}{2} \\
&= \sup_{t\geq 0} \Big(t|\xi^*| - \frac{t^2}{2} + \phi(t)\Big) - \frac{|\xi^*|^2}{2}
\end{aligned}$$

and since ϕ is even,

$$\psi(\xi^*) = \sup_{t\in I\!R} \Big(t|\xi^*| - \frac{t^2}{2} + \phi(t)\Big) - \frac{|\xi^*|^2}{2}.$$

Hence, with (3.3.3), we have

$$\begin{aligned}
-B - \frac{|\xi^*|^2}{2} + \sup_{z\in I\!R} \Big(C|z| + |\xi^*|z - \frac{z^2}{2}\Big) \leq \psi(\xi^*) \\
\leq B - \frac{|\ \xi^*\ |^2}{2} + \sup_{z\in I\!R} \Big(C|z| + |\xi^*|z - \frac{z^2}{2}\Big).
\end{aligned} \tag{3.3.12}$$

The supremum in the right-hand-side of (3.3.12) is achieved for $\overline{z} = C + |\xi^*|$ and its value is $\frac{1}{2}(C + |\xi^*|)^2$; hence, with (3.3.12),

$$\psi(\xi^*) \leq B - \frac{|\xi^*|^2}{2} + \frac{1}{2}(C + |\xi^*|)^2 = B + \frac{1}{2}C^2 + C_2 \mid \xi^*|,$$

from which we obtain the second inequality of (3.3.11), with $C' = C$ and $B' = B + \frac{1}{2}C^2$. The first inequality of (3.3.11) can be proved in the same way. The proof of (3.3.10) follows from (3.3.4) and by a classical argument of convex analysis. From (3.3.4), we have that the function $\xi \to l(\xi)$ is strictly convex, hence, $l^*(\xi^*)$ is of class C^2 (by Rockafellar [48], Dacorogna [23]) and

$$\begin{aligned}
\nabla\psi(\xi^*) &= \nabla l^*(\xi^*) - \xi^*, \\
\nabla^2\psi(\xi^*) &= \nabla^2 l^*(\xi^*) - I.
\end{aligned}$$

Moreover, since $l(\xi)$ is strictly convex, $\nabla l(\xi)$ is strictly monotonic and then, for each $\xi^* \in I\!R^2$, there is an unique $\xi_0 \in I\!R^2$, such that $\xi^* = \nabla l(\xi_0)$, or equivalently $\nabla l^*(\xi^*) = \xi_0$. Hence, we get

$$\nabla\psi(\xi^*) = \xi_0 - \nabla l(\xi_0) = -\nabla(\phi(|\xi|))_{\xi=\xi_0} = -\frac{\phi'(|\xi_0\ |)}{|\xi_0|}\xi_0 \tag{3.3.13}$$

and (see Crouzeix [22], for computational details)

$$\nabla^2\psi(\xi^*) = \left(\nabla^2 l(\xi_0)\right)^{-1} - I, \tag{3.3.14}$$

which can be also written, from the definition of $l(\xi)$ as:

$$\nabla^2\psi(\xi^*) = \left(I - \nabla^2(\phi(|\xi|)_{\xi=\xi_0}\right)^{-1} - I.$$

Thanks to (3.3.4), it is then clear that the matrix $\nabla^2\psi(\xi^*)$ is symmetric and positive-definite, consequently ψ is strictly convex. ■

Remark 3.3.5

In the Lemma 3.3.4, ξ_0 is, in fact, the unique point realizing the supremum $\sup_\xi(\xi\cdot\xi^* - l(\xi)) = l^*(\xi^*)$.

Provided with the properties of the function ψ, we can return now to the study of the problem (3.3.9):

$$\inf_{b\in L^2(\Omega)^2}\left\{T(b) = \int_\Omega \left((u_b-u_0)^2 + \frac{|b-Du_b|^2}{2} + \psi(b)\right)dxdy\right\}.$$

If b_n is a minimizing sequence, then it is simple to deduce from (3.3.8) and (3.3.11) that b_n and u_{b_n} verify the estimates:

$$\|u_{b_n}\|_{L^2(\Omega)} \le c,$$
$$\|b_n\|_{L^1(\Omega)^2} \le c,$$

where c is a constant which only depends on the data. But, we cannot obtain an $H^1(\Omega)$-estimate for u_{b_n} and a $L^2(\Omega)^2$-estimate for b_n, hence we must work on the non-reflexive space $L^1(\Omega)$ or on $\mathcal{M}_b(\Omega)$, the space of bounded measures. To overcome this difficulty, we regularize the problem, by making a slight modification on the potential ϕ. We introduce the function

$$\phi_\varepsilon(z) = \phi(z) + \frac{\varepsilon}{2}z^2, \quad \varepsilon > 0,$$

at which we associate

$$l_\varepsilon(\xi) = \frac{|\xi|^2}{2} - \phi_\varepsilon(|\xi|),$$
$$\psi_\varepsilon(\xi^*) = l_\varepsilon^*(\xi) - \frac{|\xi|^2}{2}.$$

The function ψ_ε has the same properties like ψ, if we modify and replace (3.3.4) by

$(3.3.4)_\varepsilon$ There is ε_0, with $0 < \varepsilon_0 < 1$ such that $0 < \phi''(t) < 1 - \varepsilon_0$, for all $t \in I\!R$.

The assumption $(3.3.4)_\varepsilon$ is not restrictive, because we can always change the weighting parameter in the energy $F_\alpha(u)$, to have $(3.3.4)_\varepsilon$ verified.

By proceeding as in Lemma 3.3.4, it's easy to see , for $\varepsilon \leq \varepsilon_0$, that

$$(3.3.10)_\varepsilon \qquad \xi^* \to \psi_\varepsilon(\xi^*) \text{ is strictly convex,}$$

$$(3.3.11)_\varepsilon \qquad -B + \frac{C^2}{2(1-\varepsilon)} + \frac{C}{1-\varepsilon}|\xi^*| + \frac{\varepsilon}{2(1-\varepsilon)}|\xi^*|^2 \leq \psi_\varepsilon(\xi^*)$$

$$\leq B + \frac{C^2}{2(1-\varepsilon)} + \frac{a_2}{1-\varepsilon}|\xi^*| + \frac{\varepsilon}{2(1-\varepsilon)}|\xi^*|^2,$$

and the regularized problem associated to $T(b)$ is

$$(3.3.15) \qquad \inf_{b\in L^2(\Omega)^2} \Big\{ T_\varepsilon(b) = \int_\Omega \Big((u_b - u_0)^2 + \frac{|b - Du_b|^2}{2} + \psi_\varepsilon(b) \Big) dxdy \Big\},$$

for which we state the following proposition:

Proposition 3.3.6 *Under the assumptions (3.3.2), (3.3.3) and $(3.3.4)_\varepsilon$, the problem (3.3.15) has an unique solution b_ε and there exists a constant c, independent of ε such that*

$$(3.3.16) \qquad \begin{aligned} \varepsilon\|b_\varepsilon\|^2_{L^2(\Omega)^2} &\leq c, \\ \varepsilon\|Du_{b_\varepsilon}\|^2_{L^2(\Omega)^2} &\leq c, \\ \|b_\varepsilon\|_{L^1(\Omega)^2} &\leq c, \\ \|u_{b_\varepsilon}\|_{L^2(\Omega)} &\leq c. \end{aligned}$$

Proof. The functional $T_\varepsilon(b)$ is strictly convex, since, from (3.3.6), the map $b \to u_b$ is affine from $L^2(\Omega)$ to $H^1(\Omega)$ and the function ψ_ε is strictly convex (from $(3.3.10)_\varepsilon$).

Moreover, from $(3.3.11)_\varepsilon$, for each $b \in L^2(\Omega)^2$, there are some coefficients $C' > 0$ and B', such that

$$\frac{\varepsilon}{2}\|b\|^2_{L^2(\Omega)^2} + C'\|b\|_{L^1(\Omega)^2} - B' \leq T_\varepsilon(b),$$

hence, for fixed ε, the minimizing sequences b_ε^n from (3.3.15) are bounded in $L^2(\Omega)^2$; therefore, there exist $b_\varepsilon \in L^2(\Omega)^2$ and a subsequence denoted also by b_ε^n such that $b_\varepsilon^n \overset{w}{\rightharpoonup} b_\varepsilon$ in $L^2(\Omega)$-weak. Thanks to the strictly convexity of T_ε, b_ε is unique, all the sequence b_ε^n converges to b_ε and

$$T_\varepsilon(b_\varepsilon) \leq \lim_{n\to\infty} T_\varepsilon(b_\varepsilon^n) = \inf_b T_\varepsilon(b) \leq T_\varepsilon(b), \qquad \forall b \in L^2(\Omega)^2,$$

that is to say b_ε is the unique solution of (3.3.15):

$$(3.3.17) \qquad \int_\Omega \Big((u_{b_\varepsilon} - u_0)^2 + \frac{|b_\varepsilon - Du_{b_\varepsilon}|^2}{2} + \psi_\varepsilon(b_\varepsilon) + \frac{\varepsilon}{2}|b_\varepsilon|^2 \Big) dxdy$$

$$\leq \int_\Omega \Big((u_b - u_0)^2 + \frac{|b - Du_b|^2}{2} + \psi_\varepsilon(b) + \frac{\varepsilon}{2}|b|^2 \Big) dxdy, \quad \forall b \in L^2(\Omega)^2.$$

Choosing , for example, $b = 0$ in (3.3.17), it is clear that there is a constant c, independent of ε, such that (3.3.16) is verified. ∎

The following theorem examines the optimality condition satisfied by b_ε.

Theorem 3.3.7 *The solution b_ε of the problem (3.3.15) verifies the optimality condition*

$$(1+\varepsilon)b_\varepsilon + D\psi_\varepsilon(b_\varepsilon) - Du_{b_\varepsilon} = 0, \qquad a.e.\ (x,y)\in\Omega. \tag{3.3.18}$$

Proof. To simplify the notations, we denote $u_\varepsilon = u_{b_\varepsilon}$; then let us consider a variation of b_ε of the form $b_\theta = b_\varepsilon + \theta q$, where $\theta \in I\!R$ and $q \in L^2(\Omega)^2$. Denoting $u_\theta = u_{b_\theta}$, it is clear, thanks to the linearity of formula (3.3.6) that

$$u_\theta = u_\varepsilon + \theta h, \tag{3.3.19}$$

where h verifies

$$-\triangle h + 2h = -\text{div} q \ \text{ in } \ H^1(\Omega)' \quad (\text{ the dual of } H^1(\Omega)). \tag{3.3.20}$$

With this remark

$$\frac{T_\varepsilon(b_\theta) - T_\varepsilon(b_\varepsilon)}{\theta} =$$

$$\int_\Omega (2u_\varepsilon - 2u_0 + \theta h)h\,dxdy + \int_\Omega (2b_\varepsilon - 2Du_\varepsilon + \theta(q - Dh))\cdot(q - Dh)\,dxdy \tag{3.3.21}$$

$$+\frac{1}{\theta}\int_\Omega (\psi(b_\varepsilon + \theta q) - \psi(b_\varepsilon))\,dxdy.$$

Now, as $\theta \to 0$, the sum of the two first integrals converges to

$$2\int_\Omega \Big((u_\varepsilon - u_0)h + (b_\varepsilon - Du_\varepsilon)\cdot(q - Dh)\Big)dxdy.$$

For the third integral, according to a result of convex analysis of R. Tahraoui [56], we have from $(3.3.10)_\varepsilon$ and $(3.3.11)_\varepsilon$, that there exist two constants $a(\varepsilon), b(\varepsilon) \geq 0$, such that

$$|\nabla\psi_\varepsilon(b)| \leq a(\varepsilon)|b| + b(\varepsilon),\ \ \forall b \in I\!R^2. \tag{3.3.22}$$

Hence, thanks to (3.3.16) and the Lebesgue's dominated convergence theorem, we can pass to the limit in the third integral and we obtain

$$\lim_{\theta\to 0} \tfrac{1}{\theta}(T_\varepsilon(b_\theta - T_\varepsilon(b_\varepsilon)) \tag{3.3.23}$$
$$= 2\textstyle\int_\Omega \Big((u_\varepsilon - u_0)h + (b_\varepsilon - Du_\varepsilon)\cdot(q - Dh) + D\psi_\varepsilon(b_\varepsilon)q\Big)dxdy.$$

But with the Euler equation of (3.3.7)

$$2\int_\Omega (u_\varepsilon - u_0)h\,dxdy = -\int_\Omega (Du_\varepsilon - b_\varepsilon)\cdot Dh\,dxdy. \tag{3.3.24}$$

Therefore, since b_ε is a critical point of T, for all $q \in L^2(\Omega)^2$:

$$\lim_{\theta\to 0}\frac{1}{\theta}(T_\varepsilon(b_\theta) - T_\varepsilon(b_\varepsilon)) = \int_\Omega \Big((b_\varepsilon - Du_\varepsilon) + D\psi_\varepsilon(b_\varepsilon)\Big)q\,dxdy = 0,$$

which implies the relation we wanted to prove:

$$b_\varepsilon - Du_\varepsilon + D\psi_\varepsilon(b_\varepsilon) = 0. \tag{3.3.25}$$

∎

In the following corollary, we express $D\psi_\varepsilon(b_\varepsilon)$ in term of Du_ε and $\phi'(|Du_\varepsilon|)$.

Corollary 3.3.8 *The optimality condition* (3.3.25) *can be written as:*

$$b_\varepsilon = \Big((1-\varepsilon) - \frac{\phi'(|Du_\varepsilon|)}{|Du_\varepsilon|}\Big) Du_\varepsilon. \tag{3.3.26}$$

Proof. We recall that

$$\psi_\varepsilon(\xi^*) = l^*_\varepsilon(\xi^*) - \frac{|\xi^*|^2}{2},$$

where

$$l_\varepsilon(\xi) = \frac{|\xi|^2}{2} - \phi(|\xi|) - \frac{\varepsilon}{2}|\xi|^2.$$

Thanks to (3.3.3) and $(3.3.4)_\varepsilon$, the function l_ε is strictly convex, hence l^*_ε and ψ_ε are differentiable and

$$D\psi_\varepsilon(\xi^*) = Dl^*_\varepsilon(\xi^*) - \xi^*. \tag{3.3.27}$$

But,

$$l^*_\varepsilon(\xi^*) = \sup_\xi \Big(\xi^* \cdot \xi - l_\varepsilon(\xi)\Big)$$

then $Dl^*_\varepsilon(\xi^*) = \xi_\varepsilon$, where ξ_ε is the unique point realizing the $\sup_\xi\Big(\xi^* \cdot \xi - l_\varepsilon(\xi)\Big)$, that is to say: $\xi^* - Dl_\varepsilon(\xi_\varepsilon) = 0$, or

$$\xi^* = (1-\varepsilon)\xi_\varepsilon - \xi_\varepsilon \frac{\phi'(|\xi_\varepsilon|)}{|\xi_\varepsilon|} = 0. \tag{3.3.28}$$

Denoting by

$$\mathcal{L}_\varepsilon(\xi) = (1-\varepsilon)\xi - \xi\frac{\phi'(|\xi|)}{|\xi|},$$

thanks to $(3.3.4)_\varepsilon$, $\mathcal{L}_\varepsilon$ is invertible and (3.3.28) is equivalent to

$$\xi_\varepsilon = \mathcal{L}^{-1}(\xi^*), \text{ and } D\psi_\varepsilon(\xi^*) = \mathcal{L}^{-1}(\xi^*) - \xi^*.$$

With the optimality condition, we have the following sequence of equalities:

$$\begin{gathered}
D\psi_\varepsilon(b_\varepsilon) + b_\varepsilon - Du_\varepsilon = 0,\\
\mathcal{L}^{-1}(b_\varepsilon) - b_\varepsilon + b_\varepsilon - Du_\varepsilon = 0,\\
\mathcal{L}^{-1}(b_\varepsilon) = Du_\varepsilon,\\
b_\varepsilon = \mathcal{L}(Du_\varepsilon) = (1-\varepsilon)Du_\varepsilon - Df_\varepsilon\tfrac{\phi'(|Du_\varepsilon|)}{|Du_\varepsilon|},\\
b_\varepsilon = \Big((1-\varepsilon) - \tfrac{\phi'(|Du_\varepsilon|)}{|Du_\varepsilon|}\Big)Du_\varepsilon.
\end{gathered}$$

■

Now, to prove the existence of a solution for the initial problem (3.2.13), it remains to study the behavior of u_ε and b_ε when $\varepsilon \to 0$. The system linking f_ε and b_ε consists of two equations, namely: (3.3.6) and (3.3.26).

From (3.3.26), we have

$$\mathrm{div} b_\varepsilon = (1-\varepsilon) \triangle u_\varepsilon - \mathrm{div}\Big(\frac{\phi'(|Du_\varepsilon|)}{|Du_\varepsilon|} Du_\varepsilon\Big).$$

Putting this equality in (3.3.6), we get:

$$\varepsilon \triangle u_\varepsilon + \mathrm{div}\Big(\frac{\phi'(|Du_\varepsilon|)}{|Du_\varepsilon|} Du_\varepsilon\Big) = 2(u_\varepsilon - u_0). \tag{3.3.29}$$

Then, (3.3.29) is exactly the Euler equation associated to the problem

$$\inf\Big\{F_\varepsilon(u); u \in H^1(\Omega)\Big\},$$

where

$$F_\varepsilon(u) = \int_\Omega (u-u_0)^2 dxdy + \int_\Omega \phi(|Du|)dxdy + \frac{\varepsilon}{2}\int_\Omega |Du|^2 dxdy. \tag{3.3.30}$$

Otherwise, we remark, thanks to (3.3.26), that

$$\begin{aligned} F_\varepsilon(u_\varepsilon) &= \int_\Omega (u_0-u_\varepsilon)^2 dxdy + \int_\Omega \Big(\phi(|Du_\varepsilon|) + \frac{\varepsilon}{2}|Du_\varepsilon|^2\Big)dxdy \\ &= \int_\Omega (u_0-u_\varepsilon)^2 dxdy + \inf_{b\in L^2(\Omega)^2} \int_\Omega \Big(\frac{|b-Du_\varepsilon|^2}{2} + \psi_\varepsilon(b)\Big)dxdy \\ &= \int_\Omega \Big((u_0-u_\varepsilon)^2 + \frac{|b_\varepsilon - Du_\varepsilon|^2}{2} + \psi_\varepsilon(b_\varepsilon)\Big)dxdy \\ &= \inf_{b\in L^2(\Omega)^2} T_\varepsilon(b) \le F_\varepsilon(u), \qquad \forall u \in H^1(\Omega). \end{aligned}$$

Thanks to classical results of regularity, the solution u_ε of (3.3.29) belongs to $C^2(\Omega)$ (see, for example, Ladyzenskaya-Uralceva [38] or Gilbarg-Trudinger [36]). Moreover, we can easily obtain an L^∞-estimate for u_ε.

Proposition 3.3.9 *If u_0 verifies (3.3.2), then the solution u_ε of (3.3.29) satisfies :*

$$0 \le u_\varepsilon(x,y) \le 1, \qquad a.e.\ (x,y) \in \Omega. \tag{3.3.31}$$

Proof. Let us show for example that $u_\varepsilon(x,y) \le 1, \forall (x,y) \in \Omega$. The other inequality can be proved in the same way.

u_ε is solution of the variational problem

$$\begin{gathered} 2\textstyle\int_\Omega (u_\varepsilon - u_0)v dxdy + \\ \textstyle\int_\Omega \frac{\phi'(|Du_\varepsilon|)}{|Du_\varepsilon|} Du_\varepsilon \cdot Dv dxdy + \varepsilon \int_\Omega Du_\varepsilon \cdot Dv dxdy = 0, \forall v \in H^1(\Omega). \end{gathered} \tag{3.3.32}$$

In (3.3.32), we choose $v = (u_\varepsilon - 1)_+ \ge 0$; according to Stamppachia [54], $v \in H^1(\Omega)$ and (3.3.32) can be written as

$$\varepsilon \int_{u_\varepsilon>1} |Du_\varepsilon|^2 dxdy + \int_{u_\varepsilon>1} \phi'(|Du_\varepsilon|)dxdy = -2\int_{u_\varepsilon>1}(u_\varepsilon - u_0)(u_\varepsilon - 1)_+. \tag{3.3.33}$$

But, by hypothesis, $\phi'(t) \geq 0$ on $I\!R^+$ ((3.3.3)) and $0 \leq u_0(x,y) \leq 1$, a.e. $(x,y) \in \Omega$. Then $(u_\varepsilon - u_0)(x,y) \geq 0$, a.e. $(x,y) \in \{(x,y); u_\varepsilon > 1\}$, which implies, from (3.3.33), that

$$\int_{u_\varepsilon > 1} |Du_\varepsilon|^2 dxdy \leq 0;$$

from this, we have that $Du_\varepsilon(x,y) = 0, \quad \forall (x,y) \in \{(x,y); u(x,y) > 1\}$, i.e. $(u_\varepsilon - 1)_+ = 0$, which is equivalent to $u_\varepsilon(x,y) \leq 1, \quad$ a.e. $(x,y) \in \Omega$. ∎

The following estimates are more delicate and are based on a very fine perturbation lemma due to R. Temam [57], [58].

Proposition 3.3.10 *If $u_0 \in W^{1,\infty}$, then for every open set $\mathcal{O}$ relatively compact in Ω, there is a constant $K = K(\mathcal{O}, \Omega, \|u_0\|_{W^{1,\infty}})$ such that*

$$\|u_\varepsilon\|_{W^{1,\infty}} \leq K, \tag{3.3.34}$$

$$\|u_\varepsilon\|_{H^2(\mathcal{O})} \leq K. \tag{3.3.35}$$

This proposition allows us to pass to the limit on u_ε and b_ε when $\varepsilon \to 0$. Besides the estimates (3.3.31), (3.3.34) and (3.3.35), we can add, thanks to (3.3.3):

$$\|u_\varepsilon\|_{W^{1,1}(\Omega)} \leq c, \qquad (c \text{ independent of } \varepsilon). \tag{3.3.36}$$

With these estimates, we can state, using classical results of compactness and the diagonal process, that there is a function u and a sequence $\varepsilon_m \to 0$ such that:

$$u_{\varepsilon_m} \rightharpoonup u \text{ in } L^\infty(\Omega) \text{ weak-star} \tag{3.3.37}$$
$$Du_{\varepsilon_m} \rightharpoonup Du \text{ in } L^\infty(\mathcal{O}) \text{ weak-star}, \forall \mathcal{O} \subset \overline{\mathcal{O}} \subset \Omega \tag{3.3.38}$$
$$u_{\varepsilon_m} \rightharpoonup u \text{ in } H^2(\mathcal{O}) \text{ weak}, \forall \mathcal{O} \subset \overline{\mathcal{O}} \subset \Omega \tag{3.3.39}$$
$$u_{\varepsilon_m} \to u \text{ in } L^1(\Omega) \text{ strong} \tag{3.3.40}$$
$$u_{\varepsilon_m}|_{\mathcal{O}} \to u|_{\mathcal{O}} \text{ in } H^1(\mathcal{O}) \text{ strong}, \forall \mathcal{O} \subset \overline{\mathcal{O}} \subset \Omega \tag{3.3.41}$$
$$u_{\varepsilon_m}(x,y) \to u(x,y), \text{ a.e. } (x,y) \tag{3.3.42}$$
$$Du_{\varepsilon_m}(x,y) \to Du(x,y), \text{ a.e. } (x,y), \tag{3.3.43}$$

and we have the following result:

Theorem 3.3.11 *Under the previous assumptions (3.3.2), (3.3.3) and (3.3.4) and if $u_0 \in W^{1,\infty}(\Omega)$, then the function u defined before belongs to $W^{1,1}(\Omega) \bigcap L^\infty(\Omega)$ and is the unique solution of the initial optimization problem:*

$$\inf\left\{F(u) = \int_\Omega (u - u_0)^2 dxdy + \int_\Omega \phi(|Du|)dxdy, u \in L^2(\Omega), Du \in L^1(\Omega)^2\right\}. \tag{3.3.44}$$

Proof. By the Fatou's lemma, (3.3.36) and (3.3.43), it is clear that $u \in W^{1,1}(\Omega) \bigcap L^\infty(\Omega)$ (we have moreover that $u|_{\mathcal{O}} \in H^2(\mathcal{O}) \bigcap W^{1,\infty}(\mathcal{O})$, for all $\mathcal{O}$ with $\mathcal{O} \subset \overline{\mathcal{O}} \subset \Omega$). u is solution of (3.3.44). In fact, u_{ε_m} is the solution of the variational problem (3.3.32). Thanks to the Tahraoui's result mentioned before, the assumptions (3.3.3) and (3.3.4) imply that there exists a constant $M > 0$ such that $|\phi'(t)| \leq M$, for all $t \in I\!R$. Therefore, with the convergences (3.3.37)-(3.3.43) and the Lebesgue's dominated convergence theorem, we can pass to the limit in (3.3.32), and we obtain

$$(3.3.45) \quad 2\int_\Omega (u-u_0)^2 v dxdy + \int_\Omega \frac{\phi'(|Du|)}{|Du|} Du \cdot Dv dxdy = 0, \qquad \forall v \in H^1(\Omega).$$

By density, (3.3.45) is true for all $v \in L^2(\Omega)$ with $Dv \in L^1(\Omega)^2$, and since the problem is strictly convex, u is the unique solution of (3.3.44); moreover, $0 \leq u(x,y) \leq 1$, a.e. $(x,y) \in \Omega$. ■

The previous results imply some convergence properties for the sequence of the dual variables b_ε. In fact, we have proved that b_ε verifies

$$(3.3.46) \qquad b_\varepsilon = \Big((1-\varepsilon) - \frac{\phi'(|Du_\varepsilon|)}{|Du_\varepsilon|}\Big) Du_\varepsilon,$$

and

$$(3.3.47) \qquad F_\varepsilon(u_\varepsilon) = F(u_\varepsilon) + \frac{\varepsilon}{2}\int_\Omega |Du_\varepsilon|^2 dxdy = \int_\Omega (u_0 - u_\varepsilon)^2 dxdy +$$

$$+ \inf_{b \in L^2(\Omega)^2} \int_\Omega \Big(\frac{|b - Du_\varepsilon|^2}{2} + \psi_\varepsilon(b)\Big) dxdy = \inf_{b \in L^2(\Omega)^2} T_\varepsilon(b) \leq F_\varepsilon(u), \forall v \in H^1(\Omega).$$

If $\varepsilon \to 0$, we deduce from (3.3.46) that $b_\varepsilon(x,y) \to b_0(x,y)$, a.e. $(x,y) \in \Omega$, where

$$b_0(x,y) = \Big(1 - \frac{\phi'(|Du(x,y)|)}{|Du(x,y)|}\Big) Du(x,y).$$

The sequence of equalities in (3.3.47) proves that u_ε is a minimizing sequence for the problem $\inf_u F(u)$ and that

$$\begin{aligned}\int_\Omega \phi(|Du|) dxdy &= \lim_{\varepsilon \to 0} \inf_{b \in L^2(\Omega)^2} \int_\Omega \Big(\frac{|b - Du|^2}{2} + \psi_\varepsilon(b)\Big) dxdy \\ &= \lim_{\varepsilon \to 0} \int_\Omega \Big(\phi(|Du_\varepsilon|) + \frac{\varepsilon}{2}|Du_\varepsilon|^2\Big) dxdy.\end{aligned}$$

We will present more precisely some convergence results for b_ε in the next section.

Remark 3.3.12

With the assumption $u_0 \in W^{1,\infty}(\Omega)$, we have obtained the regularity result:

$$u \in H^2(\mathcal{O}) \bigcap W^{1,\infty}(\mathcal{O}),$$

for all $\mathcal{O} \subset \overline{\mathcal{O}} \subset \Omega$, that is to say: *the process is regularizing.*

3.4 Description and convergence of the algorithm

In this section, we are working in the context of the theorem (3.3.11) and we assume the existence and uniqueness of a function $u \in W^{1,1}(\Omega) \bigcap L^{\infty}(\Omega)$, which is the solution of:

$$(3.4.1)\ \inf\Big\{F(u) = \int_\Omega (u-u_0)^2 dxdy + \int_\Omega \phi(|Du|)dxdy; u \in L^2(\Omega), Du \in L^1(\Omega)^2\Big\}.$$

We are going, by using the previous results, to describe the algorithm for computing u. We denote $T(b,u)$ the functional defined on $L^2(\Omega)^2 \times H^1(\Omega)$ by

$$(3.4.2) \qquad T(b,u) = \int_\Omega (u-u_0)^2 dxdy + \frac{1}{2}\int_\Omega |Du-b|^2 dxdy + \int_\Omega \psi(b)dxdy$$

(with ψ defined as before).

The *iterative algorithm* is as follows:

(i) $u^0 \in H^1(\Omega)$ is arbitrary given, with $0 \le u^0 \le 1$.

(ii) $u^n \in H^1(\Omega)$ being calculated, we compute b^{n+1} by solving the minimization problem

$$(3.4.3) \qquad T(b^{n+1}, u^n) \le T(b, u^n), \forall b \in L^2(\Omega)^2.$$

(3.4.3), which is a strictly convex problem, has an unique solution b^{n+1} satisfying the equation $b^{n+1} = Du^{n+1} - D\psi(b^{n+1})$, or, by the corollary (3.3.8):

$$(3.4.4) \qquad b^{n+1} = \Big(1 - \frac{\phi'(|Du^n|)}{|Du^n|}\Big)Du^n.$$

(iii) u^{n+1} is therefore calculated as the solution of the problem

$$(3.4.5) \qquad T(b^{n+1}, u^{n+1}) \le T(b^{n+1}, v), \forall v \in H^1(\Omega),$$

which is equivalent to solving the variational problem

$$(3.4.6) \qquad \int_\Omega (Du^{n+1} - b^{n+1}) \cdot Dv dxdy + 2\int_\Omega (u^{n+1} - u_0)v dxdy = 0, \forall v \in H^1(\Omega).$$

(3.4.6) has an unique solution u^{n+1}.

We denote by U_n the sequence $U_n = T(b^{n+1}, u^n)$.

Lemma 3.4.1 *The sequence U_n is convergent.*

Proof. We prove that U_n is decreasing and bounded below. We have:

$$U_{n-1} - U_n = T(b^n, u^{n-1}) - T(b^{n+1}, u^n),$$
$$U_{n-1} - U_n = (T(b^n, u^n) - T(b^{n+1}, u^n)) + (T(b^n, u^{n-1}) - T(b^n, u^n));$$

thanks to the definition of b^{n+1} and u^n, we have for all $n > 0$:

$$A_n = T(b^n, u^n) - T(b^{n+1}, u^n) \ge 0,$$
$$B_n = T(b^n, u^{n-1}) - T(b^n, u^n) \ge 0.$$

Therefore, $U_{n-1} - U_n = A_n + B_n \ge 0$, that is to say: U_n is decreasing and since

$$\inf_b \int_\Omega \psi(b)dxdy > -\infty,$$

the sequence U_n is bounded below, and then it is convergent. ∎

Lemma 3.4.2 *The previous sequence b^n verifies*

$$\lim_{n\to\infty} \|b^n - b^{n+1}\|_{L^2(\Omega)^2} = 0. \tag{3.4.7}$$

Proof. We study the term A_n, which can be written as:

$$\begin{aligned}
A_n &= \int_\Omega (u^n - u_0)^2 dxdy + \frac{1}{2}\int_\Omega |Du^n - b^n|^2 dxdy + \int_\Omega \psi(b^n) dxdy \\
&- \int_\Omega (u^n - u_0)^2 dxdy - \frac{1}{2}\int_\Omega |Du^n - b^{n+1}|^2 dxdy - \int_\Omega \psi(b^{n+1}) dxdy, \\
A_n &= \int_\Omega \Big(\frac{1}{2}|Du^n - b^n|^2 + \psi(b^n)\Big) dxdy - \int_\Omega \Big(\frac{1}{2}|Du^n - b^{n+1}|^2 + \psi(b^{n+1})\Big) dxdy.
\end{aligned}$$

Denoting by $h_n(b) = \frac{1}{2}|Du^n - b|^2 + \psi(b)$, then

$$A_n = \int_\Omega (h_n(b^n) - h_n(b^{n+1})) dxdy.$$

Thanks to the Taylor's formula, there exists c_n between b^n and b^{n+1} such that

$$A_n = \int_\Omega (b^n - b^{n+1}) \cdot Dh_n(b^{n+1}) dxdy + \frac{1}{2}\int_\Omega (b^n - b^{n+1}) \cdot D^2 h_n(c_n)(b^n - b^{n+1}) dxdy.$$

But $Dh_n(b^{n+1}) = b^{n+1} - Du^n + D\psi(b^{n+1}) = 0$, by the definition of b^{n+1}. Moreover, $D^2h_n(b) = I + D^2\psi(b) \geq I$, because ψ is convex (by Lemma 3.3.4). Consequently,

$$A_n \geq \int_\Omega |b^n - b^{n+1}|^2 dxdy.$$

Otherwise, $U_{n-1} - U_n = A_n + B_n \geq A_n \geq 0$, and since the sequence U_n is convergent, $\lim_{n\to\infty} A_n = 0$, which implies that

$$\lim_{n\to\infty} \int_\Omega |b^n - b^{n+1}|^2 dxdy = 0.$$

■

In general, we can not obtain a more precisely convergence theorem (for example, the convergence in $H^1(\Omega)$), without assuming more regularity on the solution.

Lemma 3.4.3 *If* $0 \leq \frac{\phi'(t)}{t} \leq 1, \forall t \geq 0$, *then the sequence u^n is bounded in $H^1(\Omega)$.*

Proof. The proof is based on a recurrence process. In (3.4.6) we choose $v = u^{n+1}$; we get:

$$\int_\Omega \Big(|Du^{n+1}|^2 + 2(u^{n+1})^2\Big) dxdy = \int_\Omega \Big(b^{n+1} \cdot Du^{n+1} + 2u_0 u^{n+1}\Big) dxdy; \tag{3.4.8}$$

with (3.4.4), and since $0 \leq \frac{\phi'(t)}{t} \leq 1$ (in fact, in the applications, $\frac{\phi'(t)}{t}$ decreases from 1 to 0 for $t \in]0, \infty[$), we have $|b^{n+1}| \leq |Du^n|$, which implies, with (3.4.8), that

$$\int_\Omega \Big(|Du^{n+1}|^2 + 2(u^{n+1})^2\Big) dxdy \leq \int_\Omega \Big(|Du^n| \cdot |Du^{n+1}| + 2|u_0||u^{n+1}|\Big) dxdy. \tag{3.4.9}$$

Denoting by $M = max(2\|u_0\|_{L^2}, \|u^0\|_{H^1})$, we have

$$\|u^n\|_{H^1(\Omega)} \leq M, \forall n. \tag{3.4.10}$$

In fact, (3.4.10) is true for $n = 0$; let suppose that (3.4.10) is true for n, and with (3.4.9),

$$\begin{aligned}
\|u^{n+1}\|^2_{H^1(\Omega)} &\leq \int_\Omega \Big(|Du^{n+1}|^2 + 2(u^{n+1})^2\Big)dxdy \\
&\leq M\|Du^{n+1}\|_{L^2(\Omega)} + 2\|u_0\|_{L^2(\Omega)}\|u^{n+1}\|_{L^2(\Omega)} \leq M\|u^{n+1}\|_{H^1(\Omega)},
\end{aligned}$$

from which $\|u^{n+1}\|_{H^1} \leq M$, and (3.4.10) is true for all n. ∎

Like a corollary of Lemma 3.4.3, we easily deduce that the sequence b^n is bounded in $L^2(\Omega)^2$. The following theorem examines the convergence of u^n to u, the solution of the problem (3.4.1); it is therefore necessary to add a slight regularity assumption on u.

Theorem 3.4.4 *If the solution u of (3.4.1) belongs to $H^1(\Omega)$, then*

i) $u^n \to u$ *in* $L^2(\Omega)$ *strong* ;

ii) $Du^n \rightharpoonup Du$ *in* $L^2(\Omega)^2$ *weak* ;

iii) $\displaystyle\lim_{n\to\infty} \int_\Omega \phi(|Du^n|)dxdy = \int_\Omega \phi(|Du|)dxdy;$

iv) $Du^n \to Du$ *in* $L^1(\Omega)^2$ *strong.*

Proof. We know that u is the unique solution, belonging to $V = \Big\{u \in L^2(\Omega), Du \in L^1(\Omega)^2\Big\}$, of the variational problem:

$$\int_\Omega \frac{\phi'(|Du|)}{|Du|} Du \cdot Dv dxdy + 2\int_\Omega (u - u_0)v dxdy = 0, \forall v \in H^1(\Omega). \tag{3.4.11}$$

With b_0 defined in the previous section, (3.4.11) is equivalent to (all the integrals have sense, because $u \in H^1(\Omega)$):

$$\int_\Omega (Du \cdot Dv - b_0 \cdot Dv)dxdy + 2\int_\Omega (u - u_0)v dxdy = 0, \forall v \in H^1(\Omega). \tag{3.4.12}$$

Otherwise, with (3.4.5), u^n is defined by

$$\int_\Omega (Du^n \cdot Dv - b^n \cdot Dv)dxdy + 2\int_\Omega (u^n - u_0)v dxdy = 0, \forall v \in H^1(\Omega). \tag{3.4.13}$$

By subtracting (3.4.13) from (3.4.12), and choosing $v = u - u^n$, we get

$$\int_\Omega |Du - Du^n|^2 dxdy + 2\int_\Omega |u - u^n|^2 dxdy - \int_\Omega (b^n - b_0)\cdot(Du^n - Du)dxdy = 0. \tag{3.4.14}$$

With the definition of b_0 and b^n, by adding and subtracting b^{n+1}, is easy to see that

$$\begin{aligned}
\int_\Omega (b^n - b_0)\cdot(Du^n &- u)dxdy = \\
&= \int_\Omega |Du^n - Du|^2 dxdy + \int_\Omega (b^n - b^{n+1})(Du^n - Du)dxdy \\
&+ \int_\Omega \Big(\frac{\phi'(|Du|)}{|Du|}Du - \frac{\phi'(|Du^n|)}{|Du^n|}Du^n\Big)\Big(Du^n - Du)dxdy.
\end{aligned}$$

If we denote by

$$j(u) = \int_\Omega \phi(|Du|)dxdy,$$

then (3.4.14) can be written as:

$$(3.4.15) \qquad 2\int_\Omega (u^n-u)^2 dxdy + \int_\Omega (b^{n+1}-b^n)(Du^n-Du)dxdy + \; < j^{'}(u)-j^{'}(u^n), u^n-u >= 0.$$

Since j is convex, the third integral in (3.4.15) is non negative and then

$$(3.4.16) \qquad 2\int_\Omega (u^n-u)^2 dxdy + \int_\Omega (b^{n+1}-b^n)\cdot(Du^n-Du)dxdy \leq 0.$$

By Lemma 3.4.2, $(b^{n+1}-b^n) \overset{n\to\infty}{\to} 0$ in $L^2(\Omega)^2$ strong and by L emma 3.4.3 and the assumption $u \in H^1(\Omega)$, we have that $Du^n - Du$ is bounded in $L^2(\Omega)^2$, hence, by passing to the limit in (3.4.16), we get

$$(3.4.17) \qquad \lim_{n\to\infty} \int_\Omega (u^n-u)^2 dxdy = 0.$$

To prove ii), we remark, thanks to Lemma 3.4.3, that there is $\tilde{u} \in H^1(\Omega)$ such that $u^n \rightharpoonup \tilde{u}$ (or for a subsequence) in $H^1(\Omega)$ weak and, with (3.4.17), that necessarily $\tilde{u} = u$, and that all the sequence converges.

To prove iii), we deduce from (3.4.15) and (3.4.17) that

$$\lim_{n\to\infty} < j^{'}(u)-j^{'}(u^n), u^n-u >= 0,$$

that is to say:

$$(3.4.18) \qquad \lim_{n\to\infty} \int_\Omega \Big(\frac{\phi^{'}(|Du|)}{|Du|}Du - \frac{\phi^{'}(|Du^n|)}{|Du^n|}Du^n\Big)\cdot(Du^n-Du)dxdy = 0,$$

and since $Du^n \rightharpoonup Du$ in $L^2(\Omega)^2$ weak, (3.4.18) implies that:

$$(3.4.19) \qquad \lim_{n\to\infty} \int_\Omega \frac{\phi^{'}(|Du^n|)}{|Du^n|}Du^n\cdot(Du^n-Du)dxdy = 0.$$

But, since ϕ is convex, we have

$$\int_\Omega (\phi(|u|)-\phi(|Du^n|))dxdy \geq \int_\Omega \frac{\phi^{'}(|Du^n|)}{|Du^n|}Du^n\cdot(Du-Du^n)dxdy,$$

from which, with (3.4.19):

$$\int_\Omega \phi(|Du|)dxdy \geq \overline{\lim}_{n\to\infty} \int_\Omega \phi(|Du^n|)dxdy.$$

And, since we have always (thanks to the convexity of ϕ):

$$\underline{\lim}_{n\to\infty} \int_\Omega \phi(|Du^n|)dxdy \geq \int_\Omega \phi(|Du|)dxdy,$$

we get:

$$(3.4.20) \qquad \lim_{n\to\infty} \int_\Omega \phi(|Du^n|)dxdy = \int_\Omega \phi(|Du|)dxdy.$$

The proof of iv) is a consequence of the following result due to Visintin:

Theorem 3.4.5 *(Visintin [60], Thm. 3) Let Φ a strictly convex function from $I\!R^2$ into $I\!R$ and let u_n a sequence from $L^1(\Omega)^2$ such that*

$$u_n \rightharpoonup u \text{ in } L^1(\Omega)^2 \text{ weak},$$
$$\int_\Omega \Phi(u_n)dxdy \to \int_\Omega \Phi(u)dxdy.$$

Then $u_n \to u$ in $L^1(\Omega)$ strong.

To prove iv), we apply the Visintin's result with $u_n = Du^n$ and $\Phi(u) = \phi(|u|)$. ■

Remark 3.4.6

(1) u, the solution of the initial reconstruction problem (3.4.1) verifies necessarily, in the sense of distribution:

$$2(u-u_0) - \text{div}\Big(\frac{\phi'(Du|)}{|Du|}Du\Big) = 0 \text{ in } \mathcal{D}'(\Omega). \tag{3.4.21}$$

Since $u_0, u \in L^\infty(\Omega)$, we deduce from (3.4.21) that

$$\text{div}\Big(\frac{\phi'(|Du|)}{|Du|}Du\Big) \in L^\infty(\Omega)$$

and then if $u \in H^1(\Omega)$, with a result of Lions-Magenes [40], we can give sense, on the boundary $\partial\Omega$ of Ω, to the co-normal derivative $\frac{\phi'(|Du|)}{|Du|}Du \cdot n$ (n is the exterior normal to $\partial\Omega$). Multiplying (3.4.21) by $v \in H^1(\Omega)$ and integrating by parts, we get, with (3.4.12):

$$\frac{\phi'(|Du|)}{|Du|}\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \tag{3.4.22}$$

and if $\lim_{t\to\infty}\frac{\phi'(t)}{t} = 0$, then with (3.4.22), we have either

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \tag{3.4.23}$$

or

$$|Du| = +\infty \text{ on } \partial\Omega. \tag{3.4.24}$$

Supposing that in a neighborhood of its boundary, the image does not present an edge, we can incorporate (3.4.23) like a boundary condition in the algorithm.

(2) If $\lim_{t\to 0}\frac{\phi'(t)}{t} = 1$, $\lim_{t\to\infty}\frac{\phi'(t)}{t} = 0$, then the function $\frac{|b_0|}{|Du|}(x,y)$ is an edge indicator which takes, roughly speaking, only the values 1 or 0. In fact,

$$\frac{|b_0|}{|Du|}(x,y) = \Big(1 - \frac{\phi'(|Du|)}{|Du|}\Big)(x,y).$$

In a neighborhood of a pixel (x,y) belonging to an edge, $|Du|$ is big and $\frac{|b_0|}{|Du|} \approx 1$, whereas, in the interior of an homogeneous region, $|Du|(x,y)$ is small and $\frac{|b_0|}{|Du|}(x,y) \approx 0$.

(3) There are other dualities to introduce an auxiliary variable. For example, if $t \to \phi(\sqrt{t})$ is strictly concave, we can prove that there is a function ψ strictly convex and decreasing such that

$$\phi(t) = \inf_b (bt^2 + \psi(b)).$$

This duality was exploited by Geman-Reynolds [33], Charb-Bl. Fer-Aub-Barl [17], to minimize directly the functional in the discrete case.

Chapter 4

The numerical approximation of the model

This chapter is devoted to the numerical approximation of the functional and of the associated Euler equation (1.1.5), written in conservative form.

We describe two numerical algorithms to compute the solution, by using the finite differences method.

The first, presented in Section 4.1, is a classical approximation by the finite differences method to the Euler equation associated to the minimization problem, written in conservative form.

The second algorithm, described in Section 4.2, is more adapted to piecewise-constant signals and images and for sublinear potentials, since is based on a remark on the minimum of the functional in the case where u_0 is piecewise-constant and ϕ is sublinear at infinity. We discretize directly the functional by finite differences. We will see that, by combining these two algorithms, we obtain in some cases a fast computation and the locations of edges are better preserved, than using only the discretization of the equation.

Finally, in the end of this chapter, in Section 4.3, we present numerical results in the one dimensional case using these two algorithms, while the numerical results in two dimensions will be all presented in Part III.

4.1 The discretization of the conservative problem

We present in this section a numerical approximation, by using the finite differences method, for the Euler equation associated to the minimization problem (1.1.1), written in conservative form:

$$K^*Ku - \alpha \mathrm{div}\Big(\frac{\phi'(|Du|)}{|Du|}Du\Big) = K^*u_0, \tag{4.1.1}$$

where $\Omega \subset I\!R^N$ is open and bounded (with $N = 1$ or 2 and Ω an interval or rectangle), and where $K : L^2(\Omega) \to L^2(\Omega)$ can be the identity I or a convolution operator, and the function $\phi : I\!R^+ \to I\!R^+$ is of the type of potentials introduced in Chapter 1, convex or non-convex. We will also show briefly how we can adapt the discretization to the evolution problem.

We essentially adopt the method of Rudin-Osher-Fatemi [50], who have considered the case $\phi(z) = z$, for approximating the divergence term, in this more general case, and we use a discrete version of an iteration algorithm (related to Remark 3.4.6 (3)), which allows us to approach the solution u by a sequence $(u^n)_{n\geq 0}$, with $u^0 = u_0$, such that u^{n+1} is solution of the following linear problem:

$$u^{n+1} - \alpha \mathrm{div}\Big(\frac{\phi'(|Du^n|)}{|Du^n|}Du^{n+1}\Big) = K^* u_0. \tag{4.1.2}$$

Remark 4.1.1

We have observed that, for numerical reasons, we do not solve exactly the minimization problem (2.2.1) on the space $BV(\Omega)$, but an approximation of it, that is we need to compute u^ε (with $\varepsilon > 0$ small enough), defined in Chapter 3, Section 3.1, which is the unique minimum of the functional $F_{i\varepsilon}$, with $i = 1$ or 2. But, if we take the approximation $F_{2\varepsilon}$ of F, then it is unnecessary to consider, in (3.1.1), the case $z \geq 1/\varepsilon$, since, in practice, in discrete images, the gradients are always bounded. Moreover, if the function ϕ is regular and "quadratic" at the origin (like, for instance, ϕ_1, ϕ_2, ϕ_3), we will also not consider the case $z \leq \varepsilon$. In the description of the algorithm, we will assume that $\phi : I\!R^+ \to I\!R^+$ is of class C^1, $\phi'(0) = 0$, with $z \mapsto \frac{\phi'(z)}{z}$ strictly positive, continuous and decreasing in $[0, +\infty[$ (this is true for all the potentials ϕ_i, $i = 1, ..., 5$). Therefore, we will discretize directly the equation (4.1.1).

Remark 4.1.2

We need to specify the boundary conditions on $\Gamma = \partial\Omega$ associated to the equation (4.1.1). From the previous chapters (see (2.3.6) and (3.4.22)), the natural condition is:

$$\frac{\phi'(|Du|)}{|Du|} Du \cdot n = 0, \ \ \partial\Gamma - a.e. \text{ on } \Gamma,$$

where n is the unit normal to Γ. But, since $\frac{\phi'(z)}{z}$ is strictly positive in $[0, +\infty[$, and the discrete gradients are bounded in the norm, we get the following classical boundary condition:

$$\frac{\partial u}{\partial n} = 0 \text{ on } \Gamma = \partial\Omega.$$

From this condition, multiplying with a test function (4.1.1) and integrating by parts, we naturally obtain that the problem is conservative, that is, if for instance $K = I$:

$$\int_\Omega u = \int_\Omega u_0.$$

For the moment, we will consider the case $K = I$ and let ψ the function defined by:

$$\psi : I\!R^+ \to I\!R^+, \quad \psi(z) = \begin{cases} \frac{\phi'(z)}{z} & \text{if } z > 0, \\ \lim_{z\to 0} \frac{\phi'(z)}{z} & \text{if } z = 0. \end{cases}$$

Then, for each type of potential, the function ψ is strictly positive and bounded on $I\!R^+$.

Now, we move to the precise description of the algorithm in dimension one.

4.1.1 The one-dimensional case

In this case, the equation (4.1.1) becomes:

$$u - \alpha\frac{\partial}{\partial x}(\psi(|u_x|)u_x) = u_0. \tag{4.1.3}$$

Assume that $\Omega =]0,1[$. Let $M \in I\!N^*$, $x_i = ih$, where $h = 1/M$ and $0 \le i \le M$, be the discrete points. Let $u_0 : [0,1] \to I\!R$ be given and $u :]0,1[\to I\!R$ a solution to the problem (4.1.3). We define the discrete approximations u_h of u and $u_{0,h}$ of u_0 by

$$u_h(x_i) = u_i \approx u(x_i),\ \text{for } 0 < i < M,\ \ u_{0,h}(x_i) = u_{0,i} \approx u_0(x_i),\ \text{for } 0 \le i \le M,$$

with the following discrete boundary conditions (corresponding to Neumann boundary conditions)

$$u_h(x_0) := u_h(x_1),\ \ u_h(x_M) := u_h(x_{M-1}). \tag{4.1.4}$$

We may assume that the initial discrete signal, the data, satisfies the following property:

$$\text{there exist } m_2 \ge m_1 \ge 0 \text{ such that } m_1 \le u_{0,i} \le m_2,\ \text{for } 0 \le i \le M,$$

which will be used to establish the so-called $L^\infty - stability$ for the solution. We also recall the usual notations for finite differences in dimension one. Let

$$\triangle_+ u_i := u_{i+1} - u_i,\ \ \triangle_- u_i := u_i - u_{i-1},\ \text{for } 0 < i < M.$$

Now, the numerical approximation of the equation (4.1.3) will be:

$$u_i - \frac{\alpha}{h}\triangle_-\Big[\psi\Big(\Big|\frac{\triangle_+ u_i}{h}\Big|\Big)\Big(\frac{\triangle_+ u_i}{h}\Big)\Big] = u_{0,i},\ \ 0 < i < M, \tag{4.1.5}$$

with the boundary conditions (4.1.4).

Remark 4.1.3

Of course, this discretization of the equation is natural and related to the conservative formulation of the problem. Indeed, let us consider the equivalent problem (with $V = H^1(\Omega)$):

$$\int_\Omega uv dx + \alpha\int_\Omega \psi(|u_x|)u_x v_x dx = \int_\Omega u_0 v dx, \quad \forall v \in V,$$

and write an approximation of it. Let, for $h > 0$, the subspace V_h of V defined by:

$$V_h = \{u_h \in V :\ u_h|_{[x_i - \frac{h}{2}, x_i + \frac{h}{2}[} \in P_0\}.$$

Then $u_h \in V_h$ can be written as:

$$u_h = \sum_{i=0}^{M} u_i\chi_i(x),\ \text{with } \chi_i(x) = \chi|_{[x_i - \frac{h}{2}, x_i + \frac{h}{2}[}.$$

We also define

$$\delta_h u_h(x) := \frac{u_h(x + \frac{h}{2}) - u_h(x - \frac{h}{2})}{h}.$$

Then the approximated solution $u_h \in V_h$ must satisfy:

$$\int_\Omega u_h(x)v_h(x)dx + \alpha \int_\Omega \psi(|\delta_h u_h(x)|)\delta_h u_h(x)\delta_h v_h(x)dx = \int_\Omega u_{0,h}(x)v_h(x),\ \forall v_h \in V_h.$$

Using the definition of u_h and v_h, this is equivalent with:

$$h\sum_i u_i v_i + \alpha h \sum_i \psi(|\frac{u_{i+1}-u_i}{h}|)\frac{u_{i+1}-u_i}{h}\frac{v_{i+1}-v_i}{h} = h\sum_i u_{0,i}v_i,\ \forall v_i \in I\!R.$$

Integrating by parts in the discrete case, we obtain:

$$\sum_i u_i v_i - \frac{\alpha}{h}\sum_i \Big[\psi(|\frac{u_{i+1}-u_i}{h}|)\frac{u_{i+1}-u_i}{h} - \psi(|\frac{u_i-u_{i-1}}{h}|)\frac{u_i-u_{i-1}}{h}\Big]v_i = \sum_i u_{0,i}v_i,\ \forall v_i \in I\!R,$$

which is equivalent with (4.1.5), with the previous boundary conditions.

Since the problem is still nonlinear, we will approach the numerical solution u_h by a sequence $(u_h^n)_{n\geq 0}$, which is obtained by a fixed point algorithm (see also [4]) as follows (sometimes, we will write u, u_0, u^n instead of $u_h, u_{0,h}, u_h^n$):

1) u^0 is arbitrary given, such that $m_1 \leq u_i^0 \leq m_2$ (for instance, $u^0 = u_0$).

2) If u^n is calculated, then we compute u^{n+1} as the solution to the discrete linear problem:

$$u_i^{n+1} - \frac{\alpha}{h}\triangle_-\Big[\psi\Big(|\frac{\triangle_+ u_i^n}{h}|\Big)\Big(\frac{\triangle_+ u_i^{n+1}}{h}\Big)\Big] = u_{0,i},\ 0 < i < M, \tag{4.1.6}$$

with the discrete boundary conditions for u^{n+1}.

In fact, (4.1.6) is an approximation of (4.1.2). Now, we multiply (4.1.6) by h^2/α, and we define $c_1(u_i^n)$, $c_2(u_i^n)$, $C_1(u_i^n)$, $C_2(u_i^n)$ and $C(u_i^n)$ by

$$c_1(u_i^n) = c_1 := \psi(|\tfrac{u_{i+1}^n - u_i^n}{h}|),\ c_2(u_i^n) = c_2 := \psi(|\tfrac{u_i^n - u_{i-1}^n}{h}|), \tag{4.1.7}$$

$$C_i = \tfrac{c_i}{(h^2/\alpha)+c_1+c_2},\ C = \tfrac{(h^2/\alpha)}{(h^2/\alpha)+c_1+c_2}. \tag{4.1.8}$$

We remark that $c_i, C_i, C > 0$, for $i = 1, 2$, and $C_1 + C_2 + C = 1$ (we note that these coefficients depend on (u_i^n)). All these properties on the coefficients will guarantee the $L^\infty - \textit{stability}$ of the scheme. With these notations, (4.1.6) becomes:

$$u_i^{n+1} = C_1(u_i^n)u_{i+1}^{n+1} + C_2(u_i^n)u_{i-1}^{n+1} + C(u_i^n)u_{0,i}. \tag{4.1.9}$$

Now, let $(E, \|\cdot\|)$ be the Banach space $E = \{u = (u_i)_{i=0,M}, u_i \in I\!R\} = I\!R^{M+1}$, with $\|u\| := \max_i |u_i|$, and the subset $U \subset E$, $U := \{u \in E, m_1 \leq u_i \leq m_2, 0 \leq i \leq M\}$.

Proposition 4.1.4 *(i) If $u^n \in U$, then there exists a unique $u^{n+1} \in U$ solution of (4.1.9).*

(ii) The following nonlinear discrete problem

$$u_i = C_1(u_i)u_{i+1} + C_2(u_i)u_{i-1} + C(u_i)u_{0,i} \text{ for } 0 < i < M, \tag{4.1.10}$$

with the boundary conditions (4.1.4), has a solution $u \in U$.

Proof.

(i) For $u \in U$, we define the linear application $Q_u : U \to E$ by

$$(Q_u(z))_i := C_1(u_i)z_{i+1} + C_2(u_i)z_{i-1} + C(u_i)u_{0,i}.$$

We will easily prove that $Q_u(U) \subset U$ and moreover, that Q_u is a contractive map on E. We have for $z \in U$:

$$(Q_u(z))_i \leq (C_1(u_i) + C_2(u_i) + C(u_i))m_2 = m_2.$$

We obtain in the same way that $m_1 \leq (Q_u(z))_i$. Hence, $Q_u(z) \in U$. Now, for $v, w \in E$, we have:

$$\begin{aligned} |(Q_u(v) - Q_u(w))_i| &\leq C_1(u_i)|v_{i+1} - w_{i+1}| + C_2(u_i)|v_{i-1} - w_{i-1}| \\ &\leq (C_1(u_i) + C_2(u_i))\|v - w\|. \end{aligned}$$

Now, since the function ψ is positive and bounded on $[0, +\infty[$, we have:

$$c_1(u_i^n) + c_2(u_i^n) \leq 2 \sup_{[0,+\infty[} \psi.$$

Since

$$C_1 + C_2 = \frac{c_1 + c_2}{(h^2/\alpha) + c_1 + c_2},$$

and the function

$$z \mapsto \frac{z}{(h^2/\alpha) + z}$$

is increasing in $[0, +\infty[$, we get:

$$C_1 + C_2 \leq c := \frac{2 \sup_{[0,+\infty[} \psi}{(h^2/\alpha) + 2 \sup_{[0,+\infty[} \psi}.$$

Of course $0 < c < 1$ and we deduce that:

$$|(Q_u(v) - Q_u(w))_i| \leq c\|v - w\|.$$

So, by the classical Banach fixed point theorem, we deduce that there is an unique $u^{n+1} \in U$, such that $u^{n+1} = Q_{u^n}(u^{n+1})$, this being equivalent with (4.1.9).

(ii) To prove (ii), we define the application $T : U \to U$ by $T(u) = u^*$, where u^* is the unique fixed point of Q_u. We will prove that this application is continuous from the compact and convex set U into itself. Therefore, there exists a fixed point of T, which will be solution of (4.1.10), equivalent with (4.1.5).

So, let $u_n, u \in U$ such that $\lim_{n\to\infty} \|u_n - u\| = 0$ and $u_n^* = T(u_n), u^* = T(u)$. We have the following equalities and inequalities:

$$\begin{aligned} \|u_n^* - u^*\| = \|Q_{u_n}(u_n^*) - Q_u(u^*)\| &= \|Q_u(u_n^*) - Q_u(u^*) + Q_{u_n}(u_n^*) - Q_u(u_n^*)\| \\ &\leq \|Q_u(u_n^*) - Q_u(u^*)\| + \|Q_{u_n}(u_n^*) - Q_u(u_n^*)\| \\ &\leq c\|u_n^* - u^*\| + \|Q_{u_n}(u_n^*) - Q_u(u_n^*)\|. \end{aligned}$$

Then, we get the follows:

$$(1-c)\|u_n^* - u^*\| \le \|Q_{u_n}(u_n^*) - Q_u(u_n^*)\| \le \|u_n^*\| \sup_i \Big(|C_1(u_{ni}) - C_1(u_i)| + \\ + |C_2(u_{ni}) - C_2(u_i)| + |C(u_{ni}) - C(u_i)|\Big).$$

Now, since $\|u_n^*\| \le m_2$, for all $n > 0$ and since the functions C_i, C are continuous (because the function ψ is continuous), we obtain that the right-hand-side of the last inequality converges to 0 for $n \to \infty$, hence $\|u_n^* - u^*\| \to 0$, that is to say: the application T is continuous. ■

Remark 4.1.5

The conclusion (i) of Proposition 4.1.4 says that the algorithm is unconditionally stable. Moreover, to compute u^{n+1} as solution of the linear system (4.1.6), since Q_{u^n} is a contractive map, we can use the iterative method:

$$v_0 := u^n, \quad v_{k+1} := Q_{u^n}(v_k) \quad \text{and} \quad u^{n+1} := \lim_{k\to\infty} v_k.$$

Practically, in order to accelerate the convergence to the solution u of (4.1.5), by a combination of these two iterative methods, we use a scheme based on the Gauss-Seidel algorithm: for $i = 1, 2, ..., M-1$ in this order, we let

$$u_i^{n+1} = C_1 u_{i+1}^n + C_2 u_{i-1}^{n+1} + C u_{0,i},$$

where for the computation of C_1, C_2 and C, we replace u_{i-1}^n by u_{i-1}^{n+1}.

This algorithm is quite stable, and convergent in the convex case. In the nonconvex case, in practice, the difference in the sup norm of two consecutive iterations decreases, but this is no convergent to zero.

In the case of the **evolution equation**, which is:

$$\frac{\partial u(t)}{\partial t} - \alpha \frac{\partial}{\partial x}(\psi(|u_x|)u_x) = u_0, \ u(0) = u_0,$$

in $]0, T[\times\Omega$, we can use the same approximation for the divergence term, and we have the choice between an explicit or implicit scheme. Let $\triangle t > 0$, $n \in I\!N$ and define $u_h(n \triangle t, x_i) = u_i^n \approx u(n \triangle t, x_i)$, with $u_h^0 = u_{0,h}$.

The explicit scheme is, for $1 \le i \le M-1$:

$$\frac{u_i^{n+1} - u_i^n}{\triangle t} + u_i^n - \frac{\alpha}{h^2}[c_1(u_i^n)(u_{i+1}^n - u_i^n) + c_2(u_i^n)(u_i^n - u_{i-1}^n)] = u_{0,i},$$

or

$$u_i^{n+1} = [1 - \frac{\alpha \triangle t}{h^2}(c_1 + c_2) - \triangle t]u_i^n + \frac{\alpha \triangle t}{h^2} c_1 u_{i+1}^n + \frac{\alpha \triangle t}{h^2} c_2 u_{i-1}^n + \triangle t u_{0,i}.$$

We have that

$$\{m_1 \le u_i^n \le m_2, \text{ for } 0 \le i \le M\} \Rightarrow \{m_1 \le u_i^{n+1} \le m_2, \text{ for } 0 \le i \le M\},$$

under the following stability condition:

$$1 - \triangle t(\frac{2\alpha}{h^2}\sup_{[0,+\infty[}\psi - 1) \geq 0.$$

Now, **the implicit scheme** is naturally, with $0 < i < M$:

$$\frac{u_i^{n+1} - u_i^n}{\triangle t} + u_i^{n+1} - \frac{\alpha}{h^2}[c_1(u_i^n)(u_{i+1}^{n+1} - u_i^{n+1}) + c_2(u_i^n)(u_i^{n+1} - u_{i-1}^{n+1})] = u_{0,i}.$$

As in the stationary case, we can prove that this last scheme is unconditionally stable and we can obtain a result equivalent to Proposition 4.1.4.

4.1.2 The two-dimensional case

In this section, we describe the extension of the previous approximation to the two dimensional problem, which is, for $N = 2$, with $Du = (u_x, u_y)$:

$$u - \alpha\frac{\partial}{\partial x}(\psi(|Du|)u_x) - \alpha\frac{\partial}{\partial y}(\psi(|Du|)u_y) = u_0, \tag{4.1.11}$$

in $\Omega \subset I\!R^2$ and with $\frac{\partial u}{\partial n} = 0$ on $\Gamma = \partial\Omega$.

Assume $\Omega =]0,1[\times]0,1[$, $h > 0$ and $x_i = ih, y_j = jh, h = 1/M$, for $0 \leq i,j \leq M$, be the discrete points. As in the one-dimensional case, we recall the following usual notations:

1^0) $u_h(x_i, y_j) = u_{ij} \approx u(x_i, y_j), u_{0,h}(x_i, y_j) = u_{0,ij} \approx u_0(x_i, y_j)$;

2^0) $m(a,b) = \text{minmod}(a,b) = \dfrac{\text{signa} + \text{sign}b}{2} min(|a|, |b|)$;

3^0) $\triangle_{\mp}^x u_{ij} = \mp(u_{i\mp 1,j} - u_{ij})$ and $\triangle_{\mp}^y u_{ij} = \mp(u_{i,j\mp 1} - u_{ij})$.

So, $(u_{0,ij})_{i,j=0,M}$ is the initial discrete image, the data, such that $m_1 \leq u_{0,ij} \leq m_2$, where $m_2 \geq m_1 \geq 0$. We will approach the numerical solution $(u_{ij})_{i,j=0,M}$ by a sequence $(u_{ij}^n)_{i,j=0,M}$ for $n \to \infty$, which is obtained as follows:

1) u^0 is arbitrary given, such that $m_1 \leq u_{ij}^0 \leq m_2$ (we can take $u^0 = u_0$).

2) If u^n is calculated, then we compute u^{n+1} as solution of the linear discrete problem

$$\begin{aligned} u_{ij}^{n+1} &- \frac{\alpha}{h}\triangle_-^x\left[\psi\Big(\Big(\big(\frac{\triangle_+^x u_{ij}^n}{h}\big)^2 + \Big(m\big(\frac{\triangle_+^y u_{ij}^n}{h}, \frac{\triangle_-^y u_{ij}^n}{h}\big)\Big)^2\Big)^{\frac{1}{2}}\Big)\Big(\frac{\triangle_+^x u_{ij}^{n+1}}{h}\Big)\right] \\ &- \frac{\alpha}{h}\triangle_-^y\left[\psi\Big(\Big(\big(\frac{\triangle_+^y u_{ij}^n}{h}\big)^2 + \Big(m\big(\frac{\triangle_+^x u_{ij}^n}{h}, \frac{\triangle_-^x u_{ij}^n}{h}\big)\Big)^2\Big)^{\frac{1}{2}}\Big)\Big(\frac{\triangle_+^y u_{ij}^{n+1}}{h}\Big)\right] \\ &= u_{0,ij}, \end{aligned} \tag{4.1.12}$$

for $i,j = 1, ..., M-1$, and with the boundary conditions:

$$u_{0,j}^{n+1} = u_{1,j}^{n+1}, u_{Mj}^{n+1} = u_{M-1,j}^{n+1}, u_{i0}^{n+1} = u_{i1}^{n+1}, u_{i,M}^{n+1} = u_{i,M-1}^{n+1}.$$

We use here the minmod function, in order to reduce the oscillations and to get the correct values of the derivatives in the case of local maxima and minima. We observe that, to approach, for instance, the term $\frac{\partial}{\partial x}(\psi(|(u_x, u_y)|)u_x)$, we do not use the minmod function for u_x, since we wish to obtain a five point finite difference scheme.

We multiply (4.1.12) by (h^2/α) and then we denote $c_1(u^n_{ij})$, $c_2(u^n_{ij})$, $c_3(u^n_{ij})$ and $c_4(u^n_{ij})$, the coefficients of $u^{n+1}_{i+1,j}$, $u^{n+1}_{i-1,j}$, $u^{n+1}_{i,j+1}$ and $u^{n+1}_{i,j-1}$ respectively. We remark, for $i = 1, ..., 4$, that $c_i > 0$, since the function ψ is strictly positive. Now, for u^n_{ij}, let $C_i(u^n_{ij})$ and $C(u^n_{ij})$ be defined by $(i = 1, ..., 4)$

$$C_i = \frac{c_i}{(h^2/\alpha) + c_1 + c_2 + c_3 + c_4}, \quad C = \frac{(h^2/\alpha)}{(h^2/\alpha) + c_1 + c_2 + c_3 + c_4}.$$

Then, we have that $C_i, C > 0$ and $C_1 + C_2 + C_3 + C_4 + C = 1$ (we recall that these coefficients depend on u^n_{ij}).

Hence, we write (4.1.12) as:

$$\text{(4.1.13)} \quad u^{n+1}_{ij} = C_1(u^n_{ij})u^{n+1}_{i+1j} + C_2(u^n_{ij})u^{n+1}_{i-1j} + C_3(u^n_{ij})u^{n+1}_{ij+1} + C_4(u^n_{ij})u^{n+1}_{ij-1} + C(u^n_{ij})u_{0,ij}.$$

The other details of the algorithm are the obvious extension of the one dimensional case. We can also prove a result equivalent to Proposition 4.1.4.

We will not describe the approximation for the two-dimensional evolution problem, this being similar with the one-dimensional case.

In the end of this section, we briefly consider the case $K \neq I$. In many cases, the degradation operator K, the blur, is a convolution type integral operator.

In the numerical approximations, $(K_{mn})_{m,n=0,d}$ is a symmetric matrix with

$$\sum_{m,n=1}^{d} K_{mn} = 1$$

and an approximation of Ku can be:

$$Ku_{ij} = \sum_{m,n=1}^{d} K_{mn} u_{i+\frac{d}{2}-m, j+\frac{d}{2}-n}.$$

Since K is symmetric, then $K^* = K$ and $K^*Ku = KKu$ is approximated by

$$KKu_{ij} = \sum_{m,n=1}^{d} \sum_{r,t=1}^{d} K_{mn}K_{rt} u_{i+d-r-m, j+d-t-n}.$$

Then, we use the same approximation of the divergence term and the same iterative algorithm, with a slight modification. The equation, in this case, is approximated by (using the same notations c_i as before)

$$\begin{aligned} &(h^2/\alpha)KKu^{n+1}_{ij} + (c_1(u^n_{ij}) + c_2(u^n_{ij}) + c_3(u^n_{ij}) + c_4(u^n_{ij}))u^{n+1}_{ij} \\ &= c_1(u^n_{ij})u^{n+1}_{i+1,j} + c_2(u^n_{ij})u^{n+1}_{i-1,j} + c_3(u^n_{ij})u^{n+1}_{i,j+1} + c_4(u^n_{ij})u^{n+1}_{i,j-1} + (h^2/\alpha)Ku_{0,ij}. \end{aligned}$$

Now, to compute u^{n+1} as the solution of this linear system, we can use, for example, the relaxation method.

Remark 4.1.6

The algorithm described in Chapter 3, Section 3.4, allows us to compute u^{n+1} as solution of the following linear equation (instead of (4.1.2)):

$$u^{n+1} - \alpha \triangle u^{n+1} = u_0 - \alpha \mathrm{div}\Big[\Big(1 - \frac{\phi'(|Du^n|)}{|Du^n|}\Big) Du^n\Big]. \tag{4.1.14}$$

But, in this way, using the same formula to discretize the divergence operator and a classical approximation for the Laplacian, we numerically obtain an algorithm which is not always stable. Therefore, to overcome this difficulty, we have replaced (4.1.14) by:

$$u^{n+1} - \alpha \mathrm{div}(Du^{n+1}) = u_0 - \alpha \mathrm{div}\Big[\Big(1 - \frac{\phi'(|Du^n|)}{|Du^n|}\Big) Du^{n+1}\Big],$$

which is equivalent with (4.1.2).

4.2 A new algorithm in the non-convex case

The aim of this section is to introduce a new algorithm, to solve the minimization problem, which is more natural in the case when the function ϕ has a sublinearly growth, for instance, if ϕ has an horizontal asymptote. As we will see, the locations of edges are better preserved by combining this algorithm with the discretization of the equation in conservative form. Moreover, the algorithm introduced here is TVD (at least in the one dimensional case) and rapidly convergent.

The outline of this section is as follows. In Subsection 4.2.1, we will recall the continuous minimization problem and we will make some basic remarks, in the case of a non-convex and sublinear potential. In Subsection 4.2.2 we describe the algorithm in dimension one, and in Subsection 4.2.3 we extend this algorithm to the two dimensional case.

4.2.1 The minimization problem in continuous variables

Let u and u_0 be two functions defined on the open and bounded domain $\Omega \subset I\!R^N$ (here, $N = 1, 2$). We will consider only the denoising case: $K = I$. The minimization problem is:

$$\inf_u \Big\{F(u)\Big\} = \inf_u \Big\{\frac{1}{2}\int_\Omega (u - u_0)^2 dx + \alpha \int_\Omega \phi(|\nabla u|) dx\Big\} = \inf_u \Big\{\frac{1}{2}F_1(u) + \alpha F_2(u)\Big\}. \tag{4.2.1}$$

We are going, for the moment, to consider functions $z \mapsto \phi(z)$ which are not convex and growth sublinearly as $z \to \infty$, and we have explained in Chapter 1, why there is an interest for this type of functions, to denoise images. For instance, we will recall the two non-convex and sublinear functions ϕ_4 and ϕ_5, considered in this thesis. With the notations of Chapter 1, let us introduce

$$\phi_4(t) := \frac{1}{2}\log(1 + t^2) \text{ and } \phi_5(t) := \frac{t^2}{2(1 + t^2)}.$$

Of course, with these non-convex functions, the minimization problem (4.2.1) is ill-posed. However, these methods give good results and, as we mentioned in the introduction, the algorithms introduced here, using one of these two functions, are essentially TVD and rapidly convergent.

In particular, we will consider the function $\phi(z) = \phi_5(z) = \frac{z^2}{2(1+z^2)}$, which has an horizontal asymptote at infinity. Due to this property of ϕ, the presence of edges adds no contribution in the energy. For instance, if u_0 is piecewise-constant, then

$$F_2(u_0) = \int_\Omega \phi(|\nabla u_0|)dx = 0.$$

Indeed, for instance in dimension one, let u_0 be a Heaviside function and $\Omega =]-1,+1[$. Then consider any regularizing sequence (ρ_ε), with $supp(\rho_\varepsilon) \subset [-\varepsilon, +\varepsilon]$, and approximate u_0 by $u_{0,\varepsilon} = \rho_\varepsilon * u_0$, and let $\varepsilon \to 0_+$. Then, we formally have: $F_2(u_0) = \lim_{\varepsilon\to 0^+} F_2(u_{0,\varepsilon})$ and:

$$0 \leq F_2(u_{0,\varepsilon}) = \int_{-1}^{+1} \phi_5(|\nabla u_{0,\varepsilon}|)dx = \int_{-\varepsilon}^{+\varepsilon} \phi_5(|\nabla u_{0,\varepsilon}|)dx \leq 2\varepsilon\frac{1}{2} = \varepsilon.$$

Therefore, if $\phi(z) = \phi_5(z) = \frac{z^2}{2(1+z^2)}$, the above basic observation implies that for any piecewise-constant function u_0,

$$F(u_0) = \frac{1}{2}F_1(u_0) + \alpha F_2(u_0) = 0 = \inf_u F(u).$$

We can simply verify that the same result is also satisfied by $\phi_4(z) = \frac{1}{2}\log(1+z^2)$. Indeed, if we define $u_{0,\varepsilon}$ in $[-\varepsilon, +\varepsilon]$ for instance by:

$$u_{0,\varepsilon}(x) = \begin{cases} -\frac{x^2}{2\varepsilon^2} + \frac{x}{\varepsilon} + \frac{1}{2} \text{ if } x \in [0, +\varepsilon], \\ \frac{x^2}{2\varepsilon^2} + \frac{x}{\varepsilon} + \frac{1}{2} \text{ if } x \in [-\varepsilon, 0], \end{cases}$$

then

$$0 \leq F_2(u_{0,\varepsilon}) \leq 2\varepsilon \log\left(1 + \frac{1}{\varepsilon^2}\right),$$

and

$$\lim_{\varepsilon\to 0}\left(2\varepsilon \log\left(1 + \frac{1}{\varepsilon^2}\right)\right) = 0.$$

Then, $F_2(u_0) = 0$ also for $\phi = \phi_4$ and the infimum of F will formally be u_0 in this case.

Remark 4.2.1

The above basic observation is related to the fact that, if we formally compute the asymptote function ϕ^∞ for ϕ_4 and ϕ_5, this will be equal to zero. Then, for $u \in BV(\Omega)$ piecewise-constant, with $Du = \nabla u dx + D_s u$, we will have (only formally, since the notion of function of measures is defined, for instance, only for convex functions):

$$\phi(|Du|)(\Omega) = \int_\Omega \phi(|Du|) = \int_\Omega \phi(|\nabla u|)dx + \phi^\infty(1)\int_\Omega |D_s u| = 0,$$

and then, for piecewise-constant data u_0, the solution to (4.2.1) will be u_0.

Now, if u_0 is no longer piecewise-constant, the principle of the algorithm introduced here is the following, for instance, in dimension one: at each step $n \geq 0$, assume that u^n is known (with $u^0 := u_0$); then let the functional F decrease by comparing on each interval $F(u^n)$ with $F(\bar{u^n})$, where $\bar{u^n}$ is locally constant and therefore, has locally no contribution to F_2. We therefore define u^{n+1} as the signal which minimizes F, between u^n and $\bar{u^n}$.

In the next subsection, we describe the precise algorithm in dimension one.

4.2.2 The algorithm in dimension one

Assume that $\Omega =]0,1[$. Let $x_i = ih$, where $h = 1/M$, $M \in \mathbb{N}^*$, $0 \leq i \leq M$, be the discrete points. Let u be a solution to problem (4.2.1) (we assume the existence of such a solution u) and define a discrete approximation u_h of u (and $u_{0,h}$ of u_0) by

$$u_h(x_i) = u_i \approx u(x_i), \; u_{0,i} = u_0(x_i), \quad \text{for } 0 \leq i \leq M.$$

We want to construct a sequence of such discrete functions $\{u_h^n\}_{n\geq 0}$, with $u_h^0 = u_{0,h}$, such that $F(u_h^{n+1}) \leq F(u_h^n)$, with a relatively easy computation of u_h^{n+1}.

A priori, we could approximate the solution either by piecewise-constant functions or by piecewise-linear functions. Due to the above basic observation in Subsection 4.2.1, the first choice would always lead to taking $F_2 = 0$. Therefore we turn to the second choice: on each interval the numerical solution u_h is a P_1 approximation, as in the classical finite element method:

Let $V_h \subset L^2(\Omega)$ the space of functions u_h such that

$$u_h = \sum_{i=0}^{M} u_i w_i,$$

where $w_i(\cdot)$ is the unique piecewise-linear continuous function on Ω, with

$$w_i(x_j) = \delta_{i,j}, \; \text{for } 0 \leq i, j \leq M.$$

Then, u_h, $u_{0,h} \in V_h$ can be written, for each $x \in [x_i, x_{i+1}]$, as:

$$\begin{aligned} u_h(x) &= u_i + (x - x_i)\frac{u_{i+1} - u_i}{h}, \\ u_{0,h}(x) &= u_{0,i} + (x - x_i)\frac{u_{0,i+1} - u_{0,i}}{h}, \end{aligned}$$

and in this way, u_h and $u_{0,h}$ are piecewise-linear continuous functions on Ω.

We note that for any positive h, $u_h \in H^1(\Omega) = \{v \in L^2(\Omega) | \int_\Omega |\nabla v|^2 dx < +\infty\}$.

The corresponding approximations of $F_1(u_h)$ and $F_2(u_h)$ are:

$$\begin{aligned} F_1(u_h) &= \frac{h}{3}\sum_{i=0}^{M-1}[(u_i - u_{0,i})^2 + (u_{i+1} - u_{0,i+1})^2 + (u_i - u_{0,i})(u_{i+1} - u_{0,i+1})], \\ F_2(u_h) &= h\sum_{i=0}^{M-1}\phi(\frac{|u_{i+1} - u_i|}{h}). \end{aligned}$$

Now, the expression of $F_1(u_h)$ is not simple enough, especially in 2D. Therefore, we choose to replace it by

$$\begin{aligned}\bar{F}_1(u_h) &= \frac{h}{2}\sum_{i=0}^{M-1}[(u_i-u_{0,i})^2+(u_{i+1}-u_{0,i+1})^2] \\ &= \frac{h}{2}(u_0-u_{0,0})^2+h\sum_{i=1}^{M-1}(u_i-u_{0,i})^2+\frac{h}{2}(u_M-u_{0,M})^2.\end{aligned}$$

This is nothing but replacing the L^2 scalar product on V_h by the L^2 scalar product on the space of piecewise-constant functions on each interval $[x_i-\frac{h}{2}, x_i+\frac{h}{2}[$. Clearly, the associated norms are equivalent, for each $h>0$, since we have

$$\frac{1}{\sqrt{3}}\|u_h\|_0 \le \|u_h\|_{L^2(\Omega)} \le \|u_h\|_0,$$

where

$$\|u_h\|_0^2 = \frac{h}{2}\sum_{i=0}^{M-1}(u_i^2+u_{i+1}^2).$$

This is similar to the classical mass-lumping used in the finite element method, to simplify the calculation of the mass-matrix. This procedure is very often used to compute steady flows as limits for large time of the unsteady solution, and therefore does not change the stationary numerical solutions.

Here, replacing F_1 by $\bar{F}_1$ slightly changes the problem. However, the resulting algorithm is much simpler and faster, especially in 2D.

Remark 4.2.2

We wish to compare $F_1(u_h)$ and $\bar{F}_1(u_h)$. We have:

$$\begin{aligned}\bar{F}_1(u_h) &= F_1(u_h)+\frac{h^2}{6}\Big\{h\sum_{i=0}^{M-1}\Big[\frac{(u_{i+1}-u_{0,i+1})-(u_i-u_{0,i})}{h}\Big]^2\Big\} \\ &= F_1(u_h)+\frac{h^2}{6}\int_\Omega|\nabla(u_h-u_{0,h})|^2dx.\end{aligned}$$

Therefore, replacing $F_1(u_h)$ by $\bar{F}_1(u_h)$, amounts to replacing $F(u_h)=\frac{1}{2}F_1(u_h)+\alpha F_2(u_h)$ by

$$\bar{F}(u_h)=\frac{1}{2}F_1(u_h)+\alpha F_2(u_h)+\frac{h^2}{12}\int_\Omega|\nabla(u_h-u_{0,h})|^2dx.$$

Since $h \ll 1$, and since we are only going to perform very few iterations, this very small amount of diffusion will help to stabilize the numerical solution, without smearing it too much. We also note that the term $\frac{h^2}{6}\int_\Omega|\nabla(u_h-u_{0,h})|^2dx$ plays the role of a numerical viscosity, but with a small coefficient of order of $O(h^2)$ (generally, this coefficient is only of order $O(h)$).

Now, we can describe the precise algorithm in 1D.

For any n and any fixed k, with $0 \leq k \leq M-1$, we consider the interval $[x_k, x_{k+1}]$. Then, we introduce two discrete signals v and w, defined by:

$$v = u^n \text{ and } w = (w_i)_i, \; w_i = \begin{cases} v_i, & \text{if } i \neq k \text{ and } i \neq k+1, \\ \frac{v_k + v_{k+1}}{2} & \text{if } i = k \text{ or } i = k+1. \end{cases}$$

Then we compare $F(v)$ and $F(w)$, we define $u^{n,k}$ as the signal which minimizes F between v and w:

$$F(u^{n,k}) = \min\{F(v), F(w)\},$$

and we consider the next interval $[x_{k+1}, x_{k+2}]$. In fact, to preserve the symmetry of the problem, we first let k increase from 0 to $(M-1)$ and then decrease back from $(M-1)$ to 0. This whole process is an iteration of our algorithm. The final result is denoted by u^{n+1}.

Remark 4.2.3

In order to compare $F(v)$ and $F(w)$, we only need to compare the terms which involve the values of v and w at points x_k and x_{k+1}.

This algorithm generates no oscillations. More precisely:

Proposition 4.2.4 *The above algorithm is TVD (total variation diminish), that is:*

$$TV(u^{n,k+1}) \leq TV(u^{n,k}),$$

where $TV(u_h) = h \sum_{i=0}^{M-1} |u_{i+1} - u_i|$.

Proof. To show this, we only need to verify the following inequality:

$$|v_{k+2} - \frac{v_k + v_{k+1}}{2}| + |\frac{v_k + v_{k+1}}{2} - v_{k-1}| \leq |v_{k+2} - v_{k+1}| + |v_{k+1} - v_k| + |v_k - v_{k-1}|,$$

which can be simply checked by considering the all possible cases for the order of v_{k-1}, v_k, v_{k+1}, and v_{k+2}.

4.2.3 The two-dimensional algorithm

In this section, we describe the extension of the algorithm to the two dimensional problem. In fact, we will apply the algorithm in 1D, by working on lines and columns (to avoid grid effects).

Assume $\Omega =]0,1[^2$, for simplicity, and let $x_i = ih$, $y_j = jh$, $h = 1/M$, for $0 \leq i, j \leq M$, be the discrete points. We define the discrete approximations u_h, $u_{0,h}$, of u and u_0, for $0 \leq i, j \leq M$, by:

$$\begin{aligned} u_h(x_i, y_j) &= u_{ij} \approx u(x_i, y_j), \\ u_{0,h}(x_i, y_j) &= u_{0,ij} \approx u_0(x_i, y_j). \end{aligned}$$

As in dimension one, we construct a sequence of discrete functions $\{u_h^n\}_{n\geq 0}$, with $u_h^0 = u_{0,h}$, such that $F(u_h^{n+1}) \leq F(u_h^n)$. We wish to obtain a simpler and faster algorithm. Therefore, we consider the simpler formula of discretization for the functional F (based on piecewise-constant approximations). The algorithm will be well adapted to reconstruct piecewise-constant images, from the above basic remark.

To simplify the notations, let $p_{ij} = u_{ij} - u_{0,ij}$ for each $0 \leq i,j \leq M$.

Then we use the following formula to approximate the first term in the energy:

$$F_1(u_h) = \frac{h^2}{4}\sum_{i,j=0}^{M-1}(p_{ij}^2 + p_{i+1,j}^2 + p_{i,j+1}^2 + p_{i+1,j+1}^2) \approx h^2 \sum_{i,j} p_{ij}^2,$$

by using the above notations.

Now, for the second term, we let

$$(Du_h)_{ij} = (\frac{u_{i+1,j} - u_{ij}}{h}, \frac{u_{i,j+1} - u_{ij}}{h}),$$

and then $F_2(u)$ will be approximated by:

$$F_2(u_h) \approx h^2 \sum_{i,j} \phi(|(Du_h)_{ij}|).$$

(This approximation of the functional is used in many papers, for instance in [32], [33], [16].)

Now, we pass to the precise description of the algorithm in two dimensions.

For any $n \geq 0$ and any fixed (k,l), with $0 \leq k,l \leq M-1$, we consider the pixels (k,l) and $(k,l+1)$ of the image and we introduce two discrete functions v and w, defined by:

$$v = u^n \text{ and } w = (w_{ij})_{ij},\ w_{i,j} = \begin{cases} v_{i,j}, & \text{if } (i,j) \neq (k,l) \text{ and } (i,j) \neq (k,l+1), \\ \frac{v_{k,l} + v_{k,l+1}}{2} & \text{if } (i,j) = (k,l) \text{ or } (i,j) = (k,l+1). \end{cases}$$

Then, we compare, as in the one-dimensional case, $F(v)$ and $F(w)$. In fact, we only need to compare the terms of $F(v)$ and $F(w)$ which depend on $v_{k,l}$ and $v_{k,l+1}$. We define $u^{n,k,l}$ as the signal which minimizes F between v and w:

$$F(u^{n,k,l}) = \min\{F(v), F(w)\},$$

and then we consider the pixels $(k,l+1)$ and $(k,l+2)$. When $k = l = M-1$, the result $u^{n,k,l}$ is denoted by u^{n+1}.

In order to respect the symmetry of the problem and not to privilegiate any direction, we consider all combinations of two neighbors pixels: for any fixed k, with $0 \leq k \leq M-1$, we consider the two pixels $(k,l), (k,l+1)$ and we let increase l from 0 to $(M-1)$, then, for any fixed l, with $0 \leq l \leq M-1$, we consider the pixels $(k,l), (k+1,l)$ and we let increase k from 0 to $(M-1)$. This process is considered an iteration of the algorithm.

4.3 One dimensional reconstruction

In this section we present and discuss several numerical results to denoise various signals, by using the two algorithms described before. We are going to compare the quality of the results, and the time calculation. We will try to give an answer to the question: for a given special class of signals or images, which potential we must use ?

We recall here the three potentials which will be considered in this section, and defined in Chapter I, Introduction.

$$\phi_2(z) = \sqrt{1+z^2} - 1,\ \phi_4(z) = \frac{1}{2}\log(1+z^2),\ \phi_5(z) = \frac{z^2}{2(1+z^2)}.$$

We consider two noisy signals (the first being piecewise-constant and the second piecewise-linear). A-priori, there are no optimal choice for the parameters (the regularizing parameter α, the step space h which plays a role in the behavior of the potential ϕ and the number of iterations). For the results presented in this section, we have first tested the algorithms for different values of the parameters, for each potential, and then we have retained the values for which the results have been the best. In Part III, devoted to the two-dimensional reconstruction, we will explain and show how we make the choice of parameters.

In Figure 4.1, we present the original piecewise-constant signal, represented by dots, together with the noisy signal, represented by continuous line. In the next figures, we represent each time the original signal (by dots), and the reconstructed signal, represented by continuous line, superposed, and we mention the used potential ϕ.

In Tables 4.1 and 4.2, we give, for each result, the used potential ϕ, the values of parameters α and h, the number of iterations and the time calculation, for the piecewise-constant signal and the piecewise-linear signal, respectively.

Figure	ϕ	α	Step h	Iterations	Time
4.2 (a)	ϕ_2	25	1	20	0.05
4.2 (b)	ϕ_2	142	0.7	5	0.05
4.3 (a)	ϕ_4	100	1.5	30	0.07
4.3 (b)	ϕ_5	1000	5	30	0.08

Table 4.1: Parameters for the piecewise-constant signal.

Figure	ϕ	α	Step h	Iterations	Time
4.5	ϕ_2	50	1.2	25	0.05
4.6 (a)	ϕ_4	100	2.5	30	0.07
4.6 (b)	ϕ_5	1000	5	30	0.07

Table 4.2: Parameters for the piecewise-linear signal.

The piecewise-constant signal

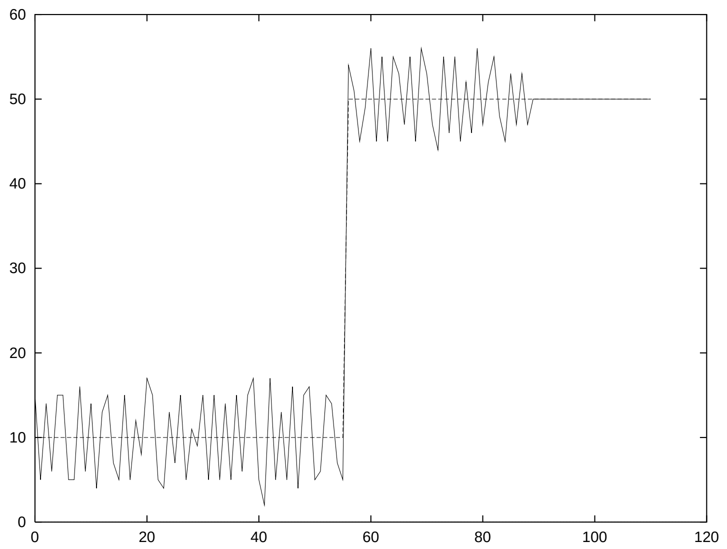


Figure 4.1: An original and noisy signal (piecewise-constant).

In Figure 4.2, we present the result using the convex potential $\phi_2(z) = \sqrt{1+z^2} - 1$. In (a) we have used the discretization of the equation, and in (b) we use the minimization algorithm. We remark that the jump has been a bit smeared in (a), while in (b) this is very well preserved and we have performed only five iterations.

(a)

(b)

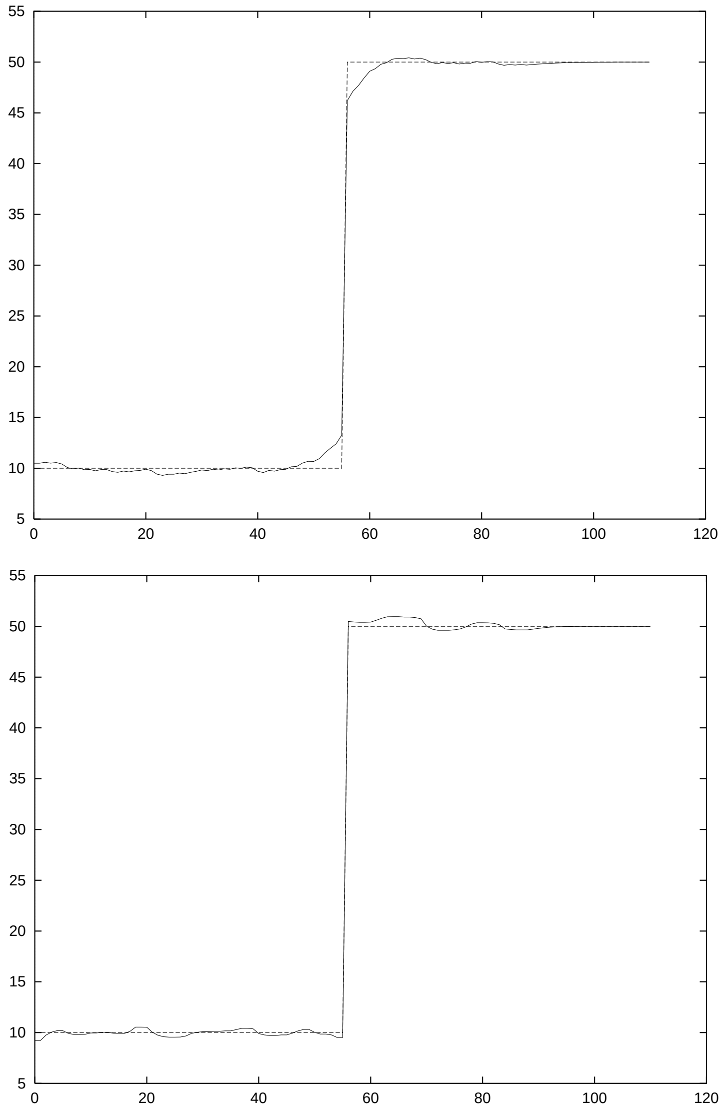


Figure 4.2: Results using $\phi_2(z)$.

In Figure 4.3, we present the results using the discretization of the equation, with the non-convex and sublinear potentials $\phi_4(z) = \frac{1}{2}\log(1+z^2)$ in (a), and $\phi_5(z) = \frac{z^2}{2(1+z^2)}$ in (b). We observe (as we have remarked in Chapter 1), that the jump is very well preserved with these two functions and that the reconstructed signal is very close to the original signal.

(a) ϕ_4

(b) ϕ_5

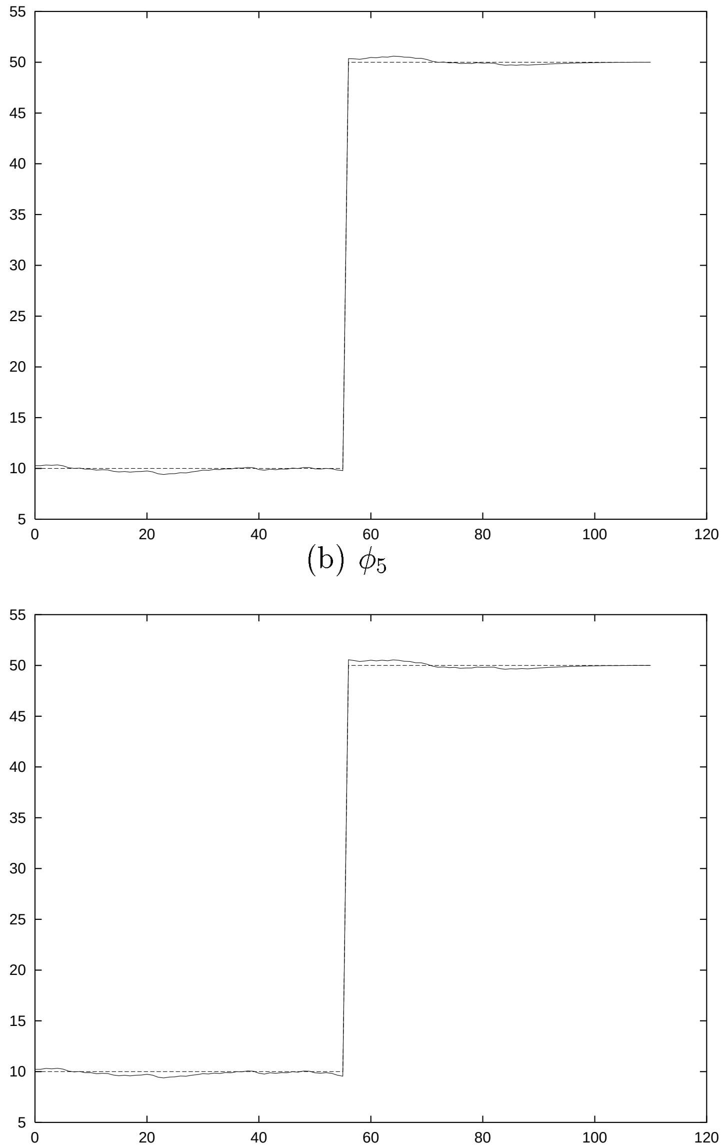


Figure 4.3: Results with ϕ_4 and ϕ_5.

The piecewise-linear signal

In Figure 4.4, we present another original and noisy signal, which is now piecewise-linear. The following results are obtained using only the discretization of the equation, since the minimization algorithm is adapted for piecewise-constant signals.

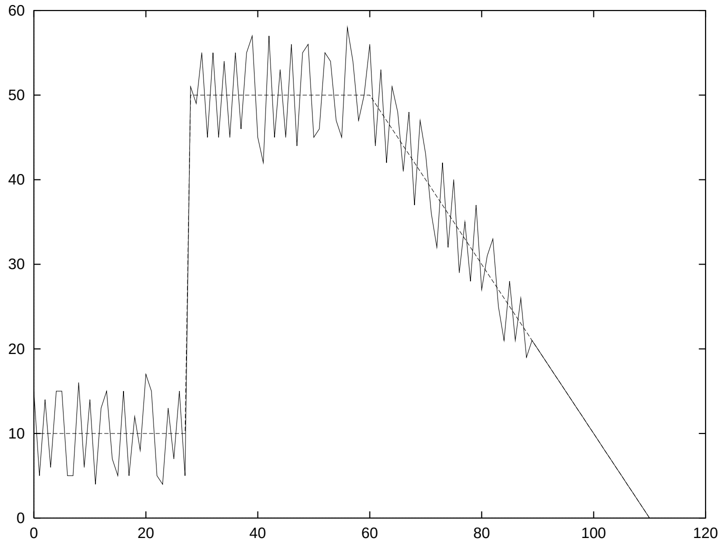


Figure 4.4: An original and noisy piecewise-linear signal.

In figure 4.5, we show the result with the convex function $\phi_2(z) = \sqrt{1+z^2} - 1$. As in the piecewise-constant case, we observe that the jump is a bit smeared.

ϕ_2

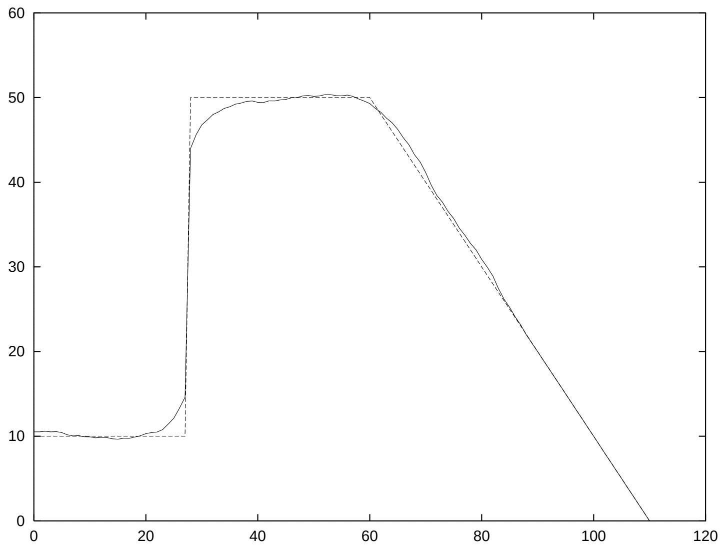
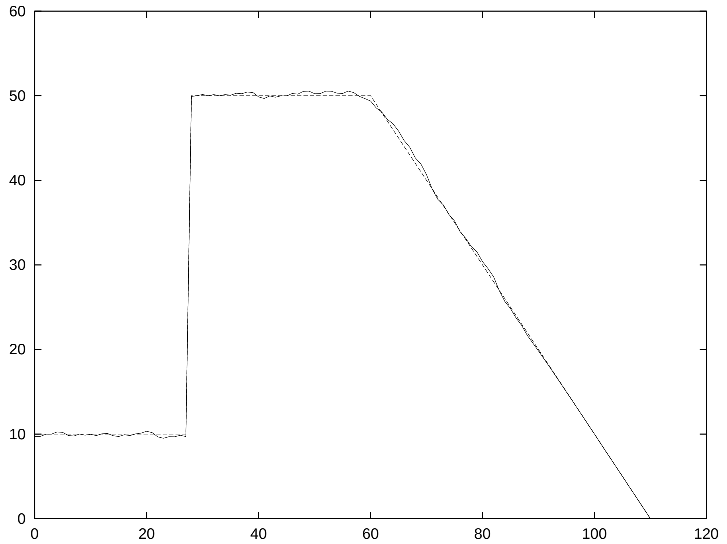


Figure 4.5:

In Figure 4.6, we show the results with $\phi_4(z) = \frac{1}{2}\log(1+z^2)$ in (a) and $\phi_5(z) = \frac{z^2}{2(1+z^2)}$ in (b). These results are surprisingly very good and we observe that the signal has been very well reconstructed in the linear region.

(a) ϕ_4



(b) ϕ_5

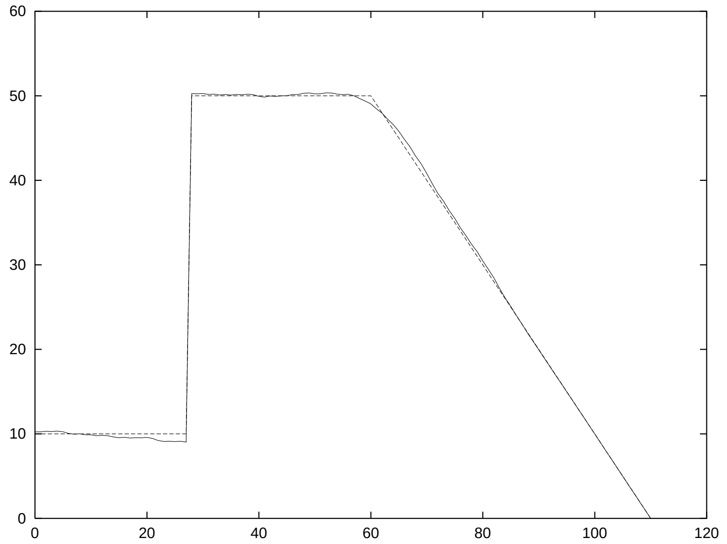


Figure 4.6: Results with ϕ_4 and ϕ_5.

Part II

Viscosity solutions and PDE's in image analysis and curves evolution

In this second part of the thesis, we consider some fully nonlinear elliptic equations of second order, in the context of viscosity solutions, with two applications.

In Chapter 5, we recall the definition, general results and properties of viscosity solutions for degenerate elliptic and parabolic equations of second order.

In Chapter 6, we consider a particular case, of quasilinear equations of second order, which arise from image analysis.

Finally, Chapter 7 is devoted to a method which allows to compute the convex envelope of a given function, by technics of curves evolution. We study the problem in the context of viscosity solutions and we develop a numerical algorithm, to convexify a function, in one and two dimensions. We present several numerical results.

Chapter 5

Viscosity solutions. Definitions and properties

In this chapter, we recall the definition, basic properties and general results of viscosity solutions for fully nonlinear elliptic and parabolic equations of second order. We essentially follow M. G. Crandall, H. Ishii, P. L. Lions ([21]), G. Barles ([6]), Y. G. Chen, Y. Giga, S. Goto ([19]). In Section 5.1, we consider stationary equations and present the general existence result based on Peron's method. In Section 5.2, we briefly consider the evolution case with associated viscosity solutions, and we interpret the initial condition in viscosity sense.

5.1 The stationary case

For the beginning, we follow [19], where discontinuous viscosity solutions are considered.

We need to recall the notions of the *lower semicontinuous (l.s.c.) relaxation* and *upper semicontinuous (u.s.c.) relaxation* (or *envelope*) of a real valued function.

For a function $h : L \to I\!\!R$ ($L \subset I\!\!R^d$, $d = 1, 2, \ldots$), we associate its l.s.c and u.s.c envelopes, defined for $x \in \overline{L}$ by:

$$h_*(x) = \liminf_{y \to x} h(y), \;\; h^*(x) = \limsup_{y \to x} h(y).$$

h_* is the greatest l.s.c function (with values in $[-\infty, +\infty]$), such that $h_* \leq h$. Similarly, h^* is the smallest u.s.c function satisfying $h \leq h^*$ and we have $h^* = -(-h)_*$.

Let now $A \subseteq I\!\!R^N$ and consider a dense subset W of $J(A) := A \times I\!\!R \times I\!\!R^N \times S^{N \times N}$, where $S^{N \times N}$ denotes the space of $N \times N$ real and symmetric matrices. Let $E = E(x, r, p, X)$ be a function from W to $I\!\!R$. Since W is dense in $J(A)$, the relaxations E_* and E^* are defined in $J(A)$ with values in $\overline{I\!\!R}$.

Definition 5.1.1 *A function $u = u(x) : A \to I\!\!R$ is called a viscosity subsolution (supersolution, respectively) of the equation*

$$E(x, u, Du, D^2u) = 0 \quad \text{in } A \tag{5.1.1}$$

if $u^* < \infty$ *(*$u_* > -\infty$*, resp.) in* A*, and for each pair* $\varphi \in C^2(A)$ *and* $x_0 \in A$ *satisfying* $max_A(u^* - \varphi) = (u^* - \varphi)(x_0)$ *(*$min_A(u_* - \varphi) = (u_* - \varphi)(x_0)$*, resp.) it holds that*

$$E_*(x_0, u^*(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq 0$$
$$(E^*(x_0, u_*(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0, \text{ resp.}).$$

If u *is both a viscosity sub- and supersolution of (5.1.1), then* u *is called a viscosity solution of (5.1.1).*

By $\varphi \in C^2(A)$ we mean that φ has a C^2 extension in an open neighborhood of A. Generally, we have $W = J(A)$, but for some particular problems, the function E can not be defined for $p = \vec{0} \in I\!R^N$, for instance. As an example, we can give the so-called mean curvature flow equation, where the solution can has zero gradients.

Definition 5.1.2 *We say that* $E : W \to I\!R$ *is degenerate elliptic, if and only if the following monotonicity condition is satisfied:*

$$E(x, r, p, X) \leq E(x, r, p, Y) \text{ in } W \text{ if } X \geq Y,$$

for all $x \in A$*,* $r \in I\!R$*,* $p \in I\!R^N$ *and* $X, Y \in S^{N\times N}$*, such that* (x, r, p, X) *and* (x, r, p, Y) *belong to* W*.*

We denote by "$\leq$" the partial order on $S^{N\times N}$, defined by:

$$Y \leq X \Leftrightarrow X - Y \geq O \Leftrightarrow ((X - Y)p, p) \geq 0, \ \forall p \in I\!R^N,$$

where $(.,.)$ denotes the usual scaler product in $I\!R^N$. This definition for $X \geq Y$ is equivalent to saying that the matrix $X - Y$ has all the eigenvalues great than or equals to zero. We also denote by $S_+^{N\times N}$ the set of all symmetric matrix X such that $X \geq O$.

Remark 5.1.3

We obtain an equivalent definition for viscosity subsolution, supersolution and solution, if in Definition 5.1.1 we replace "global maximum" by "local maximum" or "strictly local maximum", and "global minimum" by "local minimum" or "strictly local minimum".

Definition 5.1.4 *Let* $A \subset I\!R^N$ *and* $u : A \to I\!R$*. For* $x_0 \in A$*, we define the second-order "super-jet" of* u *at* x_0 *(denoted* $J_A^{2,+}u(x_0)$*) by: if* $(p, X) \in I\!R^N \times S^{N\times N}$*, then* $(p, X) \in J_A^{2,+}u(x_0)$ *if*

$$(5.1.2) \quad u(x) \leq u(x_0) + (p, x - x_0) + \frac{1}{2}(X(x - x_0), x - x_0) + o(|x - x_0|^2) \text{ as } x \to x_0.$$

In the same way, by switching the inequality sign in (5.1.2), we can define the second-order "subjet" $J_A^{2,-}u(x_0)$ *and* $J_A^{2,-}u(x_0) = -J_A^{2,+}(-u)(x_0)$*.*

Remark 5.1.5

We have the following characterization of superjets and subjets of u:

$$J_A^{2,+}u(x_0) = \{(D\varphi(x_0), D^2\varphi(x_0)) : \varphi \text{ is } C^2 \text{ and } u-\varphi \text{ has a local maximum at } x_0\},$$
$$J_A^{2,-}u(x_0) = \{(D\varphi(x_0), D^2\varphi(x_0)) : \varphi \text{ is } C^2 \text{ and } u-\varphi \text{ has a local minimum at } x_0\}.$$

In what follows we always assume that A is locally compact. We give below two basic properties of viscosity solutions and subsolutions, in order to obtain a general existence result. These results are proved in many papers, for instance in [19] in the discontinuous case. To preserve a unitary and complete exposition of this part, we also present the proofs.

Proposition 5.1.6 *Let S be a nonempty family of viscosity subsolutions of (5.1.1) and let u be a function defined on A by*

$$u(x) = \sup\{v(x);\ v \in S\} \text{ for } x \in A.$$

Suppose $u^*(x) < \infty$ for $x \in A$. Then u is a viscosity subsolution of (5.1.1).

Proof. First, we choose a function $\varphi \in C^2(A)$ and a point x_0 such that

$$\max_A (u^* - \varphi) = (u^* - \varphi)(x_0),$$

and then fix φ and x_0. Here we can assume $(u^* - \varphi)(x_0) = 0$ without loss of generality, since the function $\varphi(x)$ can be replaced by $\varphi(x) + (u^* - \varphi)(x_0)$. If $\psi(x) = \varphi(x) + |x - x_0|^4$, we see that $u^* - \psi$ attains its strict maximum in A at x_0. Then

$$(u^* - \psi)(x) + |x - x_0|^4 = (u^* - \varphi)(x) \le (u^* - \varphi)(x_0) = (u^* - \psi)(x_0) = 0$$

yields

$$(u^* - \psi)(x) \le -|x - x_0|^4 \quad \text{for } x \in A. \tag{5.1.3}$$

By the definition of u^*, there is a sequence $\{x_k\} \subset A$ such that $x_k \to x_0$ $(k \to \infty)$ and

$$\lim_{k \to \infty} a_k = (u^* - \varphi)(x_0) = 0$$

with $a_k := (u - \varphi)(x_k)$. Since $u(x) = \sup\{v(x);\ v \in S\}$, there exists a sequence $\{v_k\} \subset S$ such that $v_k(x_k) > u(x_k) - 1/k$ $(k = 1, 2, ...)$. This implies

$$(v_k^* - \varphi)(x_k) \ge (v_k - \varphi)(x_k) > a_k - 1/k.$$

Since $v_k \le u$, by (5.1.3) we get

$$(v_k^* - \psi)(x) \le -|x - x_0|^4 \quad \text{for } x \in A.$$

Since A is locally compact, there is a compact neighborhood of x_0 which we denote by B. Because the function $(v_k^* - \varphi)(x)$ is u.s.c. and has an upper bound, it attains its maximum in B at a point $y_k \in B$. We thus conclude

$$a_k - 1/k < (v_k^* - \psi)(x_k) \le (v_k^* - \psi)(y_k) \le -|y_k - x_0|^4. \tag{5.1.4}$$

This implies $y_k \to x_0$ $(k \to \infty)$. Indeed, if not, we would have

$$0 = \lim_{k\to\infty} (u - \varphi)(x_k) \le \limsup_{k\to\infty} (1/k - |y_k - x_0|^4) < 0,$$

which is a contradiction.

By (5.1.4) we can see $\lim_{k\to\infty}(v_k^* - \psi)(y_k) = 0$, and hence

$$\lim_{k\to\infty} v_k^*(y_k) = \lim_{k\to\infty} \psi(y_k) = \psi(x_0) = u^*(x_0).$$

Since v_k is a subsolution we see

$$E_*(y_k, v_k^*(y_k), D\psi(y_k), D^2\psi(y_k)) \le 0.$$

Since E_* is l.s.c and $D\psi(x_0) = D\varphi(x_0)$, $D^2\psi(x_0) = D^2\varphi(x_0)$, this yields

$$E_*(x_0, u^*(x_0), D\varphi(x_0), D^2\varphi(x_0)) \le 0,$$

which implies that u is a subsolution. ■

We note that, in Proposition 5.1.6, we have not assumed that E is degenerate elliptic. Now, to present the general existence result, based on Perron's method, we need the following lemma:

Lemma 5.1.7 *Suppose that E is degenerate elliptic. Let $g : A \to \mathbb{R}$ be a supersolution of (5.1.1). Let S be the collection of all subsolutions v of (5.1.1), satisfying $v \le g$ in A. If $v \in S$ is not a supersolution of (5.1.1), then there is a function $w \in S$ and a point $z \in A$ such that $v(z) < w(z)$.*

Proof. Since v is not a supersolution, there exist $\varphi \in C^2(A)$ and $x_0 \in A$ such that

$$\min_A (v_* - \varphi) = (v_* - \varphi)(x_0) = 0$$

and

$$E^*(x_0, v_*(x_0), D\varphi(x_0), D^2\varphi(x_0)) = E^*(x_0, \varphi(x_0), D\varphi(x_0), D^2\varphi(x_0)) < 0.$$

We may assume here

$$(v_* - \varphi)(x) \ge |x - x_0|^4 \quad \text{for } y \in A, \tag{5.1.5}$$

because the function φ can be modified as $\varphi + |x - x_0|^4$ if necessary. Evidently, we have $v_* \le g_*$ in A. We now obtain $v_*(x_0) = \varphi(x_0) < g_*(x_0)$, since, otherwise, it would contradict the fact that g is a supersolution of (5.1.1). Considering that E^* is u.s.c and $\varphi \in C^2(A)$, for sufficiently small $\delta > 0$ we have

$$E^*(x, \varphi(x) + \delta^4/2, D\varphi(x), D^2(\varphi(x))) \le 0, \tag{5.1.6}$$

$$\varphi(x) + \delta^4/2 \le g_*(x) \tag{5.1.7}$$

for $x \in B_{2\delta} \equiv B \cap \overline{B(x_0, 2\delta)}$, where B is a compact neighborhood of x_0, and $B(x, R)$ is the open ball of radius R centered at x. (Such B exists since A is locally compact.) Since

E is degenerate elliptic, the inequality (5.1.6) indicates that, by $\varphi \in C^2(A)$, the function $\varphi(x) + \delta^4/2$ is a subsolution in $B_{2\delta}$. Furthermore, by (5.1.5) we have

$$v(x) \geq v_*(x) - \delta^4/2 \geq \varphi(x) + \delta^4/2 \text{ on } B_{2\delta}\backslash B_\delta. \tag{5.1.8}$$

We now define $w(x)$ by

$$w(x) := \begin{cases} \max\{\varphi(x) + \delta^4/2, v(x)\}, & x \in B_\delta, \\ v(x), & x \in A\backslash B_\delta. \end{cases}$$

It follows from (5.1.8) that

$$w(x) = \max\{\varphi(x) + \delta^4/2, v(x)\} \text{ for } x \in B_{2\delta}.$$

According to Proposition 5.1.6, w is a subsolution of (5.1.1) over the whole of A and, thus, $w \in S$ since (5.1.7) holds.

However, we have:

$$0 = (v_* - \varphi)(x_0) = \liminf_{t\to 0}\{(v-\varphi)(x);\ x \in A \text{ and } |x - x_0| \leq t\}.$$

This implies that there is a point $z \in B_\delta$ such that $v(z) - \varphi(z) < \delta^4/2$, which yields $v(z) < w(z)$. ∎

Proposition 5.1.8 (Existence). *Suppose that E is degenerate elliptic. Let f and g from A into $I\!R$ be respectively a viscosity sub- and supersolution of (5.1.1). Assume $f \leq g$ in A. Then there exists a viscosity solution u of (5.1.1) satisfying $f \leq u \leq g$ in A.*

Proof. We appeal to Perron's method. As in Lemma 5.1.7, we set

$$S = \{v;\ v \text{ is a subsolution of (5.1.1) and } v \leq g\}.$$

Since $f \in S$, we see $S \neq \emptyset$. We define $u(x) := \sup\{v(x);\ v \in S\}$. By Proposition 5.1.6, u is a subsolution of (5.1.1), so $u \in S$ since $u \leq g$. Suppose that u were not a supersolution of (5.1.1). Then by Lemma 5.1.7 there would exist $w \in S$ such that $u(z) < w(z)$ for some $z \in A$. This is contrary to the definition of u. We thus conclude that u is a viscosity solution of (5.1.1) and $f \leq u \leq g$. ∎

5.2 The evolution problem

It is easy to extend the results of the previous section to evolution equations. Now we will use the notations Du and D^2u to denote the spatial derivatives of u.

Let us consider the following evolution problem:

$$\frac{\partial u}{\partial t} + G(t, x, u, Du, D^2u) = 0 \text{ in } (0,T) \times I\!R^N \tag{5.2.1}$$

associated to the initial condition:

$$u(0,x) = u_0(x) \text{ in } I\!R^N,$$

where u_0 is a function, not necessary continuous, but, for instance, locally bounded, in order to have: $u_0^* < \infty$ and $(u_0)_* > -\infty$.

The function $G : [0,+\infty[\times I\!R^N \times I\!R \times I\!R^N \times S^{N\times N} \to \infty$ is assumed to be degenerate elliptic. We will only precise the initial condition in viscosity sense.

A locally bounded function u is a subsolution of (5.2.1) in $[0,T) \times I\!R^N$, if for each $\varphi \in C^2([0,T) \times I\!R^N)$ and (t_0,x_0) as a local maximum of $u^* - \varphi$, it holds:

$$\begin{cases} \frac{\partial\varphi}{\partial t}(t_0,x_0) + G_*(t_0,x_0,u^*(t_0,x_0),D\varphi(t_0,x_0),D^2\varphi(t_0,x_0)) \le 0 \\ \text{if } (t_0,x_0) \in (0,T) \times I\!R^N, \end{cases}$$

and

$$\min\Big\{\Big(\frac{\partial\varphi}{\partial t}(t_0,x_0) + G_*(t_0,x_0,u^*(t_0,x_0),D\varphi(t_0,x_0),D^2\varphi(t_0,x_0))\Big), \Big(u^*(t_0,x_0) - u_0^*(x_0)\Big)\Big\} \le 0$$
$$\text{if } (t_0,x_0) \in \{0\} \times I\!R^N.$$

In the same way, a locally bounded function v is supersolution in $[0,T) \times I\!R^N$, if for each $\varphi \in C^2([0,T) \times I\!R^N)$ and (t_0,x_0) as a local minimum of $v_* - \varphi$, it holds:

$$\begin{cases} \frac{\partial\varphi}{\partial t}(t_0,x_0) + G^*(t_0,x_0,v_*(t_0,x_0),D\varphi(t_0,x_0),D^2\varphi(t_0,x_0)) \ge 0 \\ \text{if } (t_0,x_0) \in (0,T) \times I\!R^N, \end{cases}$$

and

$$\max\Big\{\Big(\frac{\partial\varphi}{\partial t}(t_0,x_0) + G^*(t_0,x_0,v_*(t_0,x_0),D\varphi(t_0,x_0),D^2\varphi(t_0,x_0))\Big), \Big(v_*(t_0,x_0) - (u_0)_*(x_0)\Big)\Big\} \ge 0$$
$$\text{if } (t_0,x_0) \in \{0\} \times I\!R^N.$$

The following result, presented in [6] for first order equations, can be easily extended to the second order case and shows that, in fact, we have a classical initial condition.

Theorem 5.2.1 *We assume that G and u_0 are locally bounded. If u (v resp.) is a locally bounded subsolution (supersolution, resp.) of (5.2.1) in $[0,T) \times I\!R^N$, then*

$$u^*(0,x) \le u_0^*(x) \text{ in } I\!R^N,$$
$$(v_*(0,x) \ge (u_0)_*(x) \text{ in } I\!R^N, \text{ resp.}).$$

Proof. We only show the property for u, for v this being similarly. Let $x \in I\!R^N$. For $\varepsilon > 0$ sufficiently small and $C > 0$ sufficiently large, the function

$$u^*(t,y) - \varphi(y,t) := u^*(t,y) - \Big(\frac{|y-x|^2}{\varepsilon} + Ct\Big),$$

will have a local maximum point (t_0,y_0) in a neighborhood of $(0,x)$. Moreover, if we fix ε and if C is sufficiently large, then the viscosity inequality:

$$C + G_*(t_0,y_0,u^*(t_0,y_0),D\varphi(t_0,y_0),D^2\varphi(t_0,y_0)) \le 0$$

can not be true. Then $t_0 = 0$ and we also have $u^*(0, y_0) \leq u_0^*(y_0)$. But, since $(0, y_0)$ is a local maximum of $u^* - \varphi$ in a neighborhood of $(0, x)$, $u^*(0, y_0) - \varphi(0, y_0) \geq u^*(0, x) - \varphi(0, x)$, and then $u^*(0, y_0) \geq u^*(0, x)$. Finally,

$$u^*(x, 0) \leq u_0^*(y_0),$$

and we conclude by making $\varepsilon \to 0$ and $C \to \infty$: then y_0 tends to x and since u_0^* is u.s.c., we have:

$$u(0, x) \leq u_0^*(y_0) \leq \limsup_{y_0 \to x} u_0^*(y_0) \leq u_0^*(x).$$

∎

Chapter 6

Quasilinear equations of second order in image analysis

In this chapter we consider some quasilinear elliptic and parabolic equations of second order. We present the study of the problems in the general case, in the context of viscosity solutions, and we show some examples arising from image analysis.

6.1 The general problem

Let the functions $a_{ij} : I\!R^N \backslash \{\vec{0}\} \to I\!R$, $1 \leq i, j \leq N$, be continuous and satisfy the following assumptions (we use the convention on repeated indices):

$$a_{ij}(p)\xi_i\xi_j \geq 0, \text{ for all } p \in I\!R^N \backslash \{\vec{0}\}, \ \xi \in I\!R^N. \tag{6.1.1}$$

$$(a_{ij})^*(0) < \infty, \ (a_{ij})_*(0) > -\infty. \tag{6.1.2}$$

We define $G : (I\!R^N \backslash \{\vec{0}\}) \times S^{N\times N} \to I\!R$ by

$$G(p, X) = -trace(A(p)X),$$

where we have denoted by $A \in M^{N\times N}$ the matrix function $(a_{ij})_{i,j=1,N}$.

We are going to consider the following quasilinear equation:

$$u + G(Du, D^2u) = u_0 \text{ in } I\!R^N, \tag{6.1.3}$$

where, for the moment, the function $u_0 : I\!R^N \to I\!R$ is assumed to be locally bounded.

In order to study this equation in the context of viscosity solutions, G need to be degenerate elliptic in $I\!R^N \backslash \{\vec{0}\} \times S^{N\times N}$, and since G is linear in X, it is then necessary to have:

$$G(p, X) \leq G(p, O) = 0, \text{ whenever } X \geq O.$$

This is true from the assumption (6.1.1) and the following property on matrices: if $A, X \in S^{N\times N}$ (i.e. $A \geq O, X \geq O$), then $trace(AX) \geq O$.

Moreover, for the l.s.c. and the u.s.c. envelope of G (at $\vec{0}$), we evidently have:

$$G_*(\vec{0}, X) = -(a_{ij})^*(\vec{0})x_{ij} \geq 0 \geq -(a_{ij})_*(\vec{0})y_{ij} = G^*(\vec{0}, Y), \tag{6.1.4}$$

if $X=(x_{ij})_{ij}\le O\le Y=(y_{ij})_{ij}$, since $a^*_{ij}(\vec{0})$ and $(a_{ij})_*(\vec{0})$ also satisfy (6.1.1), for $p=\vec{0}$.

Now, we will recall the definition of a viscosity subsolution, supersolution and solution applied to the problem (6.1.3).

A locally bounded function $u:I\!\!R^N\to I\!\!R$ is a viscosity subsolution of (6.1.3) if, for all $\varphi\in C^2(I\!\!R^N)$, the following condition holds at any local maximum point x_0 of $(u^*-\varphi)$:

$$\begin{cases} u^*(x_0)+G(D\varphi(x_0),D^2\varphi(x_0))\le u_0^*(x_0) \text{ if } D\varphi(x_0)\ne\vec{0}\\ u^*(x_0)+G_*(\vec{0},D^2\varphi(x_0))\le u_0^*(x_0) \text{ if } D\varphi(x_0)=\vec{0}.\end{cases}$$

Similarly, the locally bounded function $v:I\!\!R^N\to I\!\!R$ is a viscosity supersolution, if for all $\varphi\in C^2(I\!\!R^N)$, with x_0 a local minimum of $(u_*-\varphi)$, it holds:

$$\begin{cases} u_*(x_0)+G(D\varphi(x_0),D^2\varphi(x_0))\ge (u_0)_*(x_0) \text{ if } D\varphi(x_0)\ne\vec{0}\\ u_*(x_0)+G^*(\vec{0},D^2\varphi(x_0))\ge (u_0)_*(x_0) \text{ if } D\varphi(x_0)=\vec{0},\end{cases}$$

and u is a viscosity solution if it is both subsolution and supersolution.

Theorem 6.1.1 *(Comparison) Let $u_0:I\!\!R^N\to I\!\!R$ be a bounded and continuous function. Let u be a bounded subsolution of (6.1.3) and v a bounded supersolution of (6.1.3). Then $u^*\le v_*$ in $I\!\!R^N$. Moreover, if u is a viscosity solution of (6.1.3) with the data u_0 continuous and bounded, then u is continuous.*

Proof. Let us assume that $u^*(\bar{x})-v_*(\bar{x})=\delta>0$, for some $\bar{x}\in I\!\!R^N$ and let

$$\Phi(x,y)=u^*(x)-v_*(y)-\frac{\alpha}{4}|x-y|^4-\varepsilon(|x|^2+|y|^2),$$

where α,ε are positive parameters. For small ε, we have $\Phi(\bar{x},\bar{x})\ge\delta_1>0$, independently of ε and α.

Since u and $-v$ are u.s.c. and bounded, then Φ will have a global maximum point (x_0,y_0) (of course, (x_0,y_0) depends on α and ε).

But $\Phi(x_0,y_0)\ge\delta_1>0$, independently of α and ε.

Using Thm. 3.2 and Remark 3.8 from [21], we have the existence of $X_0,Y_0\in S^{N\times N}$ such that

$$\begin{aligned}(p_0+2\varepsilon x_0,X_0+2\varepsilon I)&\in\overline{J}^{2,+}u^*(x_0),\\ (p_0-2\varepsilon y_0,Y_0-2\varepsilon I)&\in\overline{J}^{2,-}v^*(y_0),\end{aligned}$$

where

$$p_0=\alpha|x_0-y_0|^2(x_0-y_0),$$

$\|X_0\|,\|Y_0\|\le C_1\alpha|x_0-y_0|^2$ and $X_0\le Y_0$.

We evidently have

$$\Phi(x_0,y_0)\ge\Phi(\vec{0},\vec{0})=u^*(\vec{0})-v_*(\vec{0})=C_2.$$

Then, for some $C_3>0$:

$$\frac{\alpha}{4}|x_0-y_0|^4+\varepsilon(|x_0|^2+|y_0|^2)\le u^*(x_0)-v_*(y_0)-C_2\le C_3,$$

independently of α and ε, since u and v are bounded.

Therefore, $\varepsilon(|x_0|^2+|y_0|^2)$, $\alpha|x_0-y_0|^3$, and hence p_0, are bounded independently of $\varepsilon \leq 1$, for fixed $\alpha \geq 1$. Hence, $\varepsilon x_0, \varepsilon y_0 \to \vec{0}$ as $\varepsilon \to 0^+$.

We have:

$$\begin{aligned} u^*(x_0) + G_*(p_0+2\varepsilon x_0, X_0+2\varepsilon I) &\leq u_0(x_0), \\ v_*(y_0) + G^*(p_0-2\varepsilon y_0, Y_0-2\varepsilon I) &\geq u_0(y_0), \end{aligned}$$

with $G_* = G$ and $G^* = G$ if $p_0+2\varepsilon x_0 \neq \vec{0}$, $p_0-2\varepsilon y_0 \neq \vec{0}$.

Then we obtain:

$$u^*(x_0)-v_*(y_0) \leq G^*(p_0-2\varepsilon y_0, Y_0-2\varepsilon I) - G_*(p_0+2\varepsilon x_0, X_0+2\varepsilon I) + u_0(x_0) - u_0(y_0),$$

or

$$\begin{gathered} \Phi(x_0,y_0) \\ = u^*(x_0)-v_*(y_0)-\tfrac{\alpha}{4}|x_0-y_0|^4 - \varepsilon(|x_0|^2+|y_0|^2) \\ \leq G^*(p_0-2\varepsilon y_0, Y_0-2\varepsilon I) - G_*(p_0+2\varepsilon x_0, X_0+2\varepsilon I) \\ +u_0(x_0)-u_0(y_0)-\tfrac{\alpha}{4}|x_0-y_0|^4 - \varepsilon(|x_0|^2+|y_0|^2) \\ \leq G^*(p_0-2\varepsilon y_0, Y_0-2\varepsilon I) - G_*(p_0+2\varepsilon x_0, X_0+2\varepsilon I) + u_0(x_0)-u_0(y_0)-\tfrac{\alpha}{4}|x_0-y_0|^4 \\ \leq G^*(p_0-2\varepsilon y_0, Y_0-2\varepsilon I) - G_*(p_0+2\varepsilon x_0, X_0+2\varepsilon I) + M_\alpha, \end{gathered}$$

where

$$M_\alpha = \sup_{x,y}(u_0(x)-u_0(y)-\frac{\alpha}{4}|x-y|^4).$$

Then

$$\Phi(x_0,y_0) \leq G^*(p_0-2\varepsilon y_0, Y_0-2\varepsilon I) - G_*(p_0+2\varepsilon x_0, X_0+2\varepsilon I) + M_\alpha. \tag{6.1.5}$$

For $\varepsilon \to 0^+$, (p_0, X_0, Y_0) will have a limit point (p, X, Y) with $X \leq Y$ and if $p=0$, then $X=Y=O$.

In (6.1.5), we let $\varepsilon \to 0^+$. Then

$$\limsup_{\varepsilon\to 0^+} \Phi(x_0,y_0) \leq M_\alpha \;\forall\; \alpha \geq 1.$$

Then, for $\alpha \to \infty$:

$$0 < \delta_1 \leq \limsup_{\alpha\to\infty}(\limsup_{\varepsilon\to 0^+}\Phi(x_0,y_0)) \leq \lim_{\alpha\to\infty} M_\alpha = \sup_{I\!R^N}(u_0-u_0) = 0,$$

which is a contradiction. ∎

Theorem 6.1.2 *(Existence) Let $u_0 : \Omega \to I\!R$ be bounded, where $\Omega \subset I\!R^N$ is locally compact. Then the equation (6.1.3) has a viscosity solution $u : \Omega \to I\!R$ such that:*

$$\inf_\Omega u_0 \leq u \leq \sup_\Omega u_0 \;\; in\ \Omega.$$

Proof. It suffices to produce a subsolution u^- and a supersolution u^+, with $u^- \leq u^+$ in Ω, and then to use the general existence result (Proposition 5.1.8).

Let for $x \in \Omega$, $u^-(x) := \inf_\Omega u_0$ and $u^+(x) := \sup_\Omega u_0$.

If $\varphi \in C^2(\Omega)$ and $x_0 \in \Omega$ are such that $u^- - \varphi$ has a local maximum at x_0, then $D\varphi(x_0) = \vec{0}$ and $D^2\varphi(x_0) \geq O$. Therefore, the inequality:

$$u^-(x_0) + G_*(\vec{0}, D^2\varphi(x_0)) \leq u_0^*(x_0)$$

is evidently satisfied, from the definition of u^- and the property (6.1.1) written for $a_{ij}^*(\vec{0})$. Then u^- is a subsolution of (6.1.3). In the same way, we show that u^+ is a supersolution and we conclude by applying Proposition 5.1.8. ■

Of course, the application of such an equation to image processing, considering u_0 as the initial image, require to solve (6.1.3) in a subdomain of $I\!R^N$, for instance in $[0,1]^N$ and with classical Neumann boundary conditions. So, in order to simplify the presentation and to skip some tedious and unpleasant technicalities at the corners, we prefer to work in $I\!R^N$ and with periodic solutions. Anyway, the natural extension of a picture adopted in image processing is to extend u (and u_0) by reflection across the boundary of the rectangle. Thus one sets $u(-x,y) = u(x,y)$ if $-1 \leq x \leq 0$, and $0 \leq y \leq 1$, etc... One easily sees that with this extension, u and u_0 can be assumed to be periodic on $(2Z^N)$.

Now we conclude the study of the problem (6.1.3), in the class of periodic and continuous functions on $I\!R^N$, by the following result, which is a simple consequence of the previous existence and comparison results.

Theorem 6.1.3 *Let $u_0 : I\!R^N \to I\!R$ be continuous and periodic ($u_0(x+h) = u_0(x)$, for a given $h \in I\!R^N$, with $h_i > 0$ and every $x \in I\!R^N$). Then there is a unique viscosity solution u for (6.1.3) in $I\!R^N$. Moreover, u is continuous in $I\!R^N$ and satisfies:*

$$\inf_{I\!R^N} u_0 \leq u \leq \sup_{I\!R^N} u_0 \ \text{in } I\!R^N,$$
$$u(x+h) = u(x) \ \text{in } \Omega.$$

Proof. Since u_0 is periodic and continuous on $I\!R^N$, then it is bounded. Therefore, from the existence result (Theorem 6.1.2), there is a bounded viscosity solution u of (6.1.3) in $I\!R^N$ and we can apply the comparison result (Theorem 6.1.1) for u as a subsolution and supersolution of the equation in the same time. We deduce that u is the unique viscosity solution of the problem and that u is continuous in $I\!R^N$. Moreover, since G is independent of x and u_0 is periodic, then the function w defined by $w(x) = u(x+h)$ is also a viscosity solution of (6.1.3) in $I\!R^N$. Then, by the uniqueness (or the comparison result), we deduce that $w(x) = u(x+h) = u(x)$ in $I\!R^N$, i.e. u is periodic like u_0. Of course, we will have: $\inf_{I\!R^N} u_0 \leq u \leq \sup_{I\!R^N} u_0$ in $I\!R^N$. ■

Remark 6.1.4

(1) We can also consider the associated evolution problem, as in Part I:

$$(6.1.6) \qquad \begin{cases} \frac{\partial u}{\partial t} + u + G(Du, D^2u) = u_0 \text{ in } (0,T) \times I\!R^N \\ u(0,\cdot) = u_0. \end{cases}$$

As in the stationary case, we can show the existence and the uniqueness of a continuous and periodic (in x, for fixed t) viscosity solution to this evolution problem, if u_0 is continuous and periodic. We are not going into the details to show that this is true. The proofs are similar with the stationary case, with slight modifications. We can also show, as in Part I, that $u(t,\cdot)$ converges, as $t \to \infty$, to u, the viscosity solution of the stationary problem.

(2) There are several quasilinear equations of second order, arising from image analysis and curves evolution, and studied in the context of viscosity solutions. We wish to mention here two papers, of Lions-Morel-Alvarez [41], on the anisotropic diffusion, and of Evans-Spruck [31], on the mean curvature motion. The proofs are slight different, since in the comparison, we have first considered an unbounded domain, and in the existence we use the Peron's method, while these authors use approximations by uniformly elliptic PDE's.

(3) Unfortunately, we have not considered here the case $K \neq I$. Generally, to get the uniqueness of a viscosity solution, the function $E(x, r, p, X)$ from (5.1.1), Chapter 5, need to be increasing with respect to r. In our case, it is not clear how to interpret the nonlocal term K^*Ku as an increasing function in the u-variable.

Example. Let us consider for instance the following stationary equation in divergence form, introduced in Part I, in the denoising case ($K = I$):

$$u - \alpha \mathrm{div}\Big(\frac{\phi'(|Du|)}{|Du|}Du\Big) = u_0, \tag{6.1.7}$$

where u_0, u are functions defined from the open and bounded subset Ω of $\mathbb{R}^N$ into $\mathbb{R}$, $\phi : [0,\infty) \to [0,\infty)$ is a given function and $\alpha > 0$ a regularizing parameter. The data u_0 represents the initial image, and the solution u is the reconstructed image.

We assume that ϕ is twice differentiable and we define $\psi : (0,\infty) \to (0,\infty)$ by:

$$\psi(z) = \frac{\phi'(z)}{z}.$$

We will formally develop the divergence operator (when $|Du| \neq 0$) and we will obtain in this way a quasilinear equation of second order. Let $G : (\mathbb{R}^N \backslash \{\vec{0}\}) \times S^{N\times N} \to \mathbb{R}$ defined by (for $p \neq \vec{0}$):

$$G(p, X) = -trace(A(p)X) = -a_{ij}(p)x_{ij},$$

where $A(p) \in S^{N\times N}$ is the matrix defined by:

$$(A(p))_{ii} = a_{ii}(p) = \psi(|p|) + \psi'(|p|)|p|\tfrac{p_i^2}{|p|^2},$$
$$(A(p))_{ij} = (A(p))_{ji} = a_{ij}(p) = a_{ji}(p) = \psi'(p)|p|\tfrac{p_ip_j}{|p|^2}.$$

We let, for fixed $p \in \mathbb{R}^N \backslash \{\vec{0}\}$:

$$a = a(p) = \psi(|p|),\ \ b = b(p) = \psi'(|p|)|p|,$$
$$c_i = c_i(p) = \tfrac{p_i}{|p|},\ \text{ for } 1 \leq i \leq N.$$

We wish to obtain necessary and sufficient assumptions on ϕ, such that G will be degenerate elliptic in $(I\!R^N\backslash\{\vec{0}\}) \times S^{N\times N}$, for arbitrary dimension N. We have seen that we must have: $A = A(p) \geq O$. Let for the moment $N = 2$. Then A can be written, using the previous notations, as:

$$A = \begin{pmatrix} a + bc_1^2 & bc_1c_2 \\ bc_1c_2 & a + bc_2^2 \end{pmatrix}$$

and then we have the following sequence of equivalences:

$$A \geq O \Leftrightarrow \begin{cases} a + bc_1^2 \geq 0, \ \forall c_1 \in I\!R \\ (a + bc_1^2)(a + bc_2^2) = a(a+b) \geq 0 \end{cases} \Leftrightarrow \begin{cases} a \geq 0 \\ a + b \geq 0 \end{cases} \Leftrightarrow \begin{cases} \phi' \geq 0 \\ \phi'' \geq 0 \end{cases} \text{in } (0, \infty).$$

Now, for arbitrary $N \in I\!N^*$, we assume that $\phi' \geq 0$ and $\phi'' \geq 0$ in $(0, \infty)$. Then $a \geq 0$ and $a + b \geq 0$, for all $p \in I\!R^N\backslash\{\vec{0}\}$. We will show that $A(p) \geq O$, for all $p \in I\!R^N\backslash\{\vec{0}\}$, and then G will be degenerate elliptic in $(I\!R^N\backslash\{\vec{0}\}) \times S^{N\times N}$. Indeed, we have the following:

$$a_{ij}\xi_i\xi_j = a(\sum_{i=1}^N \xi_i^2) + b(\sum_{i=1}^N c_i\xi_i)^2.$$

If $b \geq 0$, then we will evidently have that $a_{ij}\xi_i\xi_j \geq 0$. If $b < 0$, then by the Cauchy-Schwartz inequality:

$$(\sum_{i=1}^N c_i\xi_i)^2 \leq (\sum_{i=1}^N c_i^2)(\sum_{i=1}^N \xi_i^2) = \sum_{i=1}^N \xi_i^2,$$

we obtain that:

$$a_{ij}\xi_i\xi_j \geq (a+b)(\sum_{i=1}^N \xi_i^2) \geq 0.$$

Finally, for each $p \in I\!R^N\backslash\{\vec{0}\}$, we will have:

$$a_{ij}(p)\xi_i\xi_j \geq 0, \ \forall \xi \in I\!R^N.$$

For the potential $\phi_2(z) = \sqrt{1+z^2}$, the associated functions a_{ij} (and a, b respectively) are continuous and well-defined in the whole space $I\!R^N$ (in $[0, \infty)$ respectively).

6.2 The discretization of the quasilinear equation

In Part I of this thesis, Chapter 4, we have described a numerical approximation for the Euler equation, written in conservative form, naturally deduced from the continuous problem, with $L^\infty-$stability properties and convergence results in the convex case. In this section, we consider a general quasilinear operator of second order, for which we generalize a finite difference scheme, proposed for the second derivative in a given direction of a function $u(x, y)$ (see [2]) and for the curvature operator (see [42]: L. Alvarez and F. Guichard's scheme). For the particular case of the Euler equation associated to the minimization problem, this scheme gives a good estimate for the divergence term written in expanded form and has some geometric invariance properties.

We will only consider the two-dimensional case, that is, $N = 2$. The limit $\lim_{z\to 0^+} \frac{\psi'(z)}{z}$ exists and it is finite for all the potentials ϕ considered in Part I. Let G be the function, such that we formally have:

$$-\mathrm{div}\Big(\frac{\phi'(|Du|)}{|Du|}Du\Big) = -\mathrm{div}(\psi(|Du|)Du) = G(Du, D^2u).$$

Let $G : I\!R^2 \times S^{2\times 2} \to I\!R$ be defined by (for $p = (p_1, p_2) \in I\!R^2$ and $X = (x_{ij})_{i,j=1,2} \in S^{2\times 2}$):

$$\begin{aligned} G(p, X) := \quad & - \quad \Big(\psi(|p|) + \psi'(|p|)|p|\frac{p_1^2}{p_1^2+p_2^2}\Big)x_{11} - 2\Big(\psi'(|p|)|p|\frac{p_1 p_2}{p_1^2+p_2^2}\Big)x_{12} \\ & - \quad \Big(\psi(|p|) + \psi'(|p|)|p|\frac{p_2^2}{p_1^2+p_2^2}\Big)x_{22}, \text{ if } |p| > 0, \end{aligned}$$

and if $|p| = 0$: $G(\vec{0}, X) := -\psi(0)(x_{11} + x_{22})$. Then the expanded equation can be formally written as:

$$\begin{gathered} u + \alpha G(Du, D^2u) = u_0, \text{ if } |Du| > 0, \\ u - \alpha\psi(0) \bigtriangleup u = u_0, \text{ if } |Du| = 0. \end{gathered}$$

We can easily adapt and generalize the method presented in Lopez [42] and applied to the following operator:

$$|Du|\mathrm{div}\big(\frac{Du}{|Du|}\big) = |Du|\mathrm{curv}u = \frac{u_y^2}{u_x^2+u_y^2}u_{xx} - 2\frac{u_x u_y}{u_x^2+u_y^2}u_{xy} + \frac{u_x^2}{u_x^2+u_y^2}u_{yy},$$

since the two operators are quasilinear elliptic of second-order.

We have already recalled the usual notations for the finite differences method in the two-dimensional case. In the case where $|Du| = 0$, we use a classical approximation of the Laplacian.

Now, for the case $|Du| > 0$, we define the angle θ such that

$$(\cos\theta, \sin\theta) = \frac{Du}{|Du|},$$

and we assume that $Du = p$ is known. To simplify the notations, we also let:

$$\begin{aligned} a_{11}(p) &= \psi(|p|) + \psi'(|p|)|p|\cos^2\theta, \\ a_{22}(p) &= \psi(|p|) + \psi'(|p|)|p|\sin^2\theta, \\ a_{12}(p) &= \psi'(|p|)|p|\cos\theta\sin\theta, \end{aligned}$$

and then, by these notations, $G(Du, D^2u)$ becomes:

$$G(Du, D^2u) = -\Big(a_{11}(Du)u_{xx} + 2a_{12}(Du)u_{xy} + a_{22}(Du)u_{yy}\Big).$$

We want to write an approximation for $G(Du, D^2u)_{i,j}$ of second order (in $O(h^2)$), as a linear combination of the nine values of u on the fixed stencil 3×3 (that is: $u_{k,l}$, with

$i-1 \leq k \leq i+1,\ j-1 \leq l \leq j+1$). Of course, the coefficients of the linear combination will depend on Du.

From the symmetry of $G(Du, D^2u)$ and in order to have consistency of the scheme, we will seek $\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4$ (associated to the nine values of u), such that

$$\begin{aligned} (6.2.1)\quad G(Du, D^2u)_{ij} &= -\frac{1}{h^2}[-4\lambda_0 u_{ij} + \lambda_1(u_{i+1,j} + u_{i-1,j}) + \lambda_2(u_{i,j+1} + u_{i,j-1}) \\ &+ \lambda_3(u_{i-1,j-1} + u_{i+1,j+1}) + \lambda_4(u_{i-1,j+1} + u_{i+1,j-1})] + o(h^2). \end{aligned}$$

We write

$$u_{i+1,j} = u_{i,j} + h(u_x)_{i,j} + o(h^2),$$

and the same relation for the other points of the stencil (the Taylor expansions). Identifying the coefficients in (6.2.1), we obtain four relations between our five coefficients:

$$\begin{cases} 2\lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 - 4\lambda_0 = 0, \\ \lambda_1 + \lambda_3 + \lambda_4 = a_{11}, \\ \lambda_2 + \lambda_3 + \lambda_4 = a_{22}, \\ \lambda_3 - \lambda_4 = a_{12}, \end{cases}$$

or

$$\begin{cases} \lambda_1 = 2\lambda_0 - a_{22}, \\ \lambda_2 = 2\lambda_0 - a_{11}, \\ \lambda_3 = \frac{1}{2}(a_{11} + a_{22} + a_{12} - 2\lambda_0), \\ \lambda_4 = \frac{1}{2}(a_{11} + a_{22} - a_{12} - 2\lambda_0). \end{cases}$$

By these relations, the formula (6.2.1) is exact for all the polynomials of degree 0,1,2 and 3, with constant coefficients. There remains only one degree of freedom: λ_0. Therefore we can choose one more polynomial. We pick the following:

$$P(x, y) = (x^2 + y^2)(a_{11}x^2 + 2a_{12}xy + a_{22}y^2),$$

whose coefficients depend now on Du.

The above relations imply that:

$$\lambda_0 = \frac{a_{11}^2 + a_{22}^2 + a_{11}a_{22} + 2a_{12}^2}{a_{11} + a_{22}}.$$

Finally, we use an iterative scheme to solve the problem. We compute u_{ij}^{n+1} by the relation:

$$\begin{aligned} u_{ij}^{n+1}(1 + \frac{4\alpha\lambda_0}{h^2}) &= \frac{\alpha}{h^2}[\lambda_1(u_{i+1,j}^n + u_{i-1,j}^n) + \lambda_2(u_{i,j+1}^n + u_{i,j-1}^n) \\ &+ \lambda_3(u_{i-1,j-1}^n + u_{i+1,j+1}^n) + \lambda_4(u_{i-1,j+1}^n + u_{i+1,j-1}^n)] + u_{0,ij}. \end{aligned}$$

In Part III we will apply this scheme to two-dimensional reconstruction, to solve the Euler equation, written in expanded form.

Chapter 7

A method to convexify functions

In this chapter we propose a method to compute the convex envelope of a function u_0, using technics of curves evolution, like the mean curvature motion (see Osher-Sethian [45]). The function u_0 will be the initial data for an evolution equation, which has a viscosity solution $u(t,\cdot)$. The graph of the function will move in the orthogonal direction, with a speed depending on the smallest eigenvalue of the hessian matrix D^2u. When t tends to infinity, $u(t,\cdot)$ will converge to the convex envelope of u_0.

7.1 The description and study of the problem

Definition 7.1.1 *Let $u : A \to I\!R$ with $A \subseteq I\!R^N$ a convex set. The convex envelope of u in A, denoted here by $conv_A(u)$, is the greatest convex function less than or equal to u in A or, for $x \in A$:*

$$conv_A(u)(x) = \inf\{\lambda_1 u(x_1)+...+\lambda_k u(x_k) | x = \sum_{i=1}^{k} \lambda_i x_i, x_i \in A, \lambda_i > 0, \sum_{i=1}^{k} \lambda_i = 1, k \leq N+1\}.$$

If u is lower semicontinuous, then $conv_{I\!R^N}(u)$ is the conjugate of the conjugate of u, denoted by u^{**}. But, we will not use here this notation, since we use u^* to denote the u.s.c envelope of u, and not for the conjugate (or polar) of u, defined and used in Part I.

In what follows, we will naturally work with functions u for which $conv_A(u) > -\infty$, where A can be a convex bounded set or the whole space $I\!R^N$. But, since numerically we must assume that A is bounded, we will therefore make the theoretical study of the problem on bounded domains, with boundary conditions. For instance, to compute the convex envelope of $u_0 : I\!R^N \to I\!R$, we first consider the restriction of u_0 to the closed ball $\overline{B(0,R)}$ centered at the origin and of radius R, and then we make R to tend at infinity, i.e. "$B(0,R) \to I\!R^N$" as $R \to \infty$. Moreover, we have proved the following elementary result.

Theorem 7.1.2 *Let $u : I\!R^N \to I\!R$ such that $conv_{I\!R^N}(u) > -\infty$. We define, for each $R > 0$, the function $u_R : I\!R^N \to I\!R$ by:*

$$u_R(x) = \begin{cases} u(x) \ \text{if } x \notin \overline{B(0,R)} \\ conv_{\overline{B(0,R)}}(u)(x) \ \text{if } x \in \overline{B(0,R)}. \end{cases}$$

Then $u_R \to conv_{\mathbb{R}^N}(u)$ *pointwise in* $\mathbb{R}^N$.

Proof. For every fixed $x \in \mathbb{R}^N$, it is easy to see that $u_R(x)$ is decreasing in R (i.e. $u_{R_1}(x) \geq u_{R_2}(x)$ if $R_1 < R_2$). We also have

$$u_R(x) \geq conv_{\mathbb{R}^N}(u)(x) > -\infty, \ \forall x \in \mathbb{R}^N, \ \forall R > 0.$$

Then we can define the function $\hat{u} : \mathbb{R}^N \to \mathbb{R}$ by:

$$\hat{u}(x) = \inf_R u_R(x) = \lim_{R \to \infty} u_R(x).$$

Evidently, $\hat{u} \geq conv_{\mathbb{R}^N}(u)$. We will show that $\hat{u}$ is a convex function, and then we will have:

$$\hat{u} = conv_{\mathbb{R}^N}(u).$$

Indeed, let $x, y \in \mathbb{R}^N$, $\lambda \in (0,1)$. Then there is a $R_0 > 0$ (sufficiently large) such that $x, y \in \overline{B(0, R_0)}$. Then $\lambda x + (1-\lambda)y \in \overline{B(0, R_0)}$. Since the function u_R is convex on $\overline{B(0, R_0)}$, for each $R \geq R_0$, then:

$$u_R(\lambda x + (1-\lambda)y) \leq \lambda u_R(x) + (1-\lambda)u_R(y), \ \forall R \geq R_0,$$

and passing to the infimum in R (or to the limit as $R \to \infty$), we get:

$$\hat{u}(\lambda x + (1-\lambda)y) \leq \lambda \hat{u}(x) + (1-\lambda)\hat{u}(y),$$

i.e. $\hat{u}$ is convex. Then, obviously, $\hat{u} = conv_{\mathbb{R}^N}(u)$ and the sequence $\{u_R\}_{R>0}$ convergences pointwise to $conv_{\mathbb{R}^N}(u)$ in $\mathbb{R}^N$, as $R \to \infty$. ∎

Now we will describe an evolution equation which allows us to compute the convex envelope of u_0, at least in bounded domains.

Let Ω be a convex, bounded and open subset of $\mathbb{R}^N$ and $u_0 : \overline{\Omega} \to \mathbb{R}$ be locally bounded for the moment (in particular, Ω can be a ball $B(0, R)$, with $R > 0$). We introduce the following evolution problem (here, Du and D^2u stand for the spatial derivatives of u):

$$\begin{cases} \frac{\partial u}{\partial t} + G(Du, D^2u) = 0 \text{ in } (0,T] \times \Omega, \\ u(0, \cdot) = u_0 \text{ in } \{0\} \times \Omega \cup [0,T] \times \partial\Omega, \end{cases} \tag{7.1.1}$$

where

$$G : \mathbb{R}^N \times S^{N \times N} \to \mathbb{R}, \ G(p, X) := -\sqrt{1 + |p|^2} g(\lambda_{min}(X)),$$

$$g : \mathbb{R} \to \mathbb{R}, \ g(z) = \begin{cases} 0 \text{ if } z \geq 0, \\ z \text{ if } z < 0, \end{cases}$$

and $\lambda_{min}(X)$ denotes the smallest eigenvalue of X (we note that, since X is symmetric, then all its eigenvalues are reals).

Proposition 7.1.3 *If* $X, Y \in S^{N \times N}$ *such that* $X \leq Y$, *then*

$$\lambda_{min}(X) \leq \lambda_{min}(Y).$$

Moreover, $G : \mathbb{R}^N \times S^{N \times N}$ *is degenerate elliptic in* $\mathbb{R}^N \times S^{N \times N}$.

Proof. Let $y \in I\!\!R^N$ such that $Yy = \lambda_{min}(Y)y$. We have

$$\lambda_{min}(Y) \cdot |y| = (Yy, y) \geq (Xy, y) \geq \lambda_{min}(X)|y|,$$

since

$$\min_{y \in I\!\!R^N, |y|>0} \frac{(Xy, y)}{|y|} = \lambda_{min}(X),$$

and then $\lambda_{min}(X) \leq \lambda_{min}(Y)$. From this property, since g is nondecreasing, we conclude that G is degenerate elliptic in $I\!\!R^N \times S^{N\times N}$. ■

Now, we will apply the definition of a viscosity sub- and supersolution to this evolution problem, with the initial and boundary conditions.

We define $E : I\!\!R^N \times I\!\!R \times I\!\!R \times I\!\!R^N \times S^{N\times N}$ by

$$E(x, u, u_t, Du, D^2u) := \begin{cases} \frac{\partial u}{\partial t} + G(Du, D^2u) \text{ in } (0, T] \times \Omega, \\ u - u_0 \text{ in } \{0\} \times \Omega \cup [0, T] \times \partial\Omega. \end{cases}$$

A locally bounded function $u : [0, T] \times \overline{\Omega} \to I\!\!R$ is a viscosity subsolution of

$$E(x, u, u_t, Du, D^2) = 0 \text{ in } [0, T] \times \overline{\Omega},$$

if for each pair $\varphi \in C^2([0, T] \times \overline{\Omega})$ and $(t_0, x_0) \in [0, T] \times \overline{\Omega}$ as a local maximum of $u^* - \varphi$, it holds that:

$$\varphi_t(t_0, x_0) + G(D\varphi(t_0, x_0), D^2\varphi(t_0, x_0)) \leq 0 \text{ if } (t_0, x_0) \in (0, T] \times \Omega,$$

and

$$\min\{\varphi_t(t_0, x_0) + G(D\varphi(t_0, x_0), D^2\varphi(t_0, x_0)), u^*(t_0, x_0) - u_0^*(x_0)\} \leq 0$$

if $(t_0, x_0) \in \{0\} \times \Omega \cup [0, T] \times \partial\Omega$.

Similarly, u is a supersolution, if (t_0, x_0) is a local minimum of $u_* - \varphi$ with:

$$\varphi_t(t_0, x_0) + G(D\varphi(t_0, x_0), D^2\varphi(t_0, x_0)) \geq 0 \text{ if } (t_0, x_0) \in (0, T] \times \Omega,$$

and

$$\max\{\varphi_t(t_0, x_0) + G(D\varphi(t_0, x_0), D^2\varphi(t_0, x_0)), u_*(t_0, x_0) - (u_0)_*(x_0)\} \geq 0$$

if $(t_0, x_0) \in \{0\} \times \Omega \cup [0, T] \times \partial\Omega$.

(We note that $G : I\!\!R^N \times S^{N\times N} \to I\!\!R$ is continuous.)

We first give two results presented in [3], which allow us to interpret basic properties of convex functions in viscosity sense.

Lemma 7.1.4 *Let u be a l.s.c. and convex function in $\overline{\Omega}$. If $x \in \Omega$ and $(p, X) \in J_{\Omega}^{2,+}u(x)$, then u is continuously differentiable at x and $X \in S_+^{N\times N}$.*

Proof. We denote by $\partial u(x)$ the subdifferential of u (as in convex analysis and defined in Part I). Let $q \in \partial u(x)$ arbitrary. Then

$$u(x) + (q, y - x) \leq u(y) \leq u(x) + (p, y - x) + \frac{1}{2}(X(y - x), y - x) + o(|y - x|^2).$$

Sending $|y - x| \to 0$, we first obtain that $q = p$, so that u is differentiable at x, with $Du(x) = p$. Then, replacing q by p in the above inequality, we conclude that $X \geq O$. ■

It is interesting to note that the converse of this property is true.

Lemma 7.1.5 *Let Ω be a convex open set and u u.s.c. in $\overline{\Omega}$ satisfy*

$$X \geq O \text{ for every } (p,X) \in J^{2,+}_{\Omega}u(x) \text{ and every } x \in \Omega.$$

Then u is convex in Ω.

Proof. Assume that there are $y \neq z \in \Omega$ and $\lambda \in (0,1)$ such that

$$u(\lambda y + (1-\lambda)z) > \lambda u(y) + (1-\lambda)u(z).$$

Translating, rotating and dilating the coordinate system, and adding a linear function to u, we assume, without loss of a generality, that $y = 0$, $z = e_1$, $u(0,0) = u(1,0) < 0$ and $u(\lambda, 0) > 0$. We represent here a point $x \in \Omega$ as $(x_1, x^{'})$, with $x_1 = x \cdot e_1$ and $x^{'} = (x_2, ..., x_N)$. Using the upper semicontinuity of u, we may choose $\varepsilon > 0$ such that $u(0,x^{'}), u(1,x^{'}) \leq 0$ for $|x^{'}| \leq \varepsilon$ and $\Omega^{'} = \{x_1 \in (0,1), |x^{'} < \varepsilon\} \subset \Omega$. We fix C so large that $C\varepsilon^2 \geq \sup_{\Omega^{'}} u$ and $\delta > 0$ so small that $u(\lambda, 0) > \delta\lambda(1-\lambda)$. Setting

$$\varphi(x_1, x^{'}) = \delta x_1(1-x_1) + C|x^{'}|^2,$$

we observe that $u \leq \varphi$ on $\partial\Omega^{'}$, while $u(\lambda, 0) > \varphi(\lambda, 0)$. Therefore $u - \varphi$ achieves an interior maximum point in $\Omega^{'}$, while $(D^2\varphi(x)e_1, e_1) = -2\delta$ in $\Omega^{'}$. This contradicts our assumption on $J^{2,+}_{\Omega}u(x)$. ■

Theorem 7.1.6 *(Existence) Let u_0 a locally bounded function in $\overline{\Omega}$, such that $u_0 = conv_{\overline{\Omega}}u_0$ on $\partial\Omega$. Then there is a viscosity solution u to the problem (7.1.1) in $[0,T] \times \overline{\Omega}$.*

Proof. From the general existence result of Chapter 5, it suffices to give a viscosity subsolution u^- and a viscosity supersolution u^+, with $u^- \leq u^+$ in $[0,T] \times \overline{\Omega}$.

a) We let $u^+(t,x) := u_0(x)$ in $[0,T] \times \overline{\Omega}$, which is a viscosity supersolution. Indeed, let $\varphi \in C^2([0,T] \times \overline{\Omega})$ and (t_0, x_0) a local minimum of $(u_0)_* - \varphi$. Then $\varphi_t(t_0, x_0) = 0$.

If $(t_0, x_0) \in (0,T] \times \Omega$, we have of-course:

$$0 + G(D\varphi(t_0,x_0), D^2\varphi(t_0,x_0)) \geq 0,$$

since $G \geq 0$ in $I\!R^N \times S^{N \times N}$.

Also, if $t_0 = 0$ or $x_0 \in \partial\Omega$, the initial and boundary conditions in viscosity sense are satisfied.

b) We let $u^-(t,x) = conv_{\overline{\Omega}}(u_0)(x)$ if $t > 0$ and $u^-(0,x) = u_0(x)$, for $x \in \overline{\Omega}$. Then $u^- = (u^-)^*$. This is convex in x on $(0,T] \times \overline{\Omega}$ and from Lemma 7.1.4, we easy deduce that this is a subsolution, since one-yet the initial and boundary conditions are satisfied.

Then, by Proposition 5.1.8, there is a viscosity solution u of (7.1.1) in $[0,T] \times \overline{\Omega}$, with $u^- \leq u \leq u^+$, and $u(0,x) = u_0(x)$ on $\{0\} \times \Omega \cup [0,T] \times \partial\Omega$. ■

Theorem 7.1.7 *(Comparison) Let u and v be respectively sub- and supersolution of (7.1.1) in $(0,T] \times \Omega$. If $u^* \leq v_*$ on $\{0\} \times \Omega \cup [0,T] \times \partial\Omega$, then $u^* \leq v_*$ on $(0,T] \times \Omega$.*

Proof. We apply Thm. 4.1. from [19], page 765. ■

Now, the following result is obvious from the existence and comparison.

Theorem 7.1.8 *(Uniqueness) If u_0 is continuous in $\overline{\Omega}$, then there is a unique viscosity solution to (7.1.1). Moreover, $u \in C([0,T] \times \overline{\Omega})$ and of-course $u = u_0$ in $\{0\} \times \Omega \cup [0,T] \times \partial\Omega$.*

Theorem 7.1.9 *Suppose that $u_0 \in C(\overline{\Omega})$, with $\Omega \subset I\!R^N$ open, bounded and convex, such that $u_0 = conv_{\overline{\Omega}} u_0$ on $\partial\Omega$. Then, for $t \to \infty$, $u(t,\cdot)$ converges to the convex envelope of u_0.*

Proof. We show that $\hat{u}(x) := \inf_t u(t,x) = \lim_{t\to\infty} u(t,x)$ is the convex envelope of u_0, i.e. $conv_{\overline{\Omega}}(u_0)$.

It is easy to observe that $u(t,x)$ decreases, for fixed x, when $t \to \infty$, from the definition of G. First, we have seen from the previous existence result, that $conv_{\overline{\Omega}}(u_0)(x) \le u(t,x)$ for each $t > 0$. Of-course, we get that $conv_{\overline{\Omega}}(u_0)(x) \le \hat{u}(x)$ in $\overline{\Omega}$. We need to show that $\hat{u}$ is convex in Ω. To do this, we assume that there is a $x_0 \in \Omega$ and $(p,X) \in J^{2,+}\hat{u}(x_0)$ such that: $X \notin S_+^{N\times N}$. This is equivalent with: $\lambda_{min}(X) < 0$. Since $(p,X) \in J^{2,+}\hat{u}(x_0)$, then there is a function $\varphi \in C^2(\overline{\Omega})$ such that $\hat{u} - \varphi$ has a local maximum at x_0 and $(p,X) = (D\varphi(x_0), D^2\varphi(x_0))$. Since $\hat{u}(x_0) = \inf_t u(t,x_0)$, we deduce that there is a sequence $t_n \to \infty$ and $x_{t_n} \in \overline{\Omega}$ such that $x_{t_n} \to x_0$ and $u - \varphi$ has a local maximum at (t_n, x_{t_n}). Then

$$0 + G(D\varphi(x_{t_n}), D^2\varphi(x_{t_n})) \le 0,$$

which implies, by the continuity of G and the continuity of $D\varphi$, $D^2\varphi$, that

$$0 + G(D\varphi(x_0), D^2\varphi(x_0)) \le 0,$$

which can not be true, from $\lambda_{min}(D^2\varphi(x_0)) < 0$ and the definition of G. Then, applying Lemma 7.1.5, we deduce that $\hat{u}$ is convex in Ω. We have $conv_{\overline{\Omega}}(u_0) = u_0 = \hat{u}$ in $\partial\Omega$. We conclude by: $conv_{\overline{\Omega}}(u_0) = \hat{u}$ in $\overline{\Omega}$. Moreover, by the Dini's Lemma, we deduce that $u(t,\cdot)$ uniformly converges to $conv_{\overline{\Omega}}(u_0)$. ■

Remark 7.1.10

The evolution equation introduced in this chapter to convexify a function, can be written in the following form:

$$\frac{\partial u}{\partial t} = \sqrt{1 + |Du|^2}\min(\lambda_{min}(D^2u), 0),$$

where $\lambda_{min}(D^2u)$ represents the smallest eigenvalue of the hessian matrix D^2u. We wish to remark here some similarities with a variation of the mean curvature motion (see [55]):

$$\frac{\partial u}{\partial t} = |Du|\min(K, 0),$$

where K represents the mean curvature of a level curve of u. This last equation will convexify the level curves.

7.2 The numerical approximation of the equation

In this section, we present a numerical approximation, using the finite differences method, of the evolution equation studied in the previous section:

$$(7.2.1) \qquad \begin{cases} \frac{\partial u}{\partial t} = \sqrt{1 + |Du|^2} \cdot g(\lambda_{min}(D^2u)) \text{ in } (0,T] \times \Omega \\ u(t,\cdot) = u_0 \text{ in } \{0\} \times \overline{\Omega} \cup [0,T] \times \partial\Omega, \end{cases}$$

in order to compute the convex envelope of a function u_0, in one and two dimensions. Finally, we will test the model on several examples.

We recall that $g : I\!R \to I\!R$ is defined by: $g(z) = 0$ if $z \geq 0$ and $g(z) = z$ if $z < 0$.

7.2.1 The one-dimensional case

Let $\Omega = [0,1]$ for simplicity and assume that $u_0 : [0,1] \to I\!R$ is bounded. For each $T > 0$, the problem is:

$$\text{(7.2.2)} \quad \begin{cases} \frac{\partial u}{\partial t} = \sqrt{1+|u_x|^2}\, g(u_{xx}) \text{ in } (0,T)\times(0,1) \\ u(0,x) = u_0(x) \text{ in } [0,1],\ u(t,0) = u_0(0),\ u(t,1) = u_0(1) \text{ if } t \in [0,T]. \end{cases}$$

Let $M \in I\!N^*$, $x_i = ih$, where $h = 1/M$ (the step space) and $0 \leq i \leq M$, be the discrete points. Let $\triangle t > 0$, $n \in I\!N$, and define the discrete approximations $u_{0,h}$ and u_h of u_0 and u by:

$$u_{0,h}(x_i) = u_{0,i} \approx u_0(x_i) \text{ for } 0 \leq i \leq M,$$
$$u_h(n\triangle t, x_i) = u_i^n \approx u(n\triangle t, x_i) \text{ for } 0 \leq i \leq M,$$

with $u_h^0 = u_{0,h}$ and $n\triangle t \in [0,T]$.

The algorithm is as follows, for each i, $0 < i < M$ in increasing order, and for fixed $n \geq 0$: we let $u_0^{n+1} = u_0^n$, $u_M^{n+1} = u_M^n$ and then we compute

$$\lambda = (u_{xx})_i^n = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^{n+1}}{h^2},$$

since u_i^{n+1} has been calculated. If $\lambda \geq 0$, then we let $u_i^{n+1} = u_i^n$.

If not, to compute u_i^{n+1}, we use the following implicit discretization, in order to accelerate the convergence:

$$\text{(7.2.3)} \qquad \frac{u_i^{n+1} - u_i^n}{\triangle t} = \sqrt{1 + \Big(\frac{u_{i+1}^n - u_{i-1}^n}{2h}\Big)^2} \cdot \Big(\frac{u_{i+1}^n - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2}\Big).$$

Therefore, in this case, we obtain:

$$u_i^{n+1} = u_i^n + \frac{\triangle t}{2h^3}\sqrt{4h^2 + (u_{i+1}^n - u_{i-1}^n)^2} \cdot (u_{i+1}^n - 2u_i^{n+1} + u_{i-1}^{n+1}).$$

Let

$$c := \frac{\triangle t}{2h^3}\sqrt{4h^2 + (u_{i+1}^n - u_{i-1}^n)^2}$$

and then

$$u_i^{n+1}(1+2c) = u_i^n + c(u_{i+1}^n + u_{i-1}^{n+1}),$$

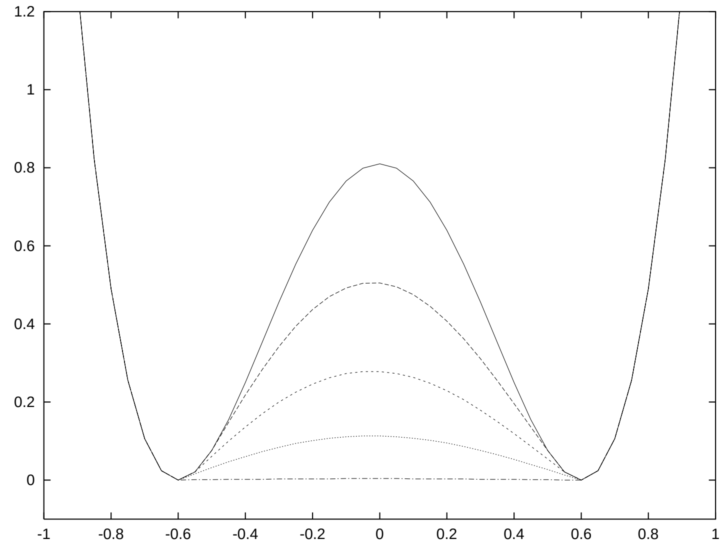
or

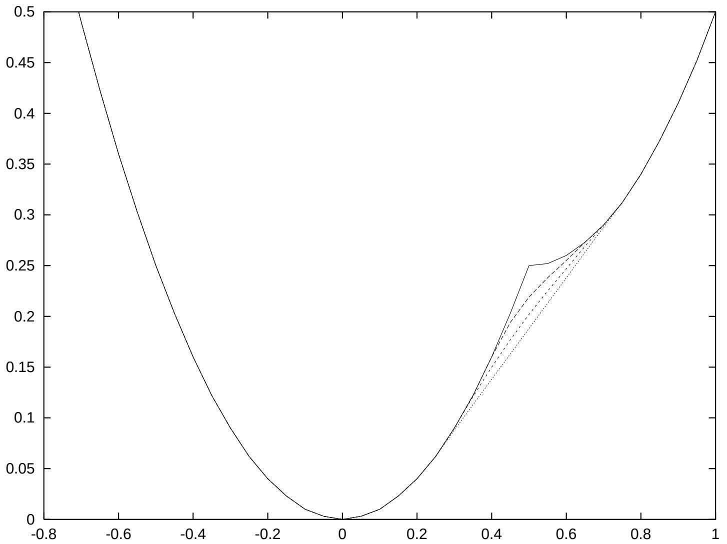
$$\text{(7.2.4)} \qquad u_i^{n+1} = \frac{u_i^n}{1+2c} + \frac{c}{1+2c}(u_{i+1}^n + u_{i-1}^{n+1}).$$

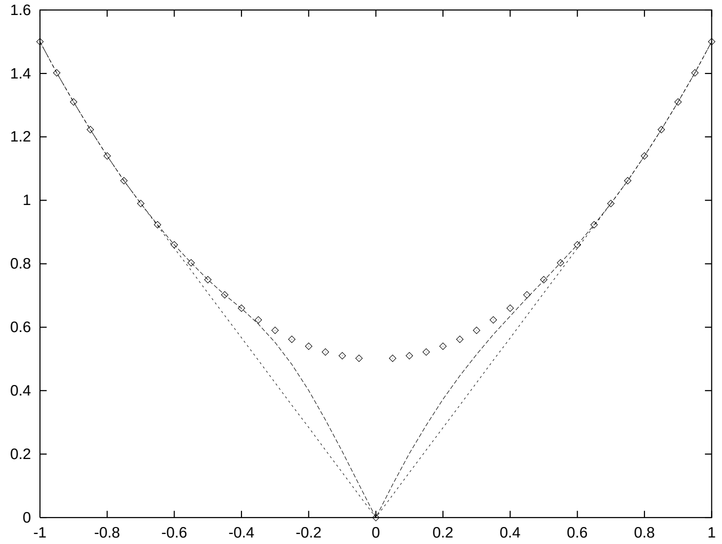
In this second case (when $\lambda < 0$), we will always have $u_i^{n+1} \leq u_i^n$ from the form of the equation and the definition of the function g, since the right-hand side of (7.2.3) will be always less than or equal to zero. Now, since $c > 0$ and from (7.2.4), if $u_i^n \geq m$, for each $0 \leq i \leq M$, then $u_i^{n+1} \geq m$, for each $0 \leq i \leq M$. In other words, the scheme is unconditionally stable.

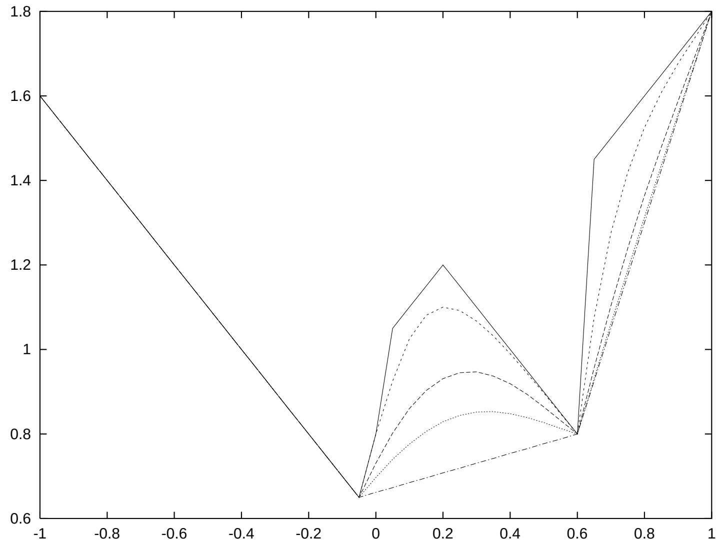
Numerical results in dimension one

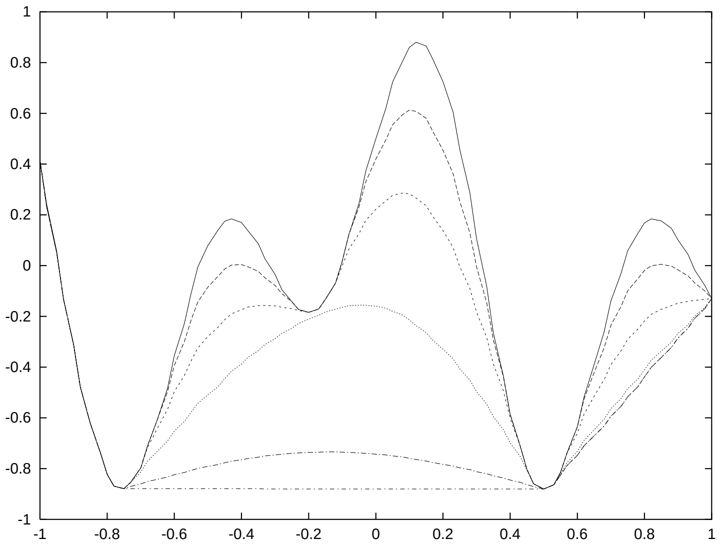
In the end of this subsection, we present some examples on five non-convex functions, using this algorithm. The third function is discontinuous at the origin, while the others are continuous.






1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1


1.8
1.6
1.4
1.2
1
0.8
0.6
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1


1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

7.2.2 The two-dimensional case

Now, for $N=2$, we let $\Omega=(0,1)^2$ and $u_0:[0,1]^2\to I\!R$ a bounded function. The smallest eigenvalue of D^2u is given by the formula

$$\lambda_{min}(D^2u)=\frac{1}{2}\Big(u_{xx}+u_{yy}-\sqrt{(u_{xx}-u_{yy})^2+4u_{xy}^2}\Big)$$

and the problem in this case is:

$$(7.2.5)\begin{cases}\frac{\partial u}{\partial t}=\sqrt{1+|u_x|^2+|u_y|^2}\cdot g(\lambda_{min}D^2u) \text{ in } (0,T]\times(0,1)^2\\ u(0,x,y)=u_0(x,y) \text{ in } [0,1]^2, u(t,x,y)=u_0(x,y) \text{ if } t\in[0,T],(x,y)\in\partial[0,1]^2.\end{cases}$$

Let $M\in I\!N^*$ and for $0\le i,j\le M$, with $h=1/M$, $x_i=ih$ and $y_j=jh$ be the discrete points. Then, the discrete approximations of u_0 and the solution u will be denoted by:

$$\begin{aligned}u_{0,h}(x_i,y_j)=u_{0,ij}&\approx u_0(x_i,y_j) \text{ for } 0\le i,j\le M,\\ u_h(n\triangle t,x_i,y_j)=u_{ij}^n&\approx u(n\triangle t,x_i,y_j) \text{ for } 0\le i,j\le M,\end{aligned}$$

with $u_h^0=u_{0,h}$ and $n\triangle t\in[0,T]$.

We use the following classical numerical approximations of the spatial derivatives: for each $0<i,j<M$, we let

$$\begin{aligned}(u_x)_{ij}&=(u_{i+1,j}-u_{i-1,j})/(2h),\\ (u_y)_{ij}&=(u_{i,j+1}-u_{i,j-1})/(2h),\\ (u_{xx})_{ij}&=(u_{i+1,j}-2u_{ij}+u_{i-1,j})/(h^2),\\ (u_{yy})_{ij}&=(u_{i,j+1}-2u_{ij}+u_{i,j-1})/(h^2),\end{aligned}$$

and for u_{xy}, we use the following formula, given in [44]:

$$\begin{aligned}(u_{xy})_{ij}&=(1/2h^2)(\triangle_-^x\triangle_-^y+\triangle_+^x\triangle_+^y u_{ij})\\ &=(1/2h^2)(2u_{ij}+u_{i+1,j+1}+u_{i-1,j-1}-u_{i+1,j}-u_{i-1,j}-u_{i,j+1}-u_{i,j-1}).\end{aligned}$$

We denote by $\triangle$ the expression

$$\triangle=(u_{xx}-u_{yy})^2+4u_{xy}^2$$

and $\triangle_{ij}^n$ will be the approximation of $\triangle$ by the previous formula, at $(n\triangle t,x_i,y_j)$.

The algorithm is as follows, for each i,j, with $0<i,j<M$ and $n\ge 0$: we first compute $\lambda=\lambda_{min}(D^2u_{ij}^n)$, using the previous approximations for the second derivatives of u at $(n\triangle t,x_i,y_j)$. If $\lambda\ge 0$, then we let $u_{ij}^{n+1}=u_{ij}^n$.

If not, as in the one-dimensional case, to compute u_{ij}^{n+1}, we use the following implicit discretization:

$$u_{ij}^{n+1}=u_{ij}^n+\frac{\triangle t}{2h^2}\cdot\sqrt{1+((u_x^n)_{ij})^2+((u_y^n)_{ij})^2}\cdot(u_{i-1,j}^{n+1}+u_{i,j-1}^{n+1}+u_{i+1,j}^n+u_{i,j+1}^n-4u_{ij}^{n+1}-h^2\sqrt{\triangle_{ij}^n})$$

(since $u_{i,j-1}^{n+1}$ and $u_{i-1,j}^{n+1}$ have already been calculated).

We then denote by c the following coefficient:

$$c = \frac{\triangle t}{2h^2} \cdot \sqrt{1 + ((u_x^n)_{ij})^2 + ((u_y^n)_{ij})^2}$$

and therefore, by this notation, we have

$$u_{ij}^{n+1}(1+4c) = u_{ij}^n + c(u_{i-1,j}^{n+1} + u_{i,j-1}^{n+1} + u_{i+1,j}^n + u_{i,j+1}^n - h^2\sqrt{\triangle_{ij}^n}).$$

Then, u_{ij}^{n+1} will be computed by

$$u_{ij}^{n+1} = \frac{u_{ij}^n}{1+4c} + \frac{c}{1+4c}(u_{i-1,j}^{n+1} + u_{i,j-1}^{n+1} + u_{i+1,j}^n + u_{i,j+1}^n - h^2\sqrt{\triangle_{ij}^n}).$$

Numerical results in two-dimensions

We now present numerical results for three non-convex functions. In each set of figures, the first represents the initial surface (or function u_0), and the others are results for different increasing numbers of iterations. The last figure represents the (discrete) convex envelope of u_0. We remark that u_0 in Figure 7.3 is discontinuous in two points.

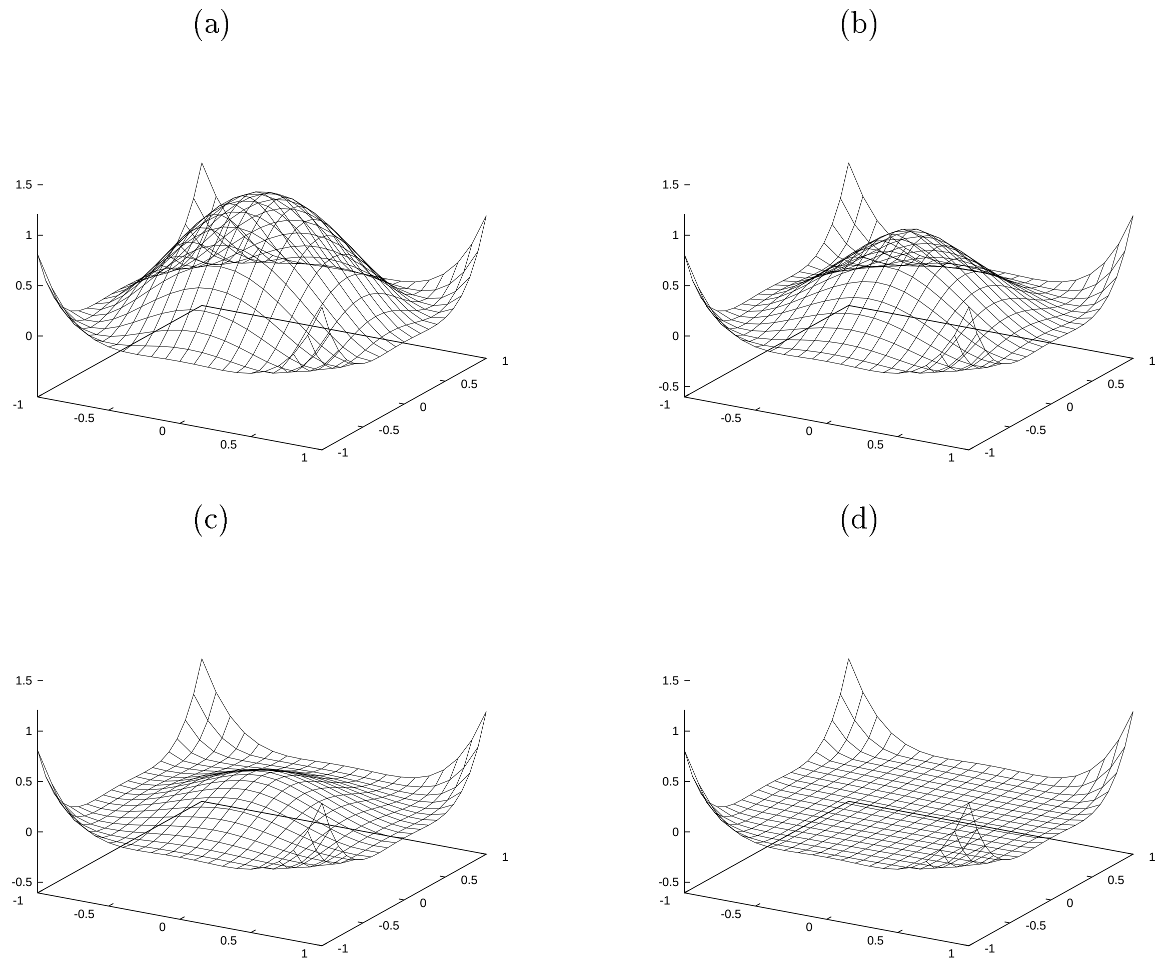


Figure 7.1: The initial function is $u_0(x, y) = (1.1 - x^2 - y^2)^2$.

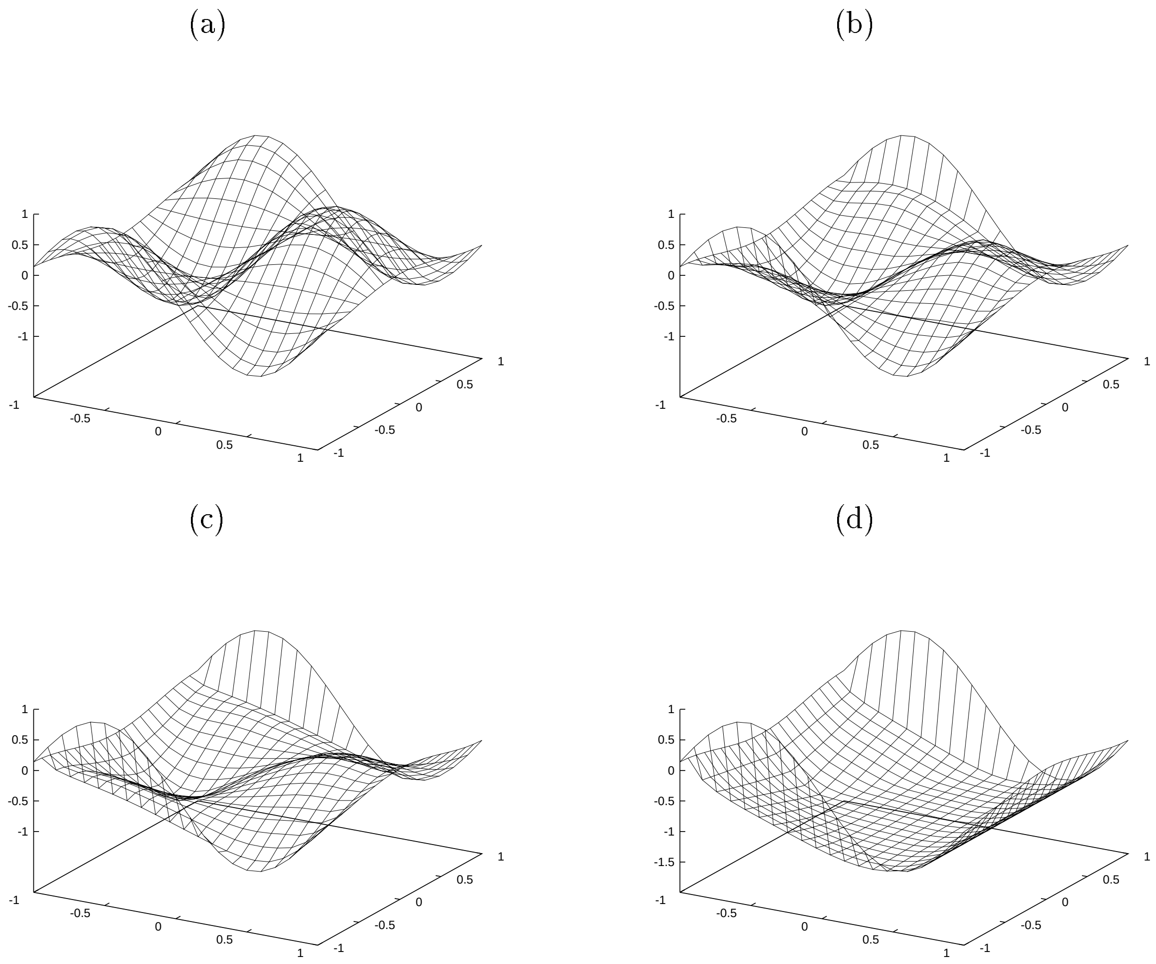


Figure 7.2: The initial function is $u_0(x, y) = \sin(3x) \cdot \cos(3y)$.

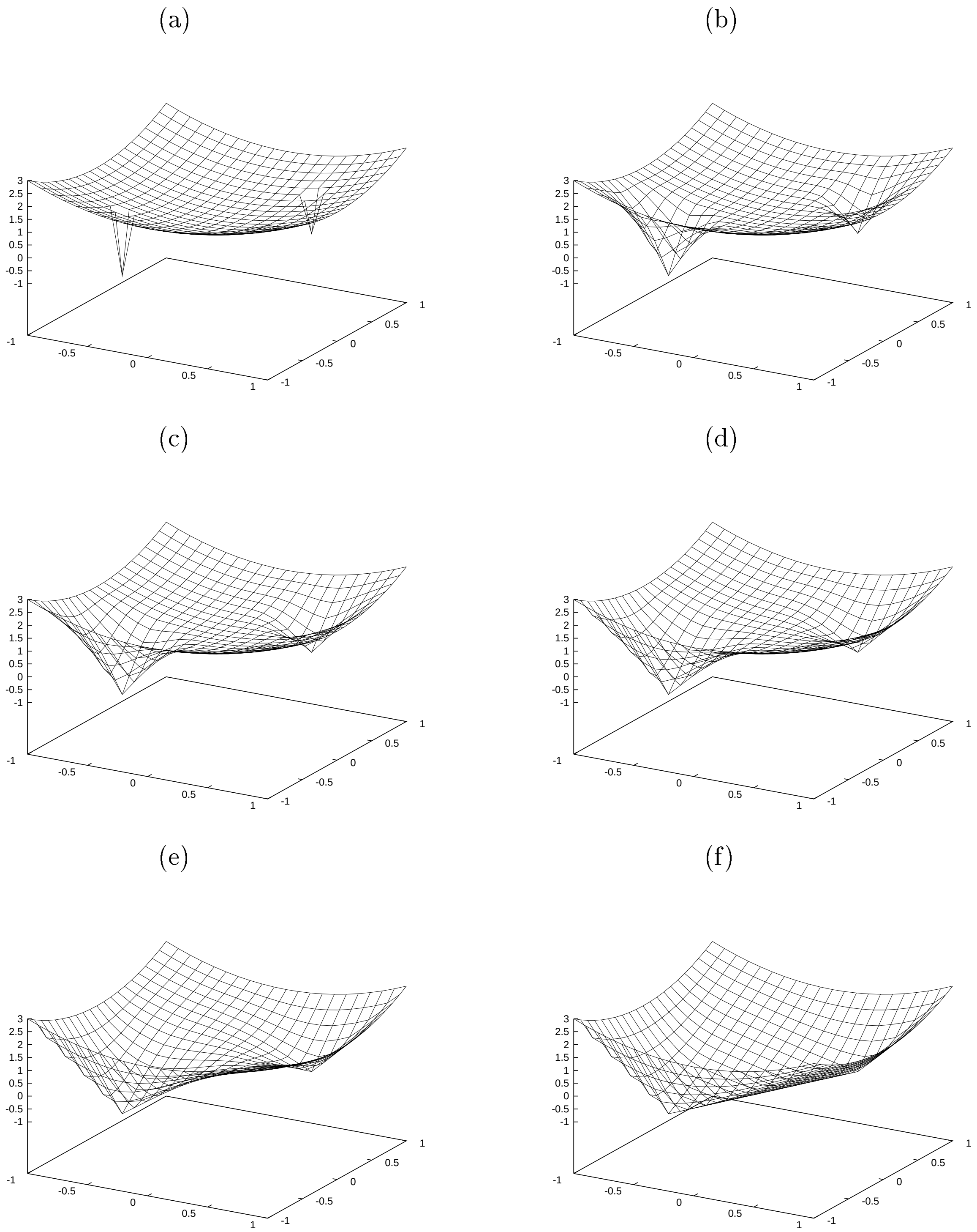


Figure 7.3: The initial function is $u_0(x,y) = x^2+y^2+1$ if $(x,y) \notin \{(-0.5,-0.5),(0.5,0.5)\}$ and $u_0(-0.5,-0.5) = -1$, $u_0(0.5,0.5) = 0$.

Part III

Two dimensional reconstruction Numerical results

In this last part of the thesis, we present several numerical results to reconstruct images, in the denoising case and the deblurring case, in order to compare the methods and potentials presented in Part I and Part II.

Chapter 8

Numerical results on images

In this last part, we present several results on synthetic and real images, and we compare the potentials ϕ and the algorithms described in Part I, Chapter 4 and in Part II, Chapter 6. We will consider images degraded by noise and blur.

We recall the definition of the potentials, introduced in Chapter I and used here.

$$\phi_1(z) = \begin{cases} \frac{1}{2}z^2 & \text{if } z \leq 1, \\ \\ z - \frac{1}{2} & \text{if } z > 1, \end{cases}$$

$$\phi_2(z) = \sqrt{1+z^2} - 1,$$

$$\phi_3 : I\!R^+ \to I\!R^+, \quad \phi_3(z) = \begin{cases} \frac{1}{2}z^2 & \text{if } z \leq 1, \\ \\ \frac{z^3}{1+z^2} & \text{if } z > 1, \end{cases}$$

$$\phi_4(z) = \frac{1}{2}\log(1+z^2) \quad \text{and} \quad \phi_5(z) = \frac{z^2}{2(1+z^2)}.$$

To simplify the presentation, we have given to each algorithm a name:

- ST.DIV.EQ. represents the discretization of the stationary equation, written in conservative form.

- EV.DIV.EQ. represents the discretization of the evolution equation, written in conservative form.

- MIN represents the minimization algorithm, by the discretization of the functional.

- EXP.EQ. represents the discretization of the stationary expanded equation.

We begin with a synthetic image, which has been noised with a variance $\sigma = 20$. The corresponding SNR (the mean signal to noise ratio, after a normalization) is 7.38 db (see Figure 8.2).

A-priori, for the moment, there is no optimal choice for the parameters (the regularizing parameter α, the step space h, which plays a role in the behavior of the potential ϕ and the number of iterations), as we have mentioned in the one dimensional case. Therefore, we have tested the discretization of the equation in conservative form (ST.DIV.EQ.), for different values of parameters for the synthetic picture. In Figure 8.1, where we represent the SNR function of α (for one, two or three values of h), we observe that we cannot use the same parameters for different potentials. So, for this synthetic picture, we have chosen the values of parameters, for each potential. The number of iterations can be considered as a parameter. With the convex potential ϕ_2, we have chosen this number by asking that the difference in the sup norm of two consecutive iterations is less than a small value (here this is 0.1). Then, we perform the same number of iterations in the discretization of the equation, for the other non-convex potentials, by which the algorithm is not convergent.

We have added the results using the algorithm MIN, combined with ST.DIV.EQ. For the synthetic picture, the computation is faster and the locations of edges are better preserved.

For the two following initial images (Figures 8.4, 8.6), we use the same parameters (as in Tables 8.2 and 8.3), since we know, a-priori, that the noise is of the same type, with the variance $\sigma = 20$. It is sufficient to perform only 10 iterations.

We observe, as we have mentioned in the Introduction, that for the synthetic image, which is piecewise constant, the results are better with a non-convex and sublinear potential. In fact, the best result is obtained by using the potential $\phi_4(z) = \frac{1}{2}\log(1+z^2)$.

On the other hand, for the real picture representing a woman, which is no longer piecewise constant, the best result is obtained with the convex potential $\phi_2(z) = \sqrt{1+z^2}$.

We consider one-yet the picture representing a woman, but more noised (Figure 8.9). It was necessary in this case to consider large α (the regularizing parameter). We present the results using the potentials ϕ_1, ϕ_2, ϕ_3 and ϕ_4, in Figure 8.10, using the algorithm ST.DIV.EQ. We observe that with the potentials ϕ_1, ϕ_2, ϕ_3, the results are very closed, and better than the result computed with ϕ_4, where a very large α was necessary. We have tested also the potential ϕ_5 on this image, but we have not obtained a satisfactory result, even if α was very large.

In Figures 8.11 and 8.13, we also consider the deblurring case or the both types of degradation (see Figure 8.11). Of course, in this case (with noise and blur in the same image), the result are less better. We present in Figure 8.12 the profiles on line, associated to the results of Figure 8.11

We have compared the results with the potential ϕ_2 and the algorithms ST.DIV.EQ. and EV.DIV.EQ., for the synthetic image (of course, using the same parameters α and h). We obtain the same result (with the SNR=18.08 db), but with ST.DIV.EQ. we need to perform 86 iterations, while in the evolution case (EV.DIV.EQ.) we need to use more iterations, more precisely, 180.

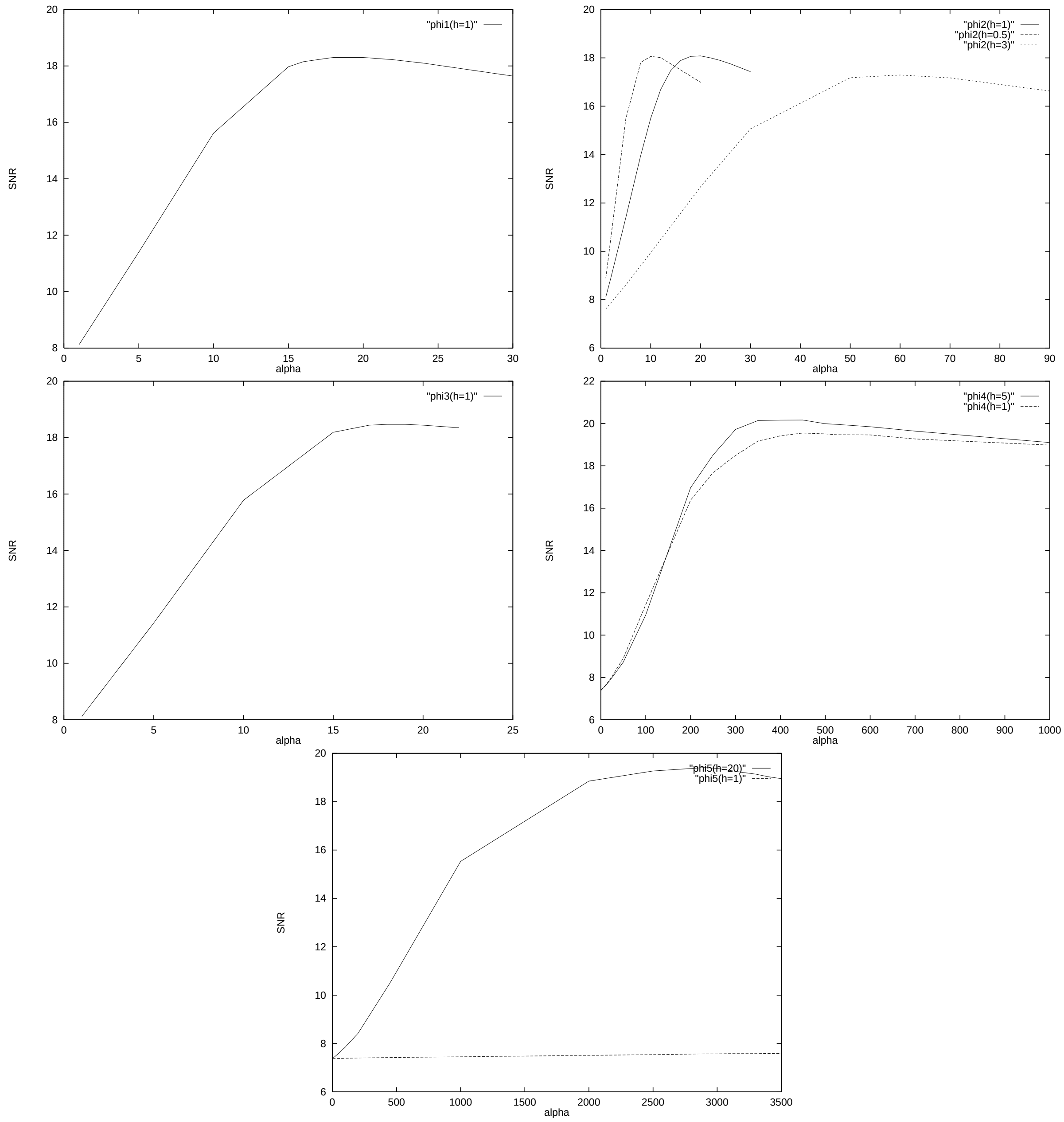


Figure 8.1: The evolutions of the SNR, for different values of α, for ϕ_1,ϕ_2, ϕ_3, ϕ_4 and ϕ_5.

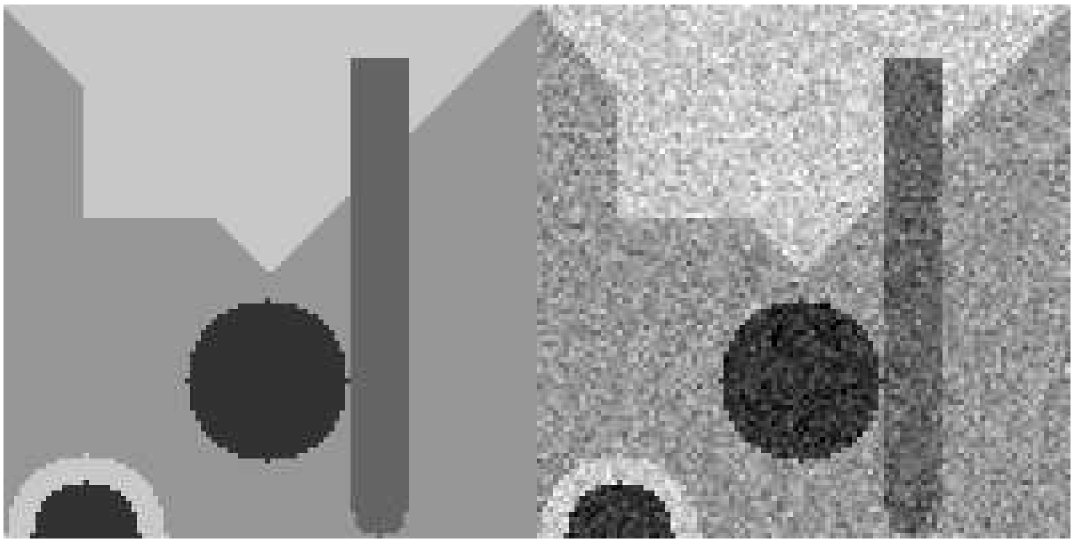


Figure 8.2: The original and noisy synthetic image (size: 100×100, $\sigma = 20$, SNR=7.38).

ST.DIV.EQ. ϕ_1 $\alpha = 20$ $h = 1$ SNR=18.30 db It=86 T=26.9 s	ST.DIV.EQ. ϕ_2 $\alpha = 20$ $h = 1$ SNR=18.08 db It=86 T=32.7 s	ST.DIV.EQ ϕ_3 $\alpha = 19$ $h = 1$ SNR=18.47 db It=86 T=26.1 s
ST.DIV.EQ. ϕ_4 $\alpha = 450$ $h = 5$ SNR=20.16 db It=86 T=24.6 s	ST.DIV.EQ. ϕ_5 $\alpha = 2900$ $h = 20$ SNR=19.41 db It=86 T=27.2 s	EXP.EQ. ϕ_2 $\alpha = 20$ $h = 1$ SNR=17.20 db It=86 T=27.9 s
MIN+ST.DIV.EQ ϕ_2 $\alpha = 20$ $h = 1$ SNR=17.75 db It=2+10 T=16.5s	MIN+ST.DIV.EQ ϕ_4 $\alpha = 450$ $h = 5$ SNR=19.45 db It=2+10 T=13.5s	MIN+ST.DIV.EQ ϕ_5 $\alpha = 2900$ $h = 20$ SNR=19.43 db It=2+10 T=11.7s

Table 8.1: Parameters for the synthetic picture, corresponding to Figure 8.3.

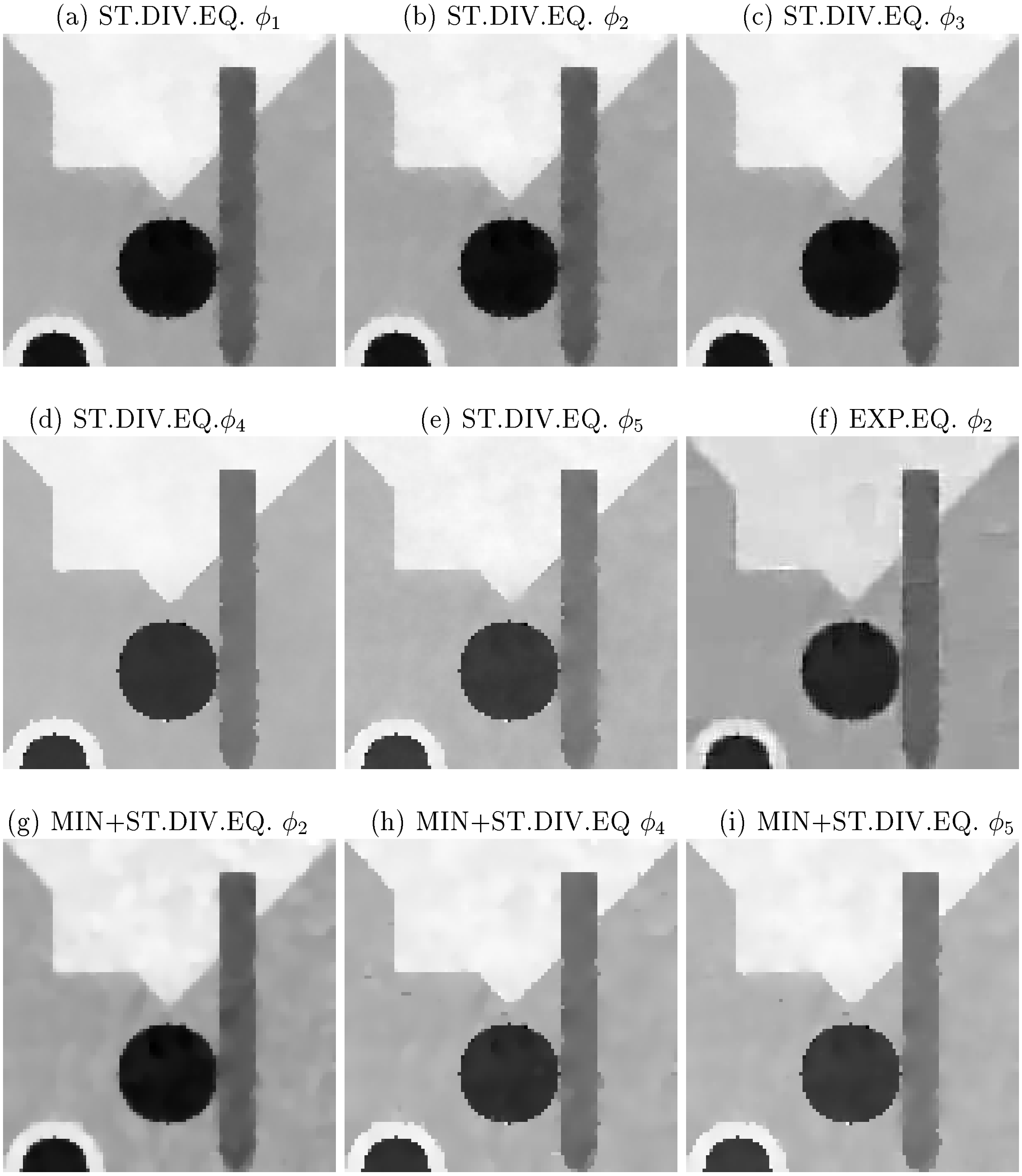


Figure 8.3: Results with the synthetic picture (corresponding to Table 8.1).

Figure 8.4: The original and noisy image: (size: 256×256, $\sigma = 20$, SNR=11.56).

ST.DIV.EQ. ϕ_1 $\alpha = 20$ $h = 1$ SNR=18.18 db It=10 T=20.1 s	ST.DIV.EQ ϕ_2 $\alpha = 20$ $h = 1$ SNR=18.17 db It=10 T=29.3 s
ST.DIV.EQ. ϕ_3 $\alpha = 19$ $h = 1$ SNR=18.20 db It=10 T=20.3 s	ST.DIV.EQ. ϕ_4 $\alpha = 450$ $h = 5$ SNR=17.26 db It=10 T=23.7 s
ST.DIV.EQ. ϕ_5 $\alpha = 2900$ $h = 20$ SNR=16.39 db It=10 T=19.3 s	EXP.EQ. ϕ_2 $\alpha = 20$ $h = 1$ SNR=18.75 db It=10 T=22.3 s

Table 8.2: Parameters for the real picture representing an office, using the discretizations of the equation, in conservative and expanded form, corresponding to Figure 8.5.

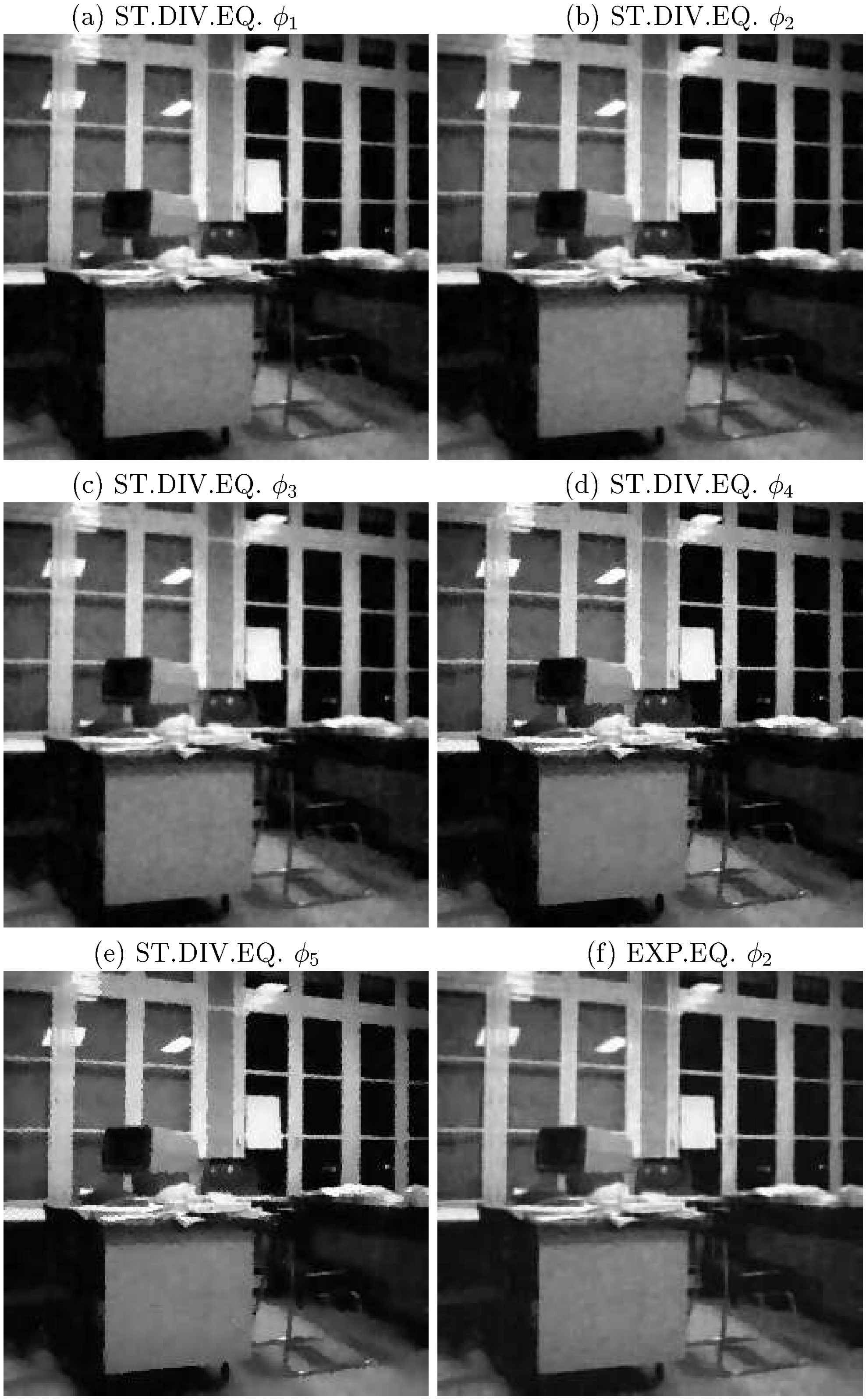


Figure 8.5: Results obtained with the office picture, corresponding to Figure 8.4 and Table 8.2.

Figure 8.6: The original and noisy image: (size: 256×256, $\sigma = 20$, SNR=11.0).

ST.DIV.EQ. ϕ_1 $\alpha = 20$ $h = 1$ SNR=18.13 db It=10 T=20.1 s	ST.DIV.EQ. ϕ_2 $\alpha = 20$ $h = 1$ SNR=18.15 db It=10 T=29.3 s
ST.DIV.EQ. ϕ_3 $\alpha = 19$ $h = 1$ SNR=18.14 db It=10 T=20.3 s	ST.DIV.EQ. ϕ_4 $\alpha = 450$ $h = 5$ SNR=17.73 db It=10 T=23.7 s
ST.DIV.EQ. ϕ_5 $\alpha = 2900$ $h = 20$ SNR=17.38 db It=10 T=19.3 s	EXP.EQ. ϕ_2 $\alpha = 20$ $h = 1$ SNR=18.01 db It=10 T=21s

Table 8.3: Parameters for the real picture representing a woman, corresponding to Figure 8.7.

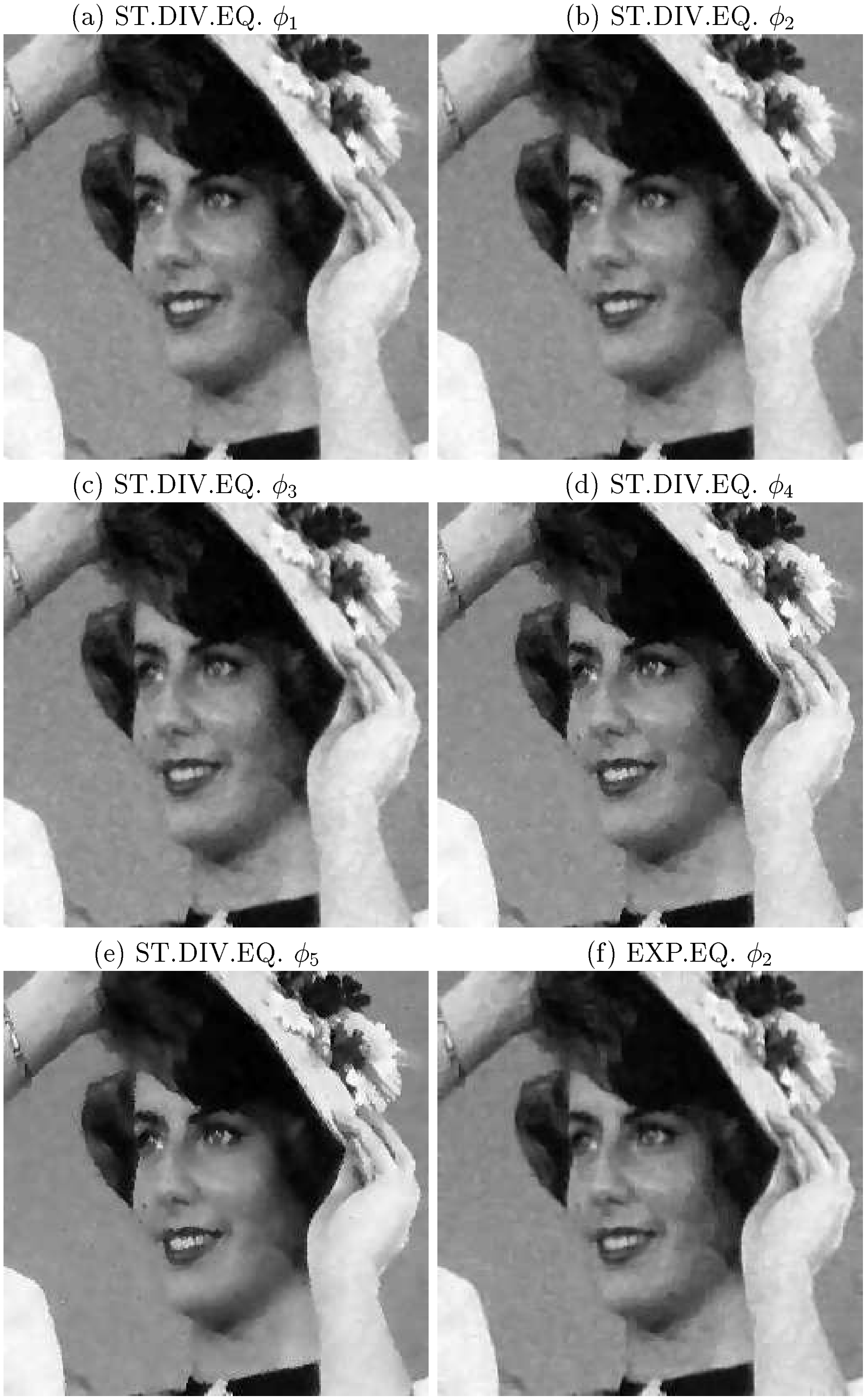


Figure 8.7: Results with the picture representing a woman corresponding to Figure 8.6 and Table 8.3.

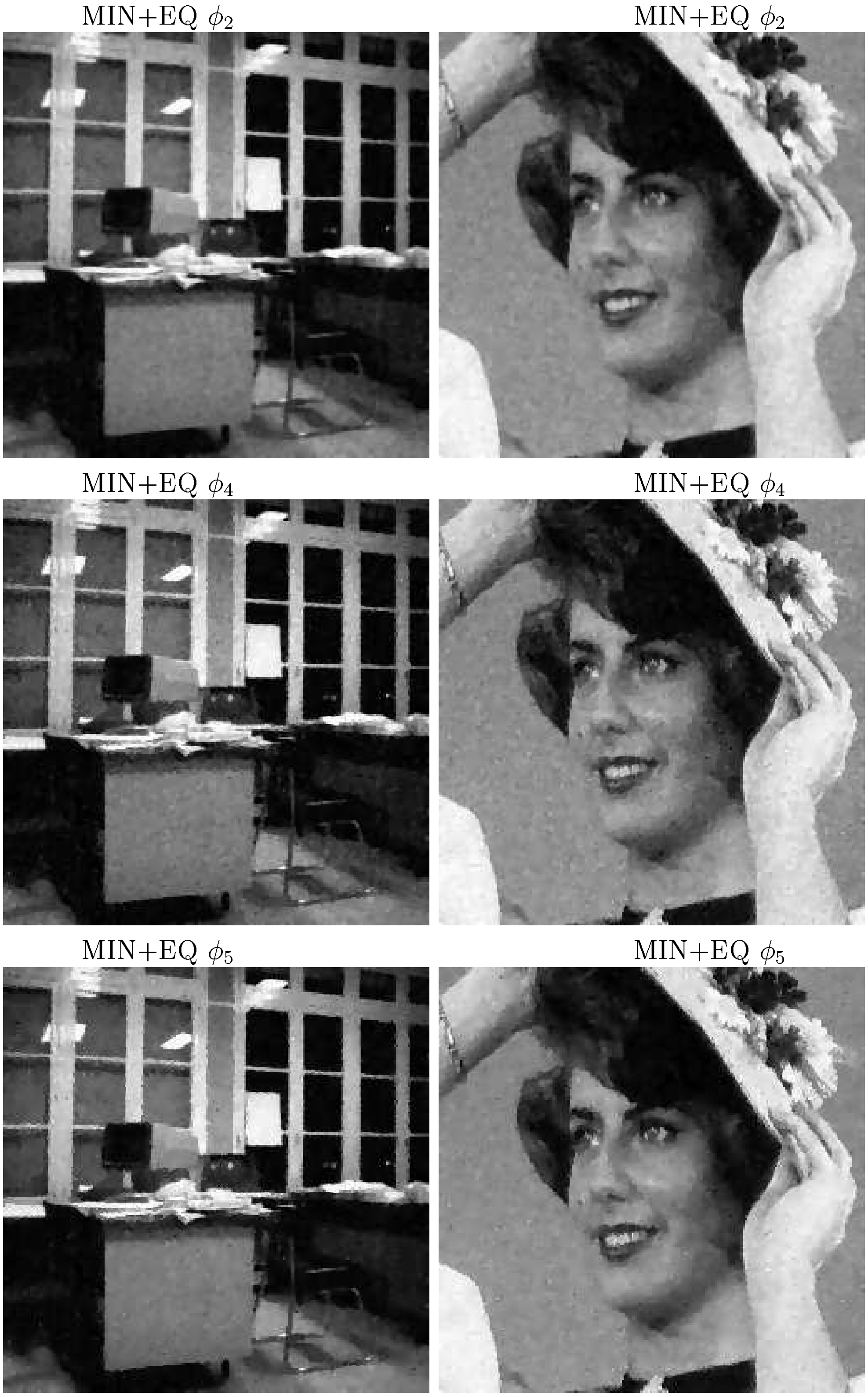


Figure 8.8: Results corresponding to Table 8.4.

MIN 1it + ST.DIV.EQ. 3it

ϕ_2 $h = 1$ $\alpha = 20$	SNR=18.30 db T=50s	SNR=17.83 db T=50s
ϕ_4 $h = 5$ $\alpha = 450$	SNR=16.70 db T=44s	SNR=16.90 db T=44s
ϕ_5 $h = 20$ $\alpha = 2900$	SNR=16.33 db T=35s	SNR=16.97 db T=35s

Table 8.4: Parameters corresponding to Figure 8.8.

Figure 8.9: A more noisy image, with the SNR=3.68 db.

ϕ_1 $\alpha = 300$ $h = 1$ SNR=15.43 db It=10 T=20.2 s		ϕ_2 $\alpha = 200$ $h = 1$ SNR=15.37 db It=10 T=29.2 s
ϕ_3 $\alpha = 180$ $h = 1$ SNR=15.42 db It=10 T= 20.4 s		ϕ_4 $\alpha = 15000$ $h = 20$ SNR=14.09 db It=10 T= 24.0 s

Table 8.5: Parameters for the real and more noisy picture representing a woman.

Figure 8.10: Results with the more noisy woman picture, corresponding to Figure 8.9 and Table 8.5.

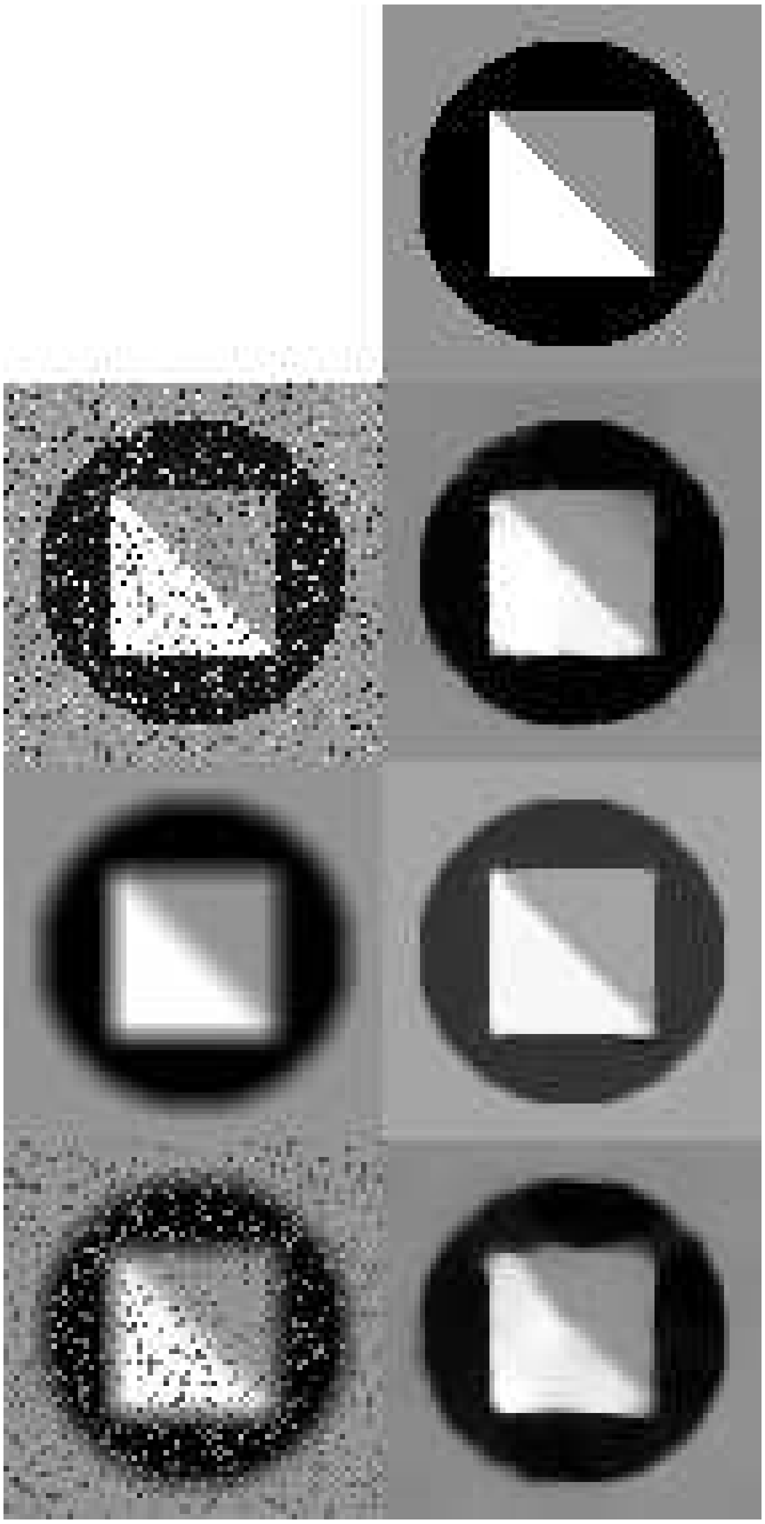
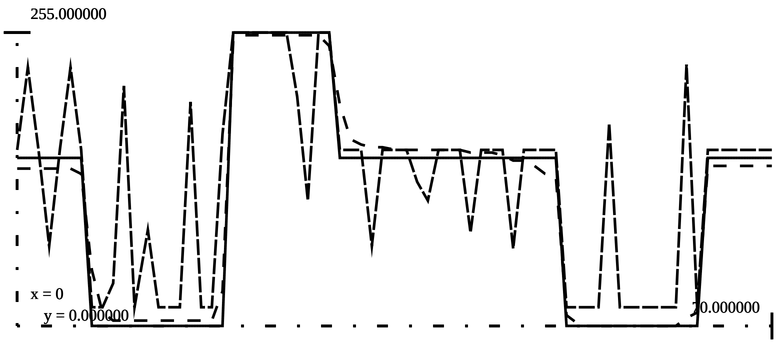
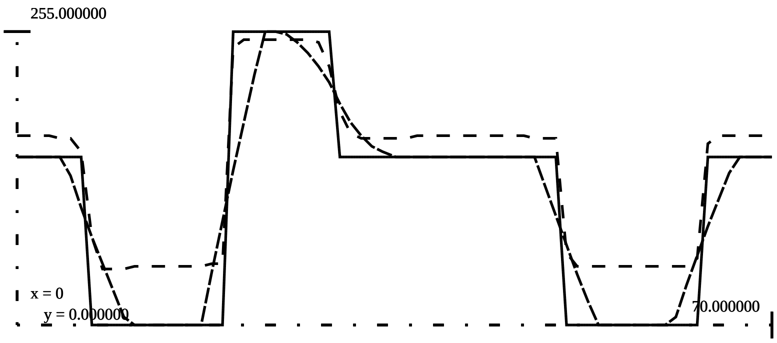


Figure 8.11: Results for another synthetic picture, obtained with the potential ϕ_2, in the denoising case, deblurring case and both, denoising-deblurring (left: the initial data, right: the result, on each line).We use the algorithm ST.DIV.EQ.

(a) denoising



(b) deblurring



(c) denoise and deblur

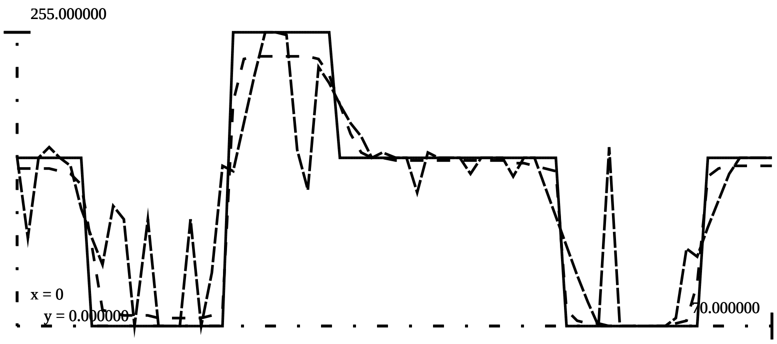


Figure 8.12: Profiles for the previous results: the original signal, the initial signal and the result are superposed.

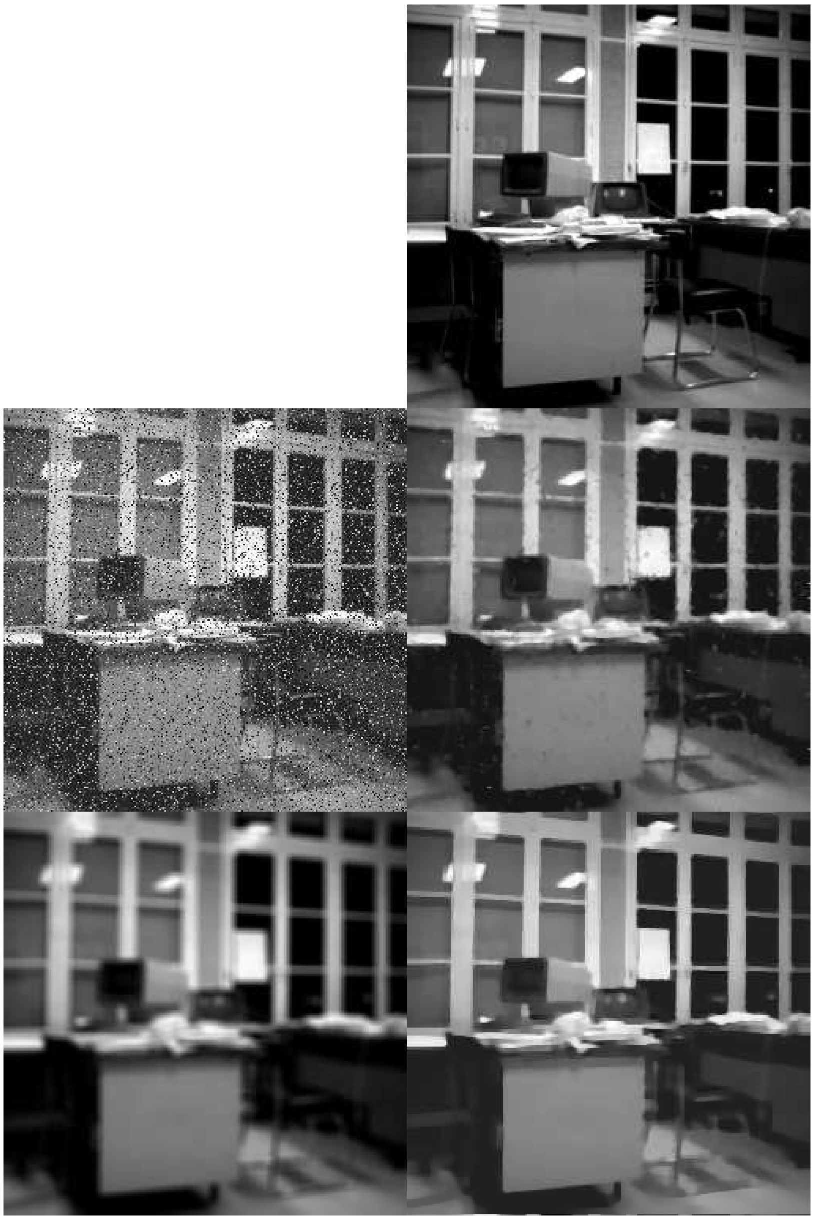


Figure 8.13: The office picture, using the potential ϕ_2. From left to right: the degraded image and the reconstructed image. From top to bottom: the denoising case and the deblurring case, with ST.DIV.EQ.

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