Multiphase Object Detection and Image Segmentation

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Abstract

This paper presents a class of techniques for object detection and image segmentation, using variational models formulated in a level set approach. We consider in particular Mumford and Shah like energies, whose minimizers are in the space of special functions of bounded variation. For such functions, all points are of two types: points where the functions have an approximate gradient, and points of discontinuities along curves or edges. The set of discontinuities is represented implicitly, using the level set method. Minimizing these energies in a level set formulation, yields coupled curve evolution and diffusion equations, which can be used for object detection and image segmentation. Finally, the proposed methods are validated by various numerical results in two dimensions.

1 Introduction

An important problem in image processing is the partition or segmentation of a given image u_0 into regions and their boundaries. Another closely related problem is the object detection by curve evolution and active contours (this is also called shape extraction or shape segmentation).

The image segmentation problem in computer vision is often posed as a variational problem. Given $u_0 : \Omega \to R$, with $\Omega \subset R^2$, the problem is to find an optimal piecewise smooth approximation u of u_0, and a set of boundaries K, such that u varies smoothly within the connected components of $\Omega \setminus K$, and rapidly or discontinuously across K.

To solve this problem, D. Mumford and J. Shah [MS88b] proposed the following minimization problem:

$$\inf_{u,K} \Big\{ F^{MS}(u,K) = \int_\Omega |u-u_0|^2 dx + \mu \int_{\Omega\setminus K} |\nabla u|^2 dx + \nu \int_K d\mathcal{H}^1 \Big\}, \qquad (1)$$

where $\mu > 0$, $\nu > 0$ are fixed parameters, to weight the different terms in the energy. For (u,K) a minimizer of the above energy, u is an "optimal" piecewise smooth approximation of the initial, possibly noisy, image u_0, and K has the role of approximating the edges of u_0; u will be smooth only outside K, i.e. on

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$\Omega \setminus K$. Here, $\mathcal{H}^1$ is the one-dimensional Hausdorff measure. Theoretical results of existence and regularity of minimizers of (1) can be found for example in [MS88b], [MS88a], [MS89], [MS94], [MS95].

A reduced case of the above model is obtained by restricting the segmented image u to piecewise constant functions, i.e. $u =$ constant c_i inside each connected component Ω_i of $\Omega \setminus K$. Then the problem is often called the "minimal partition problem", and in order to solve it, in [MS88b] it is proposed to minimize the following functional:

$$F_0^{MS}(u, K) = \sum_i \int_{\Omega_i} |u_0 - c_i|^2 dx + \nu \int_K d\mathcal{H}^1. \tag{2}$$

Again, ν is a positive parameter, having a scaling role. It is easy to see that, for a fixed closed set K (a finite union of curves), the energy (2) is minimized in the variables c_i by setting $c_i = mean(u_0)$ in Ω_i, or $c_i = \frac{\int_{\Omega_i} u_0(x)dx}{|\Omega_i|}$. Theoretical results for existence and regularity of minimizers of (2) can be found for example in [MS88b], [MT93], [Tam96], [TC96], [LT98].

It is not easy to minimize in practice the functionals (1) and (2), because of the unknown set K of lower dimension, and because these problems are not convex.

A weak formulation of (1) has been proposed in [DMMS92], where K is replaced by the set J_u of jumps of u, in order to prove the existence of minimizers. Also, it is known that a global minimizer of (1), or of the weak formulation, is not unique in general. In [MS88a], [MS89], the authors proposed a constructive existence result for the weak formulation of the Mumford and Shah problem, and in [KLM94], a practical multi-scale algorithm based on regions growing and merging is proposed in the piecewise-constant case. For a general exposition of the segmentation problem by variational methods, both in theory and practice, we refer the reader to [MS94], [MS95]. We also refer to [Amb89], [Bra98] for theoretical results on functionals defined on the appropriate space for image segmentation: the $SBV(\Omega)$ space of special functions of bounded variation, that will be introduced later.

Two elliptic approximations by Γ-convergence to the weak formulation of the Mumford and Shah functional have been proposed in [AT90], [AT92]. The authors approximated a minimizer (u, J_u) of $F^{MS}(u, J_u)$, by smooth functions (u_ρ, v_ρ), such that, as $\rho \to 0$, we have $u_\rho \to u$ and $v_\rho \to 1$ in the $L^2(\Omega)$-topology, and v_ρ is different from 1 only in a small neighborhood of J_u, which shrinks as $\rho \to 0$. The elliptic approximations lead to a coupled system of two equations in the unknowns u_ρ and v_ρ, to which standard PDE numerical methods can be applied. Related approximations and numerical results can be found in [Mar92], [Cha92], [Cha95], [Cha99], [BC00], [Bou99], [VC97]. Also, in [CM99], the authors provide an approximation by $\Gamma - convergence$ based on the finite element method, to the weak formulation of the Mumford and Shah problem. Note that, most of these methods solving the weak formulation by elliptic approximations do not explicitly compute the partition of the image and the set of curves K. In general, only an approximation to K is obtained, by a sequence of regions enclosing K, but converging in the limit to the empty set.

As it was mentioned earlier, a related problem to image segmentation is the object detection problem by snakes and active contours. An initial curve evolves in the image under some speed, and it stops on boundaries of objects, for instance where the magnitude gradient of the image is large. Some of the most well known active contour models with edge-function are: [KWT88], [CCCD93], [MSV95], [CKS97], [KKO+96], [XP98]. The active contour models using the gradient of the image for the stopping criteria are also called boundary based models.

Many active contour models use the level set method introduced by S. Osher and J. Sethian [OS88], to represent the evolving curve. This method works on a fixed rectangular grid, and allows for automatic topology changes, such as merging and breaking. A closed curve $K = \partial\omega \subset \Omega$ (with $\omega \subset \Omega$ an open subset), is represented by the zero level set of a Lipschitz-continuous function $\phi : \Omega \to R$, such that $K = \{x \in \Omega : \phi(x) = 0\}$, and $\phi > 0$ on one side of K and $\phi < 0$ on the other side of K. These properties can be expressed as follows:

$$\begin{array}{l} \phi(x) = 0 \text{ if } x \in K, \\ \phi(x) > 0 \text{ if } x \in \omega, \\ \phi(x) < 0 \text{ if } x \in \Omega \setminus \overline{\omega}. \end{array}$$

More recently, new active contour and level set methods have been proposed for image segmentation, some of them involving region based techniques, in addition to the previous boundary based techniques. Among these methods, we would like to mention those closely related with the present work.

First, the present work is a summary and generalization of the active contour model without edges and its generalizations to segmentation of images, from [CV99], [CV01b], [CV02], [CV01a] and [VC01]. These work propose a level set method for active contours and segmentation via the Mumford and Shah model [MS88b].

Other closely related variational level set methods are: “Inward and outward curve evolution using level set method”, from [ADBA99]; “A variational level set approach to multiphase motion” from [ZCMO96]; “A level set model for image classification”, from [SBFAZ99], [SBFAZ00]; “A statistical approach to snakes for bimodal and trimodal imagery”, from [YTW99b]; ”A Fully Global Approach to Image Segmentation via Coupled Curve Evolution Equations”, from [YTW02]; ”Binary Flows and Image Segmentation”, from [YTW99a]; “Geodesic Active Regions: A New Framework to Deal with Frame Partition Problems in Computer Vision”, from [PD02]; “Coupled geodesic active regions for image segmentation: a level set approach”, from [PD00]. Finally, a closely related work is “Curve evolution implementation of the Mumford-Shah functional for image segmentation, denoising, interpolation, and magnification”, from [TYW01].

Many other contributions have been proposed for image segmentation by variational or PDE methods, and it is impossible to mention all of them. However, we would like to give a few more references: “Region competition: Unifying snakes, region growing, and Bayes/MDL for multi-band image segmentation”, from [ZYL95], [ZY96]; “Normalized cuts and image segmentation”, from [SM00]; “A common framework for curve evolution, segmentation and anisotropic diffusion”, from [Sha96]; “Riemannian Drums, Anisotropic Curve Evolution and Segmentation”, from [Sha99]; “Filtering, Segmentation and Depth”, from [MNS93]; ”Codimension-Two Geodesic Active Contours for the Segmentation of Tubular Structures”, from [LFG+00].

As we have already mentioned, the proposed approach follows and generalizes the ideas from [CV99], [CV01b], [CSV00], [CV02], [CV01a], [VC01], for image segmentation and object detection by a variational level set approach. First, we extend the piecewise constant models from [CV99], [CV01b], [VC01], to piecewise linear versions. We also consider other general anisotropic Mumford and Shah like functionals, where the energy term along K also depends on the jump of u.

2 Variational models for image segmentation and image partition

We will consider here a class of functionals with solutions in the space $SBV(\Omega)$, with $\Omega \subset R^2$ an open and bounded set. For a general exposition of such functionals in a weak formulation, we refer the reader to [Bra98] and [Amb89]. By definition [Amb89], [Bra98], a function $u \in L^1(\Omega)$ is a special function of bounded variation, if its distributional derivative can be written as $Du = \nabla u dx + (u^+ - u^-) n_u \chi_K \mathcal{H}^1|_K$, where ∇u is called the approximate gradient of u at $x \in \Omega \setminus K$, and K is a set of finite 1-dimensional Hausdorff measure. So, for any point $x \in \Omega$, either u has an approximate gradient ∇u at x, or x is a jump point of u, with distinct approximate limits $u^+(x)$ and $u^-(x)$ of u on each side of K. Here, n_u denotes the exterior unit normal to K at every point $x \in K$, where $|u^+(x) - u^-(x)| > 0$, and $\mathcal{H}^1|_K$ is the restriction of the measure $\mathcal{H}^1$ to K. The space of special functions of bounded variation is denoted by $SBV(\Omega)$.

A class of functionals defined on $SBV(\Omega)$ is given by [Bra98]:

$$F(u, K) = \int_\Omega |u - u_0|^p dx + \mu \int_{\Omega \setminus K} f(\nabla u) dx + \nu \int_K g(|u^+ - u^-|) d\mathcal{H}^1, \quad (3)$$

where $p \geq 1$, $f : R^2 \to [0, +\infty)$ is a convex function satisfying

$$\lim_{|z| \to \infty} \frac{f(z)}{|z|} = +\infty,$$

and $g : [0, \infty) \to [0, \infty)$ is a sub-additive and increasing function satisfying

$$\lim_{t \to 0} \frac{g(t)}{t} = \infty.$$

The function g is sub-additive in the following sense: $g(t_1 + t_2) \leq g(t_1) + g(t_2)$, for all $t_1, t_2 \geq 0$. The assumptions on the functions f and g are necessary for existence results of minimizers on the $SBV(\Omega)$ space.

$F(u, K)$ reduces to the Mumford and Shah functional [MS88b] when $p = 2$, $f(\nabla u) = |\nabla u|^2$ and $g(|u^+ - u^-|) = 1$ is a constant function.

Other examples of functionals, in addition to the Mumford and Shah functional, can be obtained with: $f(\nabla u) = |\nabla u|^q$, with $q > 1$, and $g(|u^+ - u^-|) = \sqrt{|u^+ - u^-|}$.

Given $u_0 \in L^\infty(\Omega)$, the minimization problem

$$\inf_{u,K} F(u, K)$$

can be seen as a partitioning or segmentation problem of the given image u_0. If (u, K) is a minimizer, then the connected components of $\Omega \setminus K$ will be smooth regions or objects in the image, while the closed set K of lower dimension will represent edges, contours or boundaries of objects and of regions.

In the case when the function g is not constant and depends on the magnitude of the jump $|u^+ - u^-|$, then the quantity $g(|u^+ - u^-|)$ along the discontinuity set K of contours plays an additional anisotropic scaling role.

In the next section we will show that, by representing the set K using the level set method, and minimizing the above energy, we obtain interesting coupled curve evolution and diffusion equations. The obtained level set models can be used for object detection, denoising, image partition and segmentation.

3 Level set formulations of minimization problems on $SBV(\Omega)$

In this section, we present several extensions and generalizations of the results from [CV99], [CV01b], [CSV00], [CV02], [CV01a], [VC01]. We follow the notations and terminology from [ZCMO96], [CV01b].

In what follows, to a level set function $\phi : \Omega \to R$, we associate the Heaviside function $H(\phi)$ defined by: $H(\phi(x)) = 1$ if $\phi(x) \geq 0$ and $H(\phi(x)) = 0$ if $\phi(x) < 0$.

Let us consider various subsets of functions u having discontinuities only along $K = \{x \in \Omega : \ \phi(x) = 0\}$, defined as follows:

$$\Big\{u(x) = P^+(x)H(\phi(x)) + P^-(x)(1 - H(\phi(x)))\Big\}, \tag{4}$$

with P^+, P^- polynomials of degree at most m (in this paper, we will consider only constant and linear polynomials, i.e. $m = 0$ and $m = 1$). P^+ is defined on $\{x \in \Omega : \ \phi(x) \geq 0\}$ and P^- is defined on $\{x \in \Omega : \ \phi(x) \leq 0\}$. Another subset of functions is

$$\Big\{u(x) = u^+(x)H(\phi(x)) + u^-(x)(1 - H(\phi(x)))\Big\}, \tag{5}$$

with u^+, u^- functions, such that $u^+ \in C^1(\{x \in \Omega : \ \phi(x) \geq 0\})$, and $u^- \in C^1(\{x \in \Omega : \ \phi(x) \leq 0\})$.

As in [CV02], [CV01a], [VC01], in order to represent triple junctions and more complex topologies, we can also consider the following subsets of functions, where discontinuities are along a set defined using two level set functions ϕ_1 and ϕ_2:

$$K = \{x \in \Omega : \ \phi_1(x) = 0\} \cup \{x \in \Omega : \ \phi_2(x) = 0\}.$$

Then similarly, the corresponding subsets of functions u will be:

$$\begin{aligned} \Big\{u(x) \ &= \ P^{++}(x)H(\phi_1(x))H(\phi_2(x)) \\ &+ \ P^{+-}(x)H(\phi_1(x))(1 - H(\phi_2(x))) \\ &+ \ P^{-+}(x)(1 - H(\phi_1(x)))H(\phi_2(x)) \\ &+ \ P^{--}(x)(1 - H(\phi_1(x)))(1 - H(\phi_2(x)))\Big\}, \end{aligned} \tag{6}$$

with $P^{++}, P^{+-}, P^{-+}, P^{--}$ polynomials of degree at most m, defined respectively on $\{x \in \Omega : \ \phi_1(x) \geq 0, \phi_2(x) \geq 0\}$, $\{x \in \Omega : \ \phi_1(x) \geq 0, \phi_2(x) \leq 0\}$, $\{x \in \Omega : \ \phi_1(x) \leq 0, \phi_2(x) \geq 0\}$, $\{x \in \Omega : \ \phi_1(x) \leq 0, \phi_2(x) \leq 0\}$, and

$$
\begin{aligned}
\Big\{u(x) &= u^{++}(x)H(\phi_1(x))H(\phi_2(x)) \\
&+ u^{+-}(x)H(\phi_1(x))(1 - H(\phi_2(x))) \\
&+ u^{-+}(x)(1 - H(\phi_1(x)))H(\phi_2(x)) \\
&+ u^{--}(x)(1 - H(\phi_1(x)))(1 - H(\phi_2(x)))\Big\},
\end{aligned} \tag{7}
$$

with $u^{++}, u^{+-}, u^{-+}, u^{--}$ C^1 functions, defined respectively on the following subsets: $\{x \in \Omega : \ \phi_1(x) \geq 0, \phi_2(x) \geq 0\}$, $\{x \in \Omega : \ \phi_1(x) \geq 0, \phi_2(x) \leq 0\}$, $\{x \in \Omega : \ \phi_1(x) \leq 0, \phi_2(x) \geq 0\}$, $\{x \in \Omega : \ \phi_1(x) \leq 0, \phi_2(x) \leq 0\}$.

In what follows, we will write and minimize the energy $F(u, K)$ from (3) restricted to the subsets defined above, and we will solve some of these minimizations in a few particular cases.

For instance, the minimization of (3) restricted to the subset from (5) can be written as:

$$
\begin{aligned}
& \inf_{u^+, u^-, \phi} F(u^+, u^-, \phi) \\
= & \int_\Omega |u^+ - u_0|^p H(\phi) dx + \int_\Omega |u^- - u_0|^p (1 - H(\phi)) dx \\
+ & \ \mu \int_\Omega f(\nabla u^+) H(\phi) dx + \mu \int_\Omega f(\nabla u^-)(1 - H(\phi)) dx \\
+ & \ \nu \int_\Omega g(|u^+ - u^-|) |\nabla H(\phi)|,
\end{aligned}
$$

and similarly in the other cases. Then, writing the Euler-Lagrange equations associated with the minimization problem, interesting coupled curve evolution and diffusion equations will be obtained, with applications to object detection and image segmentation.

For the purpose of illustration, we will only consider the following particular cases: $p = 2$, $f(\nabla u) = |\nabla u|^2$, $g(|u^+ - u^-|) = 1$ and $g(|u^+ - u^-|) = \sqrt{|u^+ - u^-|}$. These particular cases yield two functionals. The first one is the isotropic classical Mumford and Shah energy [MS88b]

$$
F^{MS}(u, K) = \int_\Omega |u - u_0|^2 dx + \mu \int_{\Omega \setminus K} |\nabla u|^2 dx + \nu \int_K d\mathcal{H}^1. \tag{8}
$$

The second one is an "anisotropic" Mumford and Shah like energy

$$
E^{MS}(u, K) = \int_\Omega |u - u_0|^2 dx + \mu \int_{\Omega \setminus K} |\nabla u|^2 dx + \nu \int_K \sqrt{|u^+ - u^-|} d\mathcal{H}^1. \tag{9}
$$

Let us now consider only these last two energies, restricted to some of the sets mentioned above. We will explicitly write in each case the form of the minimization problem and the associated Euler-Lagrange equations. We will parametrize by an artificial time the descent direction in ϕ, ϕ_1, or ϕ_2. We will also regularize the Heaviside function H by $H_\varepsilon \in C^1(R)$ as $\varepsilon \to 0$, as in [CV99], [CV01b], where H_ε is defined by:

$$H_\varepsilon(s) = \frac{1}{2}\Big(1 + \frac{2}{\pi}\arctan(\frac{s}{\varepsilon})\Big).$$

We will also use the notation $\delta_\varepsilon = H'_\varepsilon$ for an approximation and regularization of the one-dimensional Dirac function δ_0, concentrated at the origin.

Minimizing the restriction of (8) to the subset

$$\Big\{u(x) = c^+ H(\phi(x)) + c^-(1 - H(\phi(x)))\Big\},$$

with c^+, c^- unknown (polynomials of degree 0), yields:

$$\inf_{c^+,c^-,\phi} \int_\Omega |u_0(x) - c^+|^2 H(\phi)dx + \int_\Omega |u_0(x) - c^-|^2(1 - H(\phi))dx + \nu \int_\Omega |\nabla H(\phi)|, \qquad (10)$$

i.e. the active contour model without edges from [CV99], [CV01b]. The minimizers have to satisfy the following coupled equations, given $\phi(0,x) = \phi_0(x)$:

$$\begin{aligned} c^+(\phi(t,\cdot)) &= \frac{\int_\Omega u_0(x)H(\phi(x))dx}{\int_\Omega H(\phi(x))dx}, \\ c^-(\phi(t,\cdot)) &= \frac{\int_\Omega u_0(x)(1 - H(\phi(x)))dx}{\int_\Omega (1 - H(\phi(x)))dx}, \\ \frac{\partial \phi}{\partial t} &= \delta_\varepsilon(\phi)\Big[\nu \mathrm{div}\Big(\frac{\nabla\phi}{|\nabla\phi|}\Big) - |u_0 - c^+|^2 + |u_0 - c^-|^2\Big]. \end{aligned}$$

A numerical result obtained with this model is shown in Figure 1. This model performs active contours, and has the following advantages, when compared with boundary-based models: it automatically detects interior contours; it detects both contours with or without gradient; the position of the initial curve can be anywhere in the image.

An extension of the previous model is introduced here, which is obtained by considering linear approximations: minimizing the restriction of (8) to the subset

$$\Big\{u(x) = \Big((a^+,b^+)\cdot x + c^+\Big)H(\phi(x)) + \Big((a^-,b^-)\cdot x + c^-\Big)(1 - H(\phi(x)))\Big\},$$

with $a^+, b^+, c^+, a^-, b^-, c^-$ unknown (as coefficients of polynomials of degree 1), yields:

$$\begin{aligned} \inf_{a^+,b^+,c^+,a^-,b^-,c^-,\phi} & \int_\Omega \Big|u_0(x) - \Big((a^+,b^+)\cdot x + c^+\Big)\Big|^2 H(\phi)dx \\ & + \int_\Omega \Big|u_0(x) - \Big((a^-,b^-)\cdot x + c^-\Big)\Big|^2 (1 - H(\phi))dx \\ + \mu \int_\Omega \Big((a^+)^2 + (b^+)^2\Big)H(\phi)dx & + \mu \int_\Omega \Big((a^-)^2 + (b^-)^2\Big)(1 - H(\phi))dx \\ & + \nu \int_\Omega |\nabla H(\phi)|. \qquad (11) \end{aligned}$$

The minimizers have to satisfy the following coupled equations, given $\phi(0,x) = \phi_0(x)$ (note the linear algebraic systems for (a^+, b^+, c^+) and (a^-, b^-, c^-)):

$$\begin{aligned}
a^+ \int_\Omega (x_1^2 + \mu) H(\phi(x))dx + b^+ \int_\Omega x_1 x_2 H(\phi(x))dx + c^+ \int_\Omega x_1 H(\phi(x))dx \\
= \int_\Omega x_1 u_0(x) H(\phi(x))dx, \\
a^+ \int_\Omega x_1 x_2 H(\phi(x))dx + b^+ \int_\Omega (x_2^2 + \mu) H(\phi(x))dx + c^+ \int_\Omega x_2 H(\phi(x))dx \\
= \int_\Omega x_2 u_0(x) H(\phi(x))dx, \\
a^+ \int_\Omega x_1 H(\phi(x))dx + b^+ \int_\Omega x_2 H(\phi(x))dx + c^+ \int_\Omega H(\phi(x))dx \\
= \int_\Omega u_0(x) H(\phi(x))dx,
\end{aligned}$$

(similarly for (a^-, b^-, c^-)), substituting $H(\phi)$ by $(1 - H(\phi))$, and

$$\begin{aligned}
\frac{\partial \phi}{\partial t} = \delta_\varepsilon(\phi) \Big[\nu \mathrm{div}\Big(\frac{\nabla \phi}{|\nabla \phi|}\Big) - \Big| u_0(x) - \Big((a^+, b^+) \cdot x + c^+ \Big) \Big|^2 \\
+ \Big| u_0(x) - \Big((a^-, b^-) \cdot x + c^- \Big) \Big|^2 \\
- \mu \Big((a^+)^2 + (b^+)^2 \Big) + \mu \Big((a^-)^2 + (b^-)^2 \Big) \Big].
\end{aligned}$$

Numerical results obtained with this model are presented in Figures 4 and 5. This linear case has also been discussed by the author with P. Hamilton, during a collaboration on medical image segmentation.

Minimizing the restriction of (8) to the subset

$$\Big\{ u(x) = u^+(x) H(\phi(x)) + u^-(x)(1 - H(\phi(x))) \Big\},$$

yields (see [CV01a], [VC01]):

$$\begin{aligned}
\inf_{u^+, u^-, \phi} \int_\Omega |u^+ - u_0|^2 H(\phi) dx + \int_\Omega |u^- - u_0|^2 (1 - H(\phi)) dx \qquad (12) \\
+ \mu \int_\Omega |\nabla u^+|^2 H(\phi) dx + \mu \int_\Omega |\nabla u^-|^2 (1 - H(\phi)) dx + \nu \int_\Omega |\nabla H(\phi)|.
\end{aligned}$$

The minimizers have to satisfy the following coupled equations, given the initial condition $\phi(0,x) = \phi_0(x)$:

$$\begin{aligned}
u^+ = u_0 + \mu \triangle u^+ \text{ on } \{\phi > 0\}, \frac{\partial u^+}{\partial \vec{n}} = 0 \text{ on } \{\phi = 0\} \cup \partial\Omega, \\
u^- = u_0 + \mu \triangle u^- \text{ on } \{\phi < 0\}, \frac{\partial u^-}{\partial \vec{n}} = 0 \text{ on } \{\phi = 0\} \cup \partial\Omega, \\
\frac{\partial \phi}{\partial t} = \delta_\varepsilon(\phi) \Big[\nu \mathrm{div}\Big(\frac{\nabla \phi}{|\nabla \phi|}\Big) - |u^+ - u_0|^2 + |u^- - u_0|^2 - \mu |\nabla u^+|^2 + \mu |\nabla u^-|^2 \Big].
\end{aligned}$$

We would like to mention that this last case was also proposed and solved independently by [TYW01]. Figure 2 shows a numerical result using this case.

Let us now consider the restriction of (8) to the subset

$$\Big\{u(x) = u^{++}(x)H(\phi_1(x))H(\phi_2(x)) + u^{+-}(x)H(\phi_1(x))(1-H(\phi_2(x))) + u^{-+}(x)(1-H(\phi_1(x)))H(\phi_2(x)) + u^{--}(x)(1-H(\phi_1(x)))(1-H(\phi_2(x)))\Big\}. \quad (13)$$

In this case, the minimization problem can be written as:

$$\inf_{u^{++},u^{+-},u^{-+},u^{--},\phi_1,\phi_2} \int_\Omega \Big[|u^{++}-u_0|^2 H(\phi_1)H(\phi_2) + |u^{+-}-u_0|^2 H(\phi_1)(1-H(\phi_2)) + |u^{-+}-u_0|^2(1-H(\phi_1))H(\phi_2) + |u^{--}-u_0|^2(1-H(\phi_1))(1-H(\phi_2))\Big]dx$$
$$+\mu \int_\Omega \Big[|\nabla u^{++}|^2 H(\phi_1)H(\phi_2) + |\nabla u^{+-}|^2 H(\phi_1)(1-H(\phi_2)) + |\nabla u^{-+}|^2(1-H(\phi_1))H(\phi_2) + |\nabla u^{--}|^2(1-H(\phi_1))(1-H(\phi_2))\Big]dx$$
$$+\nu \int_\Omega |\nabla H(\phi_1)| + \nu \int_\Omega |\nabla H(\phi_2)|\Big((1-H(-\phi_1)) + (1-H(\phi_1))\Big). \quad (14)$$

Note that the term $\Big((1-H(-\phi_1)) + (1-H(\phi_1))\Big)$ is used to avoid counting more than once the segments of curves belonging to both $\{\phi_1 = 0\}$ and $\{\phi_2 = 0\}$. The minimizers $(u^{++}, u^{+-}, u^{-+}, u^{--}, \phi_1, \phi_2)$ satisfy coupled curve evolution and diffusion equations, similar with those from the previous case. We think that in this case, based on the Four Color Theorem, defining the set of minimizers by this set with four functions u^{++}, u^{+-}, u^{-+}, u^{--} and with only two level set functions, should formally suffice to represent any case. The connection with the Four Color Theorem in image segmentation has been also made in [WKCn00]. A numerical result obtained in this case is presented in Figure 9, from [VC01].

Finally, for the purpose of illustration, let us consider one example of minimization problem involving the anisotropic energy (9). The minimization of the energy (9) restricted for instance to the subset

$$\Big\{u(x) = c^+ H(\phi(x)) + c^-(1-H(\phi(x)))\Big\},$$

with c^+, c^- unknown (polynomials of degree 0), yields:

$$\inf_{c^+,c^-,\phi} \int_\Omega |u_0(x)-c^+|^2 H(\phi(x))dx + \int_\Omega |u_0(x)-c^-|^2(1-H(\phi(x)))dx + \nu\sqrt{|c^+-c^-|}\int_\Omega |\nabla H(\phi(x))|. \quad (15)$$

The minimizers have to satisfy the following coupled equations, given the initial condition $\phi(0,x) = \phi_0(x)$:

$$c^+ = \frac{\int_\Omega u_0(x)H(\phi(x))dx}{\int_\Omega H(\phi(x))dx} - \frac{sgn(c^+-c^-)}{4\sqrt{|c^+-c^-|}}\frac{\int_\Omega |\nabla H(\phi)|}{\int_\Omega H(\phi(x))dx},$$

$$
\begin{aligned}
c^- &= \frac{\int_\Omega u_0(x)(1-H(\phi(x)))dx}{\int_\Omega (1-H(\phi(x)))dx} - \frac{sgn(c^- - c^+)}{4\sqrt{|c^+ - c^-|}} \frac{\int_\Omega |\nabla H(\phi)|}{\int_\Omega (1-H(\phi(x)))dx}, \\
\frac{\partial \phi}{\partial t} &= \delta_\varepsilon(\phi)\Big[\nu\sqrt{|c^+ - c^-|}\mathrm{div}\Big(\frac{\nabla\phi}{|\nabla\phi|}\Big) - |u_0 - c^+|^2 + |u_0 - c^-|^2\Big].
\end{aligned}
$$

A numerical result obtained using this model is presented in Figure 3. If the coefficient ν is the same as in the isotropic case (10), this new anisotropic model has a stronger constraint on the length term. In addition, in the general case, the presence of the additional factor $\sqrt{|u^+ - u^-|}$ in the energy term along K should remove the limitations on the type of edges obtained by the isotropic Mumford and Shah model. Some of these limitations are [MS88b]: junctions can be only triple junctions with angles of 120^o, and if one edge intersects the boundary $\partial\Omega$, it has to be at right angle.

Similarly, the anisotropic Mumford and Shah like energy (9) can also be written for the other cases.

The general functional in (3) has the advantage of having minimizers on the space $SBV(\Omega)$, the appropriate space for image segmentation: for $u \in SBV(\Omega)$, each point $x \in \Omega$ will be either a point in a homogeneous region, or an edge point. Unfortunately, the minimization of (3) is not convex, and it is not always guaranteed that the numerical algorithm will converge to a global minimizer. In addition, the global minimizer is not unique in general. To overcome this difficulties, an idea could be to consider convex minimization problems on the larger space $BV(\Omega)$ of functions of bounded variation, and to apply the same level set techniques. An example of such convex functional is given by the total variation minimization [ROF92]. In [VO02], a similar level set method is proposed to the minimization of this energy, again with applications to active contours and image segmentation.

4 Experimental results

We present in this section some numerical results obtained with the models from the previous section. For the details of the numerical schemes and for other numerical results, we refer the reader to [CV99], [CV01b], [CSV00], [CV01a], [CV02], [VC01].

As we will see in this section, these models have the abilities of automatic detection of interior contours, of detection of contours with or without gradient, and of detection and representation of complex topologies. The multiphase level set approach employed here has been introduced in [CV01a], [CV02], [VC01], and has the advantages of always keeping the multiple phases disjoint and with their union the entire domain, by definition. Triple junctions can be represented, with an optimal number of level set functions.

In Figure 1 we show a result obtained using the model (10). In Figure 2, we show a result obtained using the model (12). In Figures 3 and 6, the anisotropic model from (15) is used. In Figures 4 and 5, we have used the linear approximation model from (11). In Figure 7, the linear approximation model in a four phase fashion is used, with two level set functions (the obtained final four segments are shown in Figure 8). Finally, in Figure 9, a numerical result is presented using the general four-phase Mumford and Shah level set algorithm from (14), previously introduced in [CV01a], [VC01].

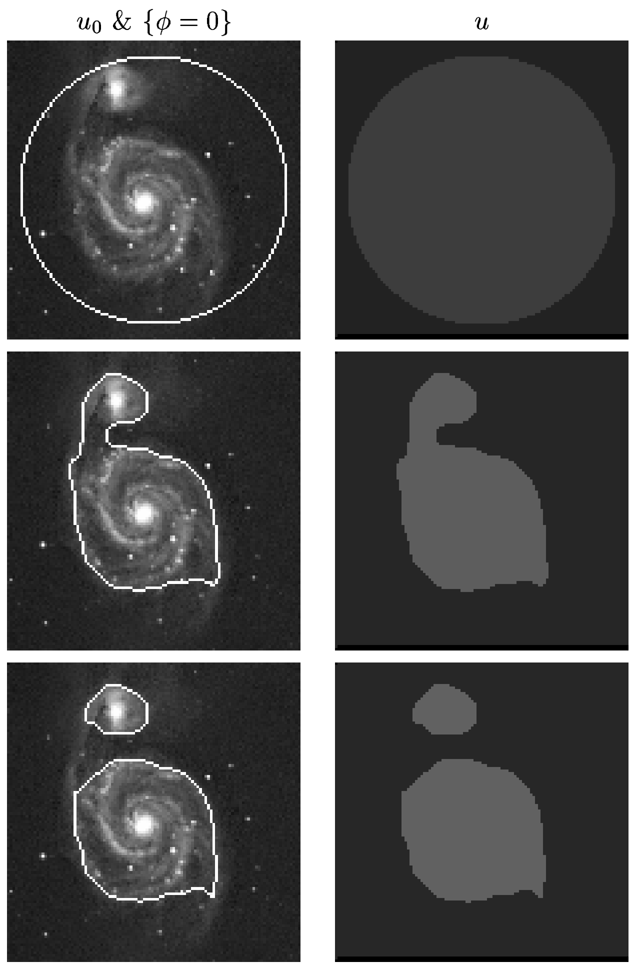

Figure 1: Numerical result using the piecewise-constant Mumford and Shah level set algorithm from (10), with $\nu = 0.09 * 255^2$.

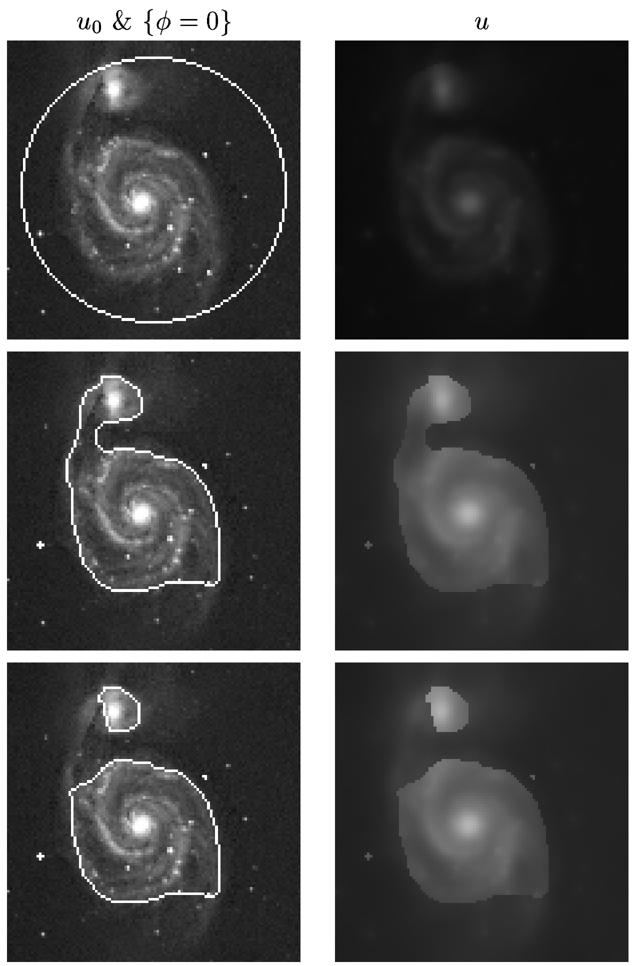

Figure 2: Numerical result using the piecewise-smooth Mumford and Shah level set algorithm from (12), with $\nu = 0.0305 * 255^2$ and $\mu = 10$.

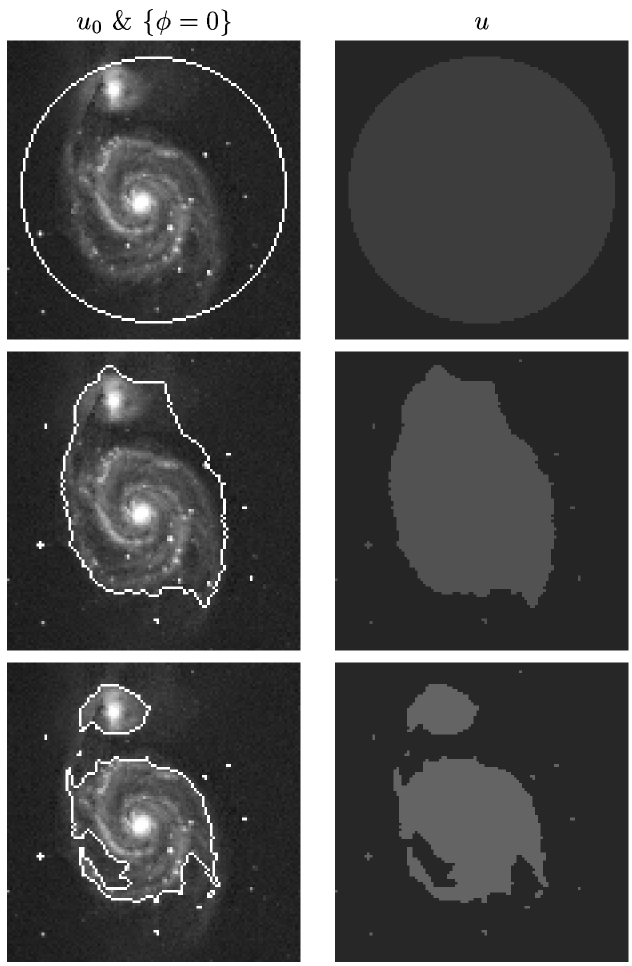

Figure 3: Numerical result using the piecewise-constant anisotropic Mumford and Shah like level set algorithm from (15), with $\nu = 0.0015 * 255^2$.

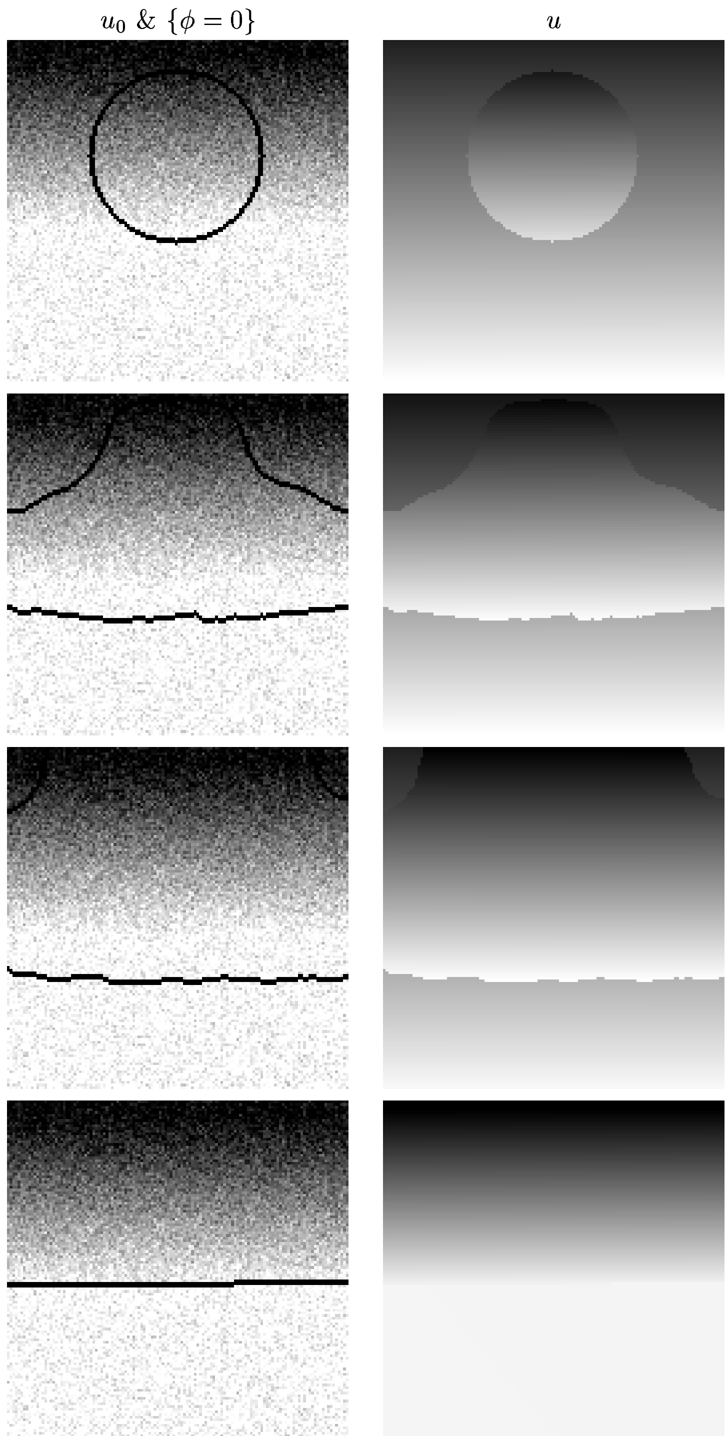

Figure 4: Numerical result using the piecewise-linear Mumford and Shah level set algorithm from (11), with $\mu = 1$, $\nu = 0.1 * 255^2$.

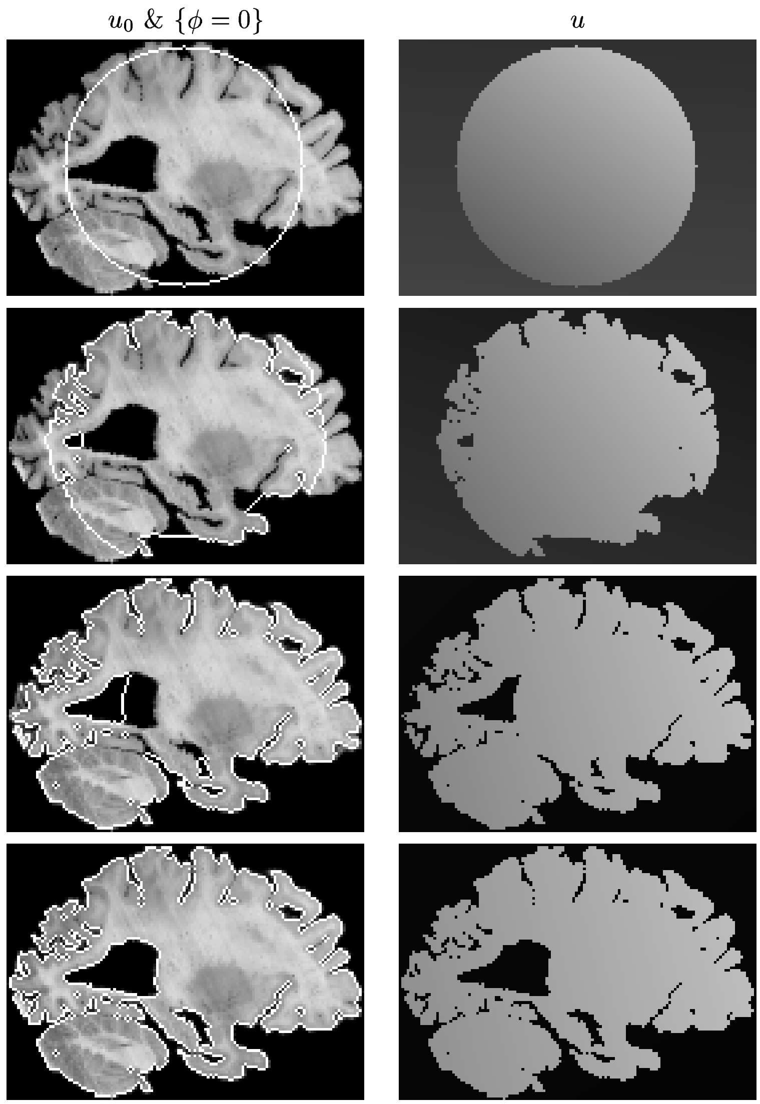

Figure 5: Numerical result using the piecewise-linear Mumford and Shah level set algorithm from (11), with $\mu = 0.001$ and $\nu = 0.001 * 255^2$.

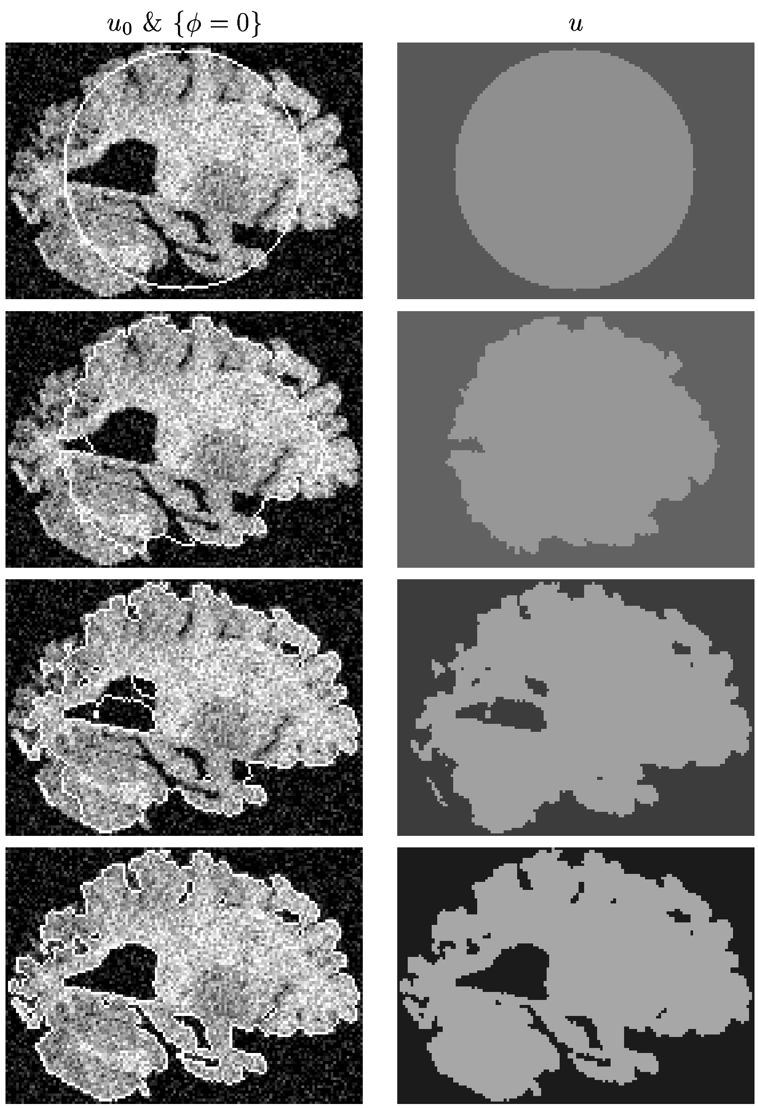

Figure 6: Numerical result using the piecewise-constant anisotropic Mumford and Shah like level set algorithm from (15), with $\nu = 0.01 * 255^2$.

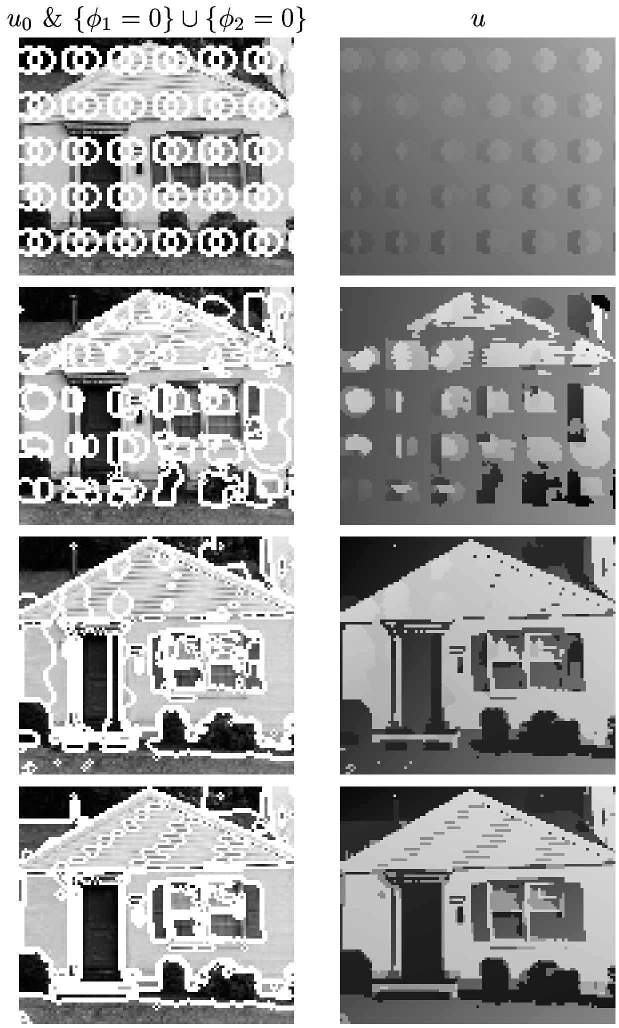

Figure 7: Numerical result using the piecewise-linear four phase Mumford and Shah level set algorithm, with $\mu = 1$, $\nu = 0.014 \cdot 255^2$.

Figure 8: Final four segments obtained for the result from Figure 7.

5 Conclusion

We have presented in this paper a level set technique for the minimization of a class of functionals defined on the space of special functions of bounded variation, arising in image segmentation. The obtained variational level set models yield coupled geometric and diffusion partial differential equations. Applications to object detection and image segmentation have been illustrated.

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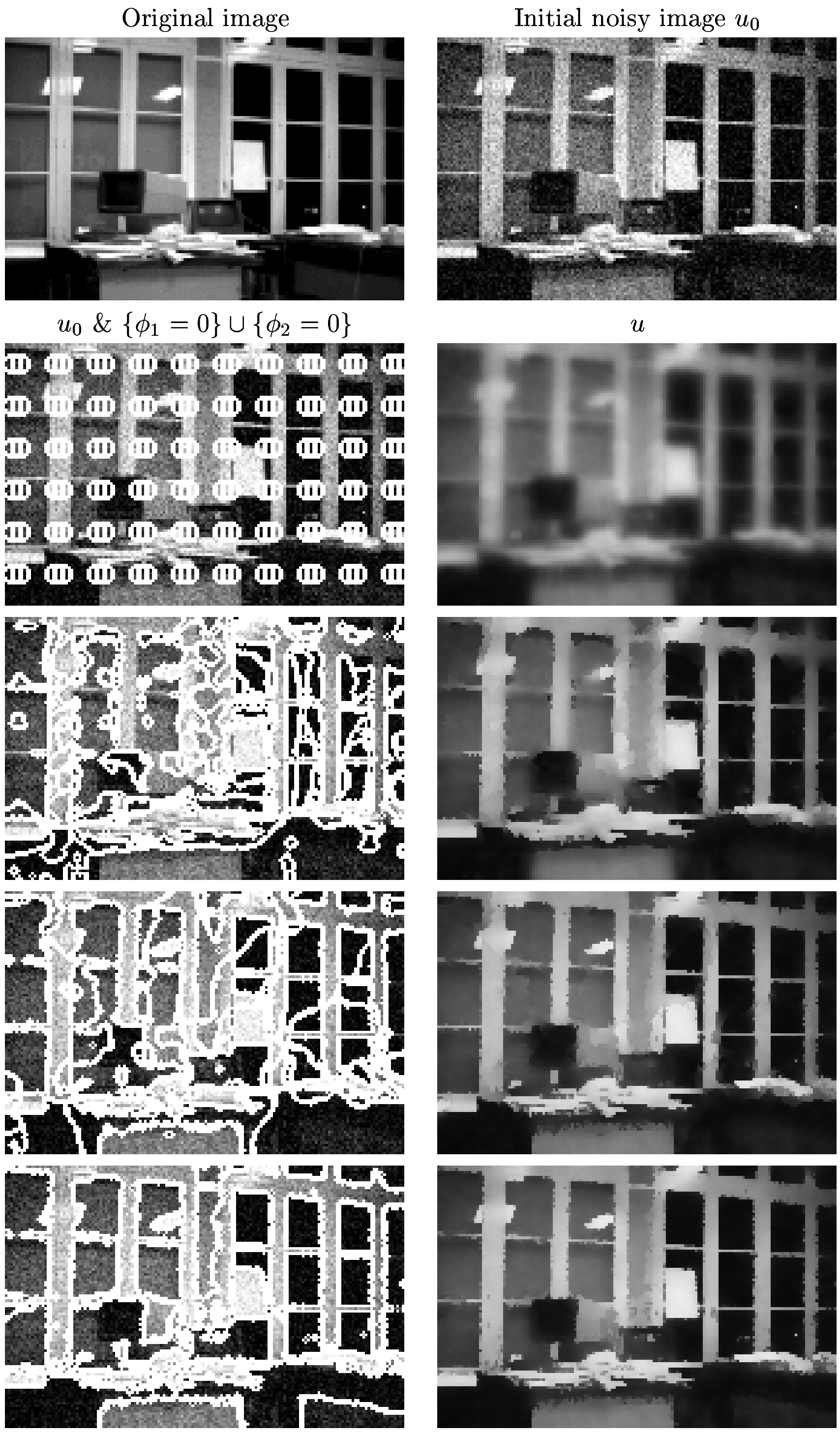

Figure 9: Numerical result by the general four phase Mumford and Shah level set algorithm from (14), with $\mu = 5$ and $\nu = 0.005 \cdot 255^2$.

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