Functional minimization problems in image processing

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ABSTRACT

In this work we wish to recover an unknown image from a blurry version. We solve this inverse problem by energy minimization and regularization. We seek a solution of the form $u+v$, where u is a function of bounded variation (cartoon component), while v is an oscillatory component (texture), modeled by a Sobolev function with negative degree of differentiability. Experimental results show that this cartoon + texture model better recovers textured details in natural images, by comparison with the more standard models where the unknown is restricted only to the space of functions of bounded variation.

Keywords: image deblurring, variational models, bounded variation, Sobolev spaces.

1. INTRODUCTION

We consider in this paper one of the classical problems in image analysis: the recovery of an unknown image from its blurry version, in the presence of a known blurring operator. Suppose that we are given a blurry (and possibly noisy) gray-scale image $f : \Omega \to \mathbb{R}$, where Ω is either $\mathbb{R}^n$ or an open and bounded subset of $\mathbb{R}^n$, and we wish to recover a clean image $\tilde{f}$ from f. Let K be the blurring operator. The standard linear degradation model that relates f to $\tilde{f}$ is

$$f = K\tilde{f} + noise.$$

By our proposed method, we do not only recover a sharp image $\tilde{f}$, but we also decompose $\tilde{f}$ into the cartoon and the texture parts, which will be denoted by u and $v = \Delta g$, respectively (here $\triangle$ is the Laplacian operator acting on the function g). We assume that K is a linear and continuous smoothing operator, for instance a convolution with the Gaussian kernel or with the average kernel.

The standard method for solving such inverse ill-posed problems is inspired from Tikhonov regularization[26] ,[27] ,[28] , which can be written as the general minimization problem

$$\inf_{\tilde{f}} \Big\{ \int_\Omega |f - K\tilde{f}|^p dx + \lambda \int_\Omega R(\tilde{f}) dx \Big\}, \tag{1}$$

where p is chosen function of the noise type (for instance, $p = 2$ for Gaussian noise, $p = 1$ for salt-and-pepper noise, etc), and $R(\tilde{f}) = r(|\nabla \tilde{f}|)$ is a regularizing potential, that usually depends on partial derivatives of $\tilde{f}$, and with at most linear growth at infinity (to recover sharp edges).

We refer in this context to an extensive work of minimization models of the form (1), with theoretical results, numerical algorithms, and experimental results:[12] ,[24] (as a generalization of[23]) ,[4] ,[29] ,[1] ,[7] ,[14] ,[6] ,[17] ,[18] ,[19] ,[10] ,[5] , among others. Also, a recent work on image deblurring using regularized locally-adaptive kernel regression is[25] .

More recently, model (1) has been generalized to cases of the form

$$\inf_{\tilde{f}} \Big\{ \|f - K\tilde{f}\|^p + \lambda \int_\Omega R(\tilde{f}) dx \Big\}, \tag{2}$$

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where $\|\cdot\|$ denotes the norm in a Banach space of generalized functions (of negative degree of differentiability), that better model oscillatory functions (such as noise or texture). This is inspired by proposals of Y. Meyer[20] and of D. Mumford - B. Gidas[21] .

Using such norms in dual spaces of distributions, for image deblurring, the work[22] computes $\tilde{f} \in BV(\Omega)$, and $f - K\tilde{f} \in \dot{H}^{-1}(\Omega)$, and this is generalized in[15] ,[16] , to the case $\tilde{f} \in BV(\Omega)$ and $f - K\tilde{f} \in \dot{H}^{-s}(\Omega)$, defined in terms of the Fourier transform. However, in these works, as in those mentioned above, the recovered image $\tilde{f}$ is still represented by a function of bounded variation, that penalizes too much oscillatory details, such as texture. It has been shown in[13] ,[2] and[3] , that natural images with finer details are not well represented by functions of bounded variation.

We propose in this paper a variational deblurring model that aims to recover the unknown image $\tilde{f}$ as the sum of two components, $u + v$, where u is a function of bounded variation, representing the cartoon component, and v is a function in a Sobolev space of negative degree of differentiability (in $\dot{W}^{-s,p}$, more general than the choice $\dot{H}^{-s}$ considered in[15] ,[16]). The space $\dot{W}^{-s,p}$ has been satisfactorily proposed and used in J. Garnett, P. Jones, T. Le and the second author[11] to model oscillatory components in natural images, in the case $K = identity$. We will make this choice to model the oscillatory component v of the recovered image, therefore the proposed deblurring model is a continuation of the work[11] . We thus recall here the main ingredient for our model, the image decomposition model $f \approx u + v$, previously proposed in[11] :

$$\Big\{ \inf_{(u,v)} \|f - (u+v)\|^2_{L^2(\Omega)} + 2\mu |u|_{BV(\Omega)} + \|v\|_{\dot{W}^{-s,p}} \Big\}.$$

A related prior work is by I. Daubechies and G. Teschke[9] , where the authors also recover an image from its blurry version by the following “cartoon + texture” minimization model

$$\Big\{ \inf_{(u,v)} \|f - K(u+v)\|^2_{L^2(\Omega)} + 2\mu |u|_{B^1_{1,1}(\Omega)} + \|v\|^2_{H^{-1}(\Omega)} \Big\},$$

in the Besov-wavelets framework. Very satisfactory results are reported in[9] , where the recovered sharp image is given by $\tilde{f} = u + v$.

Finally, we also recall the L. Rudin - S. Osher model[24] for image deblurring using the total variation (as an extension of the TV denoising model proposed by L. Rudin - S. Osher -E. Fatemi[23]): given a degradation model of the form $f = Ku + noise$, the authors[24] have proposed to recover a sharp image u, in the presence of a blurring operator K and noise, by the minimization

$$\Big\{ \inf_{u \in BV(\Omega)} \lambda \|f - Ku\|^2_{L^2(\Omega)} + |u|_{BV(\Omega)} \Big\},$$

where in practice, the total variation $|u|_{BV(\Omega)}$ is approximated by $|u|_{BV(\Omega)} \approx \int_\Omega |Du| dx$.

We will show numerical comparisons between our proposed model and the above Rudin-Osher model, that we solve using the Euler-Lagrange equation and gradient descent:

$$u(0,x) = f(x) \text{ in } \Omega, \quad \frac{\partial u}{\partial t} = 2\lambda K^*(f - Ku) + \operatorname{div}\Big(\frac{Du}{|Du|}\Big) \text{ in } (0,\infty) \times \Omega, \quad \frac{\partial u}{\partial \vec{n}} = 0 \text{ on } (0,\infty) \times \partial\Omega.$$

The outline of the paper is as follows: Section 2 is devoted to the necessary definitions and the description of the proposed deblurring model. Section 3 states theoretical results and remarks regarding the existence and the characterization of minimizers (the proofs of these results will be given in a forthcoming work). Finally, Section 4 gives the Euler-Lagrange equations associated with the optimization problem based on alternating minimization, while Section 5 presents numerical results and comparisons.

2. DESCRIPTION OF THE PROPOSED DEBLURRING MODEL

Before we introduce our proposed minimization model for deblurring, we need the necessary definitions of the function spaces that will be used.

DEFINITION 2.1. *We say that a function* $u : \Omega \to \mathbb{R}$ *is a function of bounded variation,* $u \in BV(\Omega)$*, if and only if* $u \in L^1(\Omega)$ *and*

$$\int_\Omega |Du| := \sup \Big\{ \int_\Omega u div\phi dx : \ \phi \in C_c^1(\Omega, \mathbb{R}^n), \ \|\phi\|_\infty \leq 1 \Big\} < \infty.$$

The space $W^{1,1}(\Omega)$ is a subspace of $BV(\Omega)$, and for $u \in W^{1,1}(\Omega)$, we have $\int_\Omega |Du| = \int_\Omega |\nabla u| dx$. The Banach space $BV(\Omega)$ is equipped with the following norm, which extends the classical norm in $W^{1,1}(\Omega)$:

$$\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + \int_\Omega |Du|.$$

We will also use the notation $|u|_{BV(\Omega)}$ for the semi-norm $\int_\Omega |Du|$. The space $BV(\Omega)$ will be used to model the cartoon component u of the recovered deblurred image $\tilde{f}$.

To model the texture component v, we use the Sobolev spaces $\dot{W}^{-s,p}$, $s > 0$, $p \geq 1$, that do not penalize oscillations in images. Because these spaces for $s \in \mathbb{R}$ are defined in terms of the Fourier transform, we have to assume here that the data is defined in $\mathbb{R}^n$ (obtained by extension by zero outside of the fundamental domain Ω, or by periodicity when Ω is a rectangle). Keeping this in mind, we will still write $\dot{W}^{-s,p}(\Omega)$.

DEFINITION 2.2. *The homogeneous Sobolev space* $\dot{W}^{s,p}(\Omega)$ *for* $s \in \mathbb{R}$, $1 \leq p \leq \infty$ *on a fundamental domain* Ω *is defined by*

$$\dot{W}^{s,p}(\Omega) = \Big\{ v : |\nabla|^s v \in L^p \Big\},$$

where $|\nabla|^s v(x) := ((2\pi|\xi|)^s \widehat{v}(\xi))^\vee(x)$*, with the norm on the quotient (homogeneous) space*

$$\|v\|_{\dot{W}^{s,p}(\Omega)} = \|((2\pi|\xi|)^s \widehat{v}(\xi))^\vee\|_{L^p}.$$

Notice that $|\nabla|^s v$ is defined by using the Fourier and the inverse Fourier transforms, and there is a corresponding kernel to the operator $|\nabla|^s$, which we will denote by k_s, i.e.,

$$|\nabla|^s v = k_s * v.$$

As mentioned in the introduction, in the work by J. Garnett, P. Jones, T. Le, and the second author of the present paper[11] , the authors proposed an image decomposition model

$$f \approx u + v,$$

where u is the cartoon part and $v = \Delta g$ for some $g \in \dot{W}^{-\alpha+2,p}$ is the texture or noise part. The homogeneous Sobolev spaces of functions with negative degree of differentiability turned out to be a good space to model texture. Notice that $\Delta(\dot{W}^{-\alpha+2,p}) = \dot{W}^{-\alpha,p}$. Inspired by this model, we will consider the following model for image deblurring:

$$f = K(u+v) + r = k * (u+v) + r,$$

where $v = \Delta g$ for some $g \in \dot{W}^{-\alpha+2,p}$ and r is a small residual. $u + v = u + \Delta g$ will be our recovered deblurred image, and this can be done by minimizing the following functional

$$\mathcal{F}(u,g) = |u|_{BV(\Omega)} + \mu \int_\Omega |f - k * (u + \Delta g)|^2 dx + \lambda \|g\|_{\dot{W}^{s,p}(\Omega)}, \tag{3}$$

where k is a standard convolution kernel such as the Gaussian kernel or the average kernel, that models the blurring operator.

3. MINIMIZERS OF THE FUNCTIONAL $\mathcal{F}$

In this section we only mention some theoretical results for the proposed model, but without giving all the details of the proofs (these will be included in a forthcoming paper).

3.1 Existence of a minimizer for $1 < p \leq \infty$

The problem is to minimize the following functional and we want to see if there is a minimizer of this problem when $1 < p < \infty$,

$$\begin{aligned}\mathcal{F}(u,g) &= |u|_{BV(\Omega)} + \mu \int_\Omega |f - k*u - k*\Delta g|^2 dx + \lambda \|\Delta g\|_{\dot{W}^{\alpha,p}(\Omega)} \\ &= |u|_{BV(\Omega)} + \mu \int_\Omega |f - k*u - k*\Delta g|^2 dx + \lambda \|g\|_{\dot{W}^{s,p}(\Omega)}.\end{aligned}$$

Here and in what follows, $-2 \leq \alpha < 0$, $s = \alpha + 2$ and $\Omega = [0, M] \times [0, N] \subset \mathbb{R}^2$ will be the fundamental domain of the periodic domain such as T^2 (extension to higher dimensions can be treated in the same way).

Suppose first that $1 < p < \infty$ and the kernel k is in $\dot{W}^{2-s,q}(\Omega)$, where p and q are conjugate exponents. Then we claim that $k * \Delta g$ is continuous. To see this, notice that

$$k * \Delta g = |\nabla|^{2-s} k * |\nabla|^s g.$$

Since $|\nabla|^{2-s} k \in L^q$ and $|\nabla|^s g \in L^p$, then $k * \triangle g$ is continuous. So $k * \Delta g \in L^2$. We can even impose a weaker condition on k and prove the following theorem.

THEOREM 3.1. *Let $\mu, \lambda > 0$, $-2 \leq \alpha < 0$, $s = 2 + \alpha$, $1 < p < \infty$, $k \in L^1(\Omega)$ with $\int_\Omega k(x)dx = 1$, and $f \in L^2(\Omega)$. The minimization problem*

$$\inf_{u \in BV(\Omega), g \in \dot{W}^{s,p}(\Omega)} \mathcal{F}(u,g) = |u|_{BV(\Omega)} + \mu \int_\Omega |f - k*u - k*\Delta g|^2 dx + \lambda \|g\|_{\dot{W}^{s,p}(\Omega)}$$

has a solution.

REMARK 1. *The property $\int_\Omega k(x)dx = 1$ (which is a standard normalization of the blurring kernel) is necessary to show that for a minimizing sequence (u_n, g_n), the means $\int_\Omega u_n dx$ are bounded, which enabled us to find a $BV - *$ limit u_0 using the Poincaré-Wirtinger inequality. So if $\int_\Omega k(x)dx = \beta > 0$, then we can change k to $\frac{1}{\beta}k$, f to $\frac{1}{\beta}f$ and μ to $\beta\mu$, and we can apply the above theorem.*

REMARK 2. *When $p = \infty$, the theorem also remains true since the weak-$*$ convergence in $L^\infty(\Omega)$ guarantees that we can still find the weak-$*$ limit and in the end we can pass to the limit to obtain a minimizer.*

3.2 Characterization of minimizers

By the previous Remark 1, here and in what follows we will assume that $|\Omega| = 1$, and $\int_\Omega k(x)dx = 1$.

DEFINITION 3.2. *We define the set of minimizers $\mathcal{M}$ by*

$$\mathcal{M} = \Big\{(u,g) \in BV(\Omega) \times \dot{W}^{s,p}(\Omega) : \mathcal{F}(u,g) = \inf_{(v,h) \in BV(\Omega) \times \dot{W}^{s,p}(\Omega)} \mathcal{F}(v,h)\Big\},$$

and also a subset $\mathcal{M}'$ by

$$\mathcal{M}' = \Big\{(u,g) \in \mathcal{M} : |u|_{BV(\Omega)} \neq 0 \quad or \quad \|g\|_{\dot{W}^{s,p}(\Omega)} \neq 0\Big\}.$$

We will see later that the assertion "either $\mathcal{M} = \mathcal{M}'$ or $\mathcal{M} = \mathcal{M}' \cup \{((f)_\Omega, 0)\}$", where $(f)_\Omega = \int_\Omega f(x)dx$, is true. Since the functional $\mathcal{F}$ is convex, for $(u_1, g_1), (u_2, g_2) \in \mathcal{M}'$ and $0 < t < 1$,

$$\mathcal{F}(tu_1 + (1-t)u_2, tg_1 + (1-t)g_2) \leq t\mathcal{F}(u_1, g_1) + (1-t)\mathcal{F}(u_2, g_2).$$

As a matter of fact, we have for $0 < t < 1$,

$$\mathcal{F}(tu_1 + (1-t)u_2, tg_1 + (1-t)g_2) = t\mathcal{F}(u_1, g_1) + (1-t)\mathcal{F}(u_2, g_2). \tag{4}$$

THEOREM 3.3. *Let $1 < p < \infty$. For $(u_1, g_1), (u_2, g_2) \in \mathcal{M}'$, there exists $m > 0$ such that*

$$k * (u_1 + \Delta g_1) = k * (u_2 + \Delta g_2), \tag{5}$$
$$\Delta g_1 = m\Delta g_2. \tag{6}$$

Proof. By (4), we know that for $0 < t < 1$,

$$|f - (tk * (u_1 + \Delta g_1) + (1-t)k * (u_2 + \Delta g_2))|^2 = t|f - k * (u_1 + \Delta g_1)|^2 + (1-t)|f - k * (u_2 + \Delta g_2)|^2.$$

Since the mapping $x \mapsto x^2$ is strictly convex, this implies that

$$f - k * (u_1 + \Delta g_1) = f - k * (u_2 + \Delta g_2) \quad a.e.,$$

and therefore we obtain (5). Also by the fact that the Minkowski inequality becomes equality if and only if the functions are linearly dependent, we know that there exists $m > 0$ such that

$$|\nabla|^s g_1 = m|\nabla|^s g_2 \quad a.e.,$$

which implies (6). $\square$

The following definition helps us further investigate the minimizers.

DEFINITION 3.4. *Given a function $w \in L^2(\Omega)$ and $\lambda > 0$, we define*

$$\|w\|_{*,\lambda} = \sup_{u \in BV(\Omega),\ |u|_{BV(\Omega)} \neq 0,\ g \in \dot{W}^{s,p}(\Omega),\ \|g\|_{W^{s,p}(\Omega)} \neq 0} \frac{\langle w,\ k * (u + \Delta g)\rangle}{|u|_{BV(\Omega)} + \lambda \|g\|_{\dot{W}^{s,p}(\Omega)}}, \tag{7}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\Omega)$.

REMARK 3. *Notice that if $\int_\Omega w \neq 0$, then $\|w\|_{*,\lambda} = \infty$.*

THEOREM 3.5. *Let $f \in L^2(\Omega)$ and $1 < p < \infty$. Also let $(f)_\Omega = \int_\Omega f$. Then each $(u_0, g_0) \in \mathcal{M}'$ satisfies*

$$|u_0|_{BV(\Omega)} = 2\mu \langle f - k * (u_0 + \Delta g_0), k * u_0 \rangle, \tag{8}$$
$$\|g_0\|_{\dot{W}^{s,p}(\Omega)} = \tfrac{2\mu}{\lambda} \langle f - k * (u_0 + \Delta g_0), k * \Delta g_0 \rangle. \tag{9}$$

Furthermore,

1. *$\|f - (f)_\Omega\|_{*,\lambda} \leq \frac{1}{2\mu}$ if and only if $((f)_\Omega, 0) \in \mathcal{M}$.*
2. *If $\|f - (f)_\Omega\|_{*,\lambda} > \frac{1}{2\mu}$, then $(u_0, g_0) \in \mathcal{M}'$ if and only if it satisfies the following additional condition together with (8), (9) :*

$$\left\| f - k * (u_0 + \Delta g_0) \right\|_{*,\lambda} = \frac{1}{2\mu}. \tag{10}$$

REMARK 4. *This theorem says that if $\|f - (f)_\Omega\|_{*,\lambda} \leq 1/(2\mu)$, then $\mathcal{M} = \mathcal{M}' \cup \{((f)_\Omega, 0)\}$ and if $\|f - (f)_\Omega\|_{*,\lambda} > 1/(2\mu)$, then $\mathcal{M} = \mathcal{M}'$.*

4. THE NUMERICAL MINIMIZATION ALGORITHM

For the sake of computation, we think of a given image f as part of a periodic function defined in $\mathbb{R}^2$ whose periodic domain is 2Ω, where $\Omega = [0,1] \times [0,1]$. For the practical implementation, we assume that we work with functions $u \in W^{1,1}(\Omega)$, thus $|u|_{BV(\Omega)} = \int_\Omega |\nabla u| dx$. Moreover, this restriction is not too strong, since any $BV(\Omega)$ function can be approximated by a sequence of functions in $W^{1,1}(\Omega)$, in the strong topology $L^1(\Omega)$. We will formally compute the Euler-Lagrange equations associated with the optimization problem, using alternating minimization.

If (u_0, g_0) is a minimizer of the functional $\mathcal{F}(u,g)$, then it satisfies that for any $v \in W^{1,1}(\Omega) \subset BV(\Omega)$ and for any $w \in \dot{W}^{s,p}(\Omega)$:

$$\begin{aligned} & \int_\Omega -v \cdot div\Big(\tfrac{Du_0}{|Du_0|}\Big)dx + 2\mu \int_\Omega (k * v)\cdot(k*(u_0+\Delta g_0) - f)dx \\ + \quad & 2\mu \int_\Omega (k*\Delta w)\cdot(k*(u_0+\Delta g_0)-f)dx + \lambda \int_\Omega \|k_s * g_0\|_p^{1-p}(|k_s*g_0|^{p-2}k_s*g_0)\cdot k_s * w dx = 0. \end{aligned}$$

We solve this by using a gradient descent method and a finite difference scheme, i.e., we solve the following time-dependent system of PDE's:

$$\frac{\partial u}{\partial t} = div\Big(\frac{Du}{|Du|}\Big) + 2\mu k^* * (f - k*(u+\Delta g)) \tag{11}$$

$$\frac{\partial g}{\partial t} = 2\mu \Delta k^* * (f - k*(u+\Delta g)) - \lambda \|k_s * g\|_p^{1-p} k_s * (|k_s * g|^{p-2} k_s * g), \tag{12}$$

where k^* is the transpose of k. Since the full periodic domain is 2Ω, when we compute the Sobolev norm we should use the full domain 2Ω. Notice that

$$\|g\|^p_{\dot{W}^{s,p}(\Omega)} = \frac{1}{4}\|\tilde{g}\|^p_{\dot{W}^{s,p}(2\Omega)}$$

where $\tilde{g}$ is the periodic function whose periodic domain is 2Ω and $\tilde{g}\Big|_\Omega = g$. So when we compute the second term in (12), we use $\tilde{g}$ instead of g and obtain the values on Ω. Also, the Sobolev norm will be computed using the Fast Fourier Transform (FFT) since the space itself is defined in terms of the Fourier and the inverse Fourier transformations.

5. NUMERICAL RESULTS AND COMPARISONS

Figure 1 shows the blurry data images f_1, f_2 and f_3 to be tested, and the original versions (the clean images have been artificially blurred by convolution with a blurring kernel k).

To obtain blurry image f_1, a 7×7 averaging kernel k was used, and for the blurry images f_2 and f_3, a 5×5 averaging kernel k was used.

To recover a clean image from the blurry images f_1, f_2 and f_3, the Sobolev space $\dot{W}^{0.1,1.3}$ is used, which means that the texture part $v = \Delta g$ belongs to $\dot{W}^{-1.9,1.3}$. The tuning parameters μ and λ for the three images are set to be $\mu = 0.05$, $\lambda = 10$. Using the original clean images, we compute the SNR (Signal-to-Noise-Ratio), and we compare with the Rudin-Osher model[24] . The parameters for the Rudin-Osher model are: $\triangle t = 0.01$, $\lambda = 100$, and the number of iterations is 2000, 3000 and 1500 for f_1, f_2 and f_3 respectively (the SNR increases also for the Rudin-Osher model, with the number of iterations). The following table summarizes the results and comparisons, by showing that the proposed model recovers better textured images from their blurry versions. The experimental results are shown in Figures 2 and 3.

We show in Figure 4 plots of the numerical energy decrease versus iterations for the three experiments with the proposed model, to illustrate that our numerical implementation is stable in practice.

Figure 1. Original images (top), and their blurry versions (bottom; left, data $f = f_1$; middle, data $f = f_2$; right, data $f = f_3$).

Image	SNR Blurry	SNR (TV/Sobolev)	SNR (R-O)
f_1	8.9183	22.6316	15.5319
f_2	7.8599	22.7535	14.7255
f_3	10.2671	28.1160	21.0735

Table 1. The SNR (signal-to-noise ratio) before and after reconstruction for the three image data, by the proposed model and the standard Rudin-Osher model[24] .

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Figure 2. Left - cartoon part u. Middle - texture part $v = \triangle g$. Right - recovered image $u + v$, using the proposed BV/Sobolev decomposition.

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Figure 3. Recovered images: Left - Original. Middle - using TV/Sobolev model. Right - using the Rudin-Osher model[24] .

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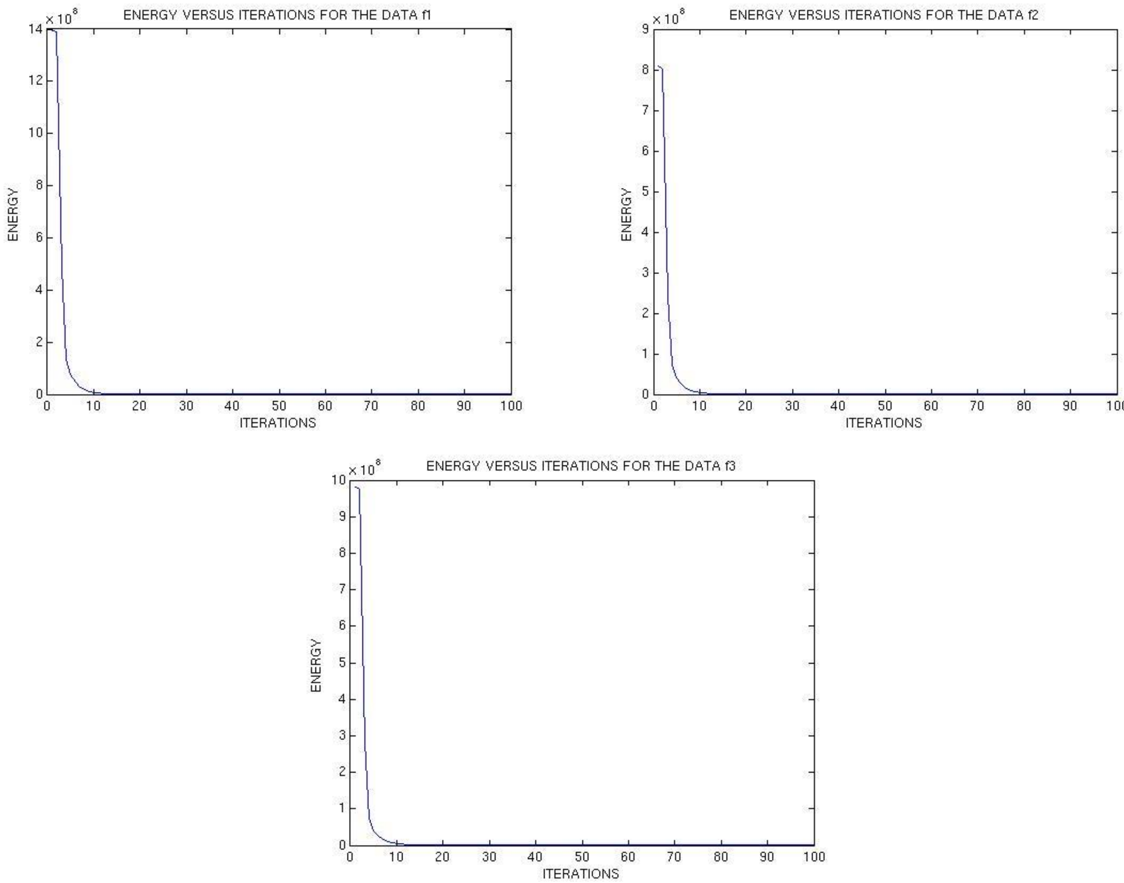

Figure 4. Numerical energy versus iterations for each of the three data f_1 (top left), f_2 (top right) and f_3 (bottom) respectively, illustrating that the numerical algorithm is stable (here only 100 iterations are shown).

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