IMAGE DECOMPOSITION, IMAGE RESTORATION, AND TEXTURE MODELING USING TOTAL VARIATION MINIMIZATION AND THE H^{-1} NORM

Stanley Osher
Maths. Department
UCLA, Los Angeles
sjo@math.ucla.edu

Andrés Solé
Departament de Tecnologia
Universitat Pompeu Fabra
Barcelona, Spain

Luminita Vese
Maths. Department
UCLA, Los Angeles
lvese@math.ucla.edu

This paper is in the memory of our special friend and co-author Andrés Solé, whose life ended too early. He will remain in our memory and thoughts.

ABSTRACT

We propose a new model for image restoration and decomposition, based on the total variation minimization of Rudin-Osher-Fatemi [8], and on some new techniques by Y. Meyer [5] for oscillatory functions. An initial image f is decomposed into a cartoon part u and a texture or noise part v. The u component is modeled by a function of bounded variation, while the v component by an oscillatory function, with bounded H^{-1} norm. After some transformation, the resulting PDE is of fourth order. The proposed model continues the ideas and techniques previously introduced by the authors in [9]. Image decomposition and denoising numerical results will be shown by the proposed new fourth order nonlinear partial differential equation.

1. INTRODUCTION AND MOTIVATIONS

An important task in image processing is the restoration or reconstruction of a true image u, from an observation f. Given an image function $f \in L^2(\Omega)$, with $\Omega \subset \mathbb{R}^2$ an open and bounded domain, the problem is to extract u from f. The observation f is usually a noisy and/or blurred version of the true image. In order to solve this inverse problem in the denoising case, one of the most well known techniques is by energy minimization and regularization. To this end, L. Rudin, S. Osher and E. Fatemi [8], [7] have proposed the following minimization problem:

$$\inf_u F(u) = \int_\Omega |\nabla u| + \lambda \int_\Omega |f-u|^2 dxdy. \qquad (1)$$

Work supported in part by grants NSF DMS-0074735, ONR N00014-97-1-0027, NIH P20MH65166, NSF ITR-0113439, ONR N00014-02-1-0015, PNPGC BFM2000-0962-C02-01.

Here, $\lambda > 0$ is a weight parameter, $\int_\Omega |f-u|^2 dxdy$ is a fidelity term, and $\int_\Omega |\nabla u|$ is a regularizing term, to remove the noise. The term $\int_\Omega |\nabla u|$ is the total variation of u. If $u \in L^1(\Omega)$ and $\int_\Omega |\nabla u| < \infty$, then $u \in BV(\Omega)$, the space of functions of bounded variation (the gradient is taken in the sense of measures). This space allows for discontinuities along curves, therefore edges and contours generally appear in the image u, which is the minimizer of this convex optimization problem.

Formally minimizing the functional (1), yields the associated Euler-Lagrange equation:

$$u = f + \frac{1}{2\lambda}\mathrm{div}\Big(\frac{\nabla u}{|\nabla u|}\Big) \text{ in } \Omega, \quad \frac{\partial u}{\partial \vec{n}} = 0 \text{ on } \partial\Omega.$$

This model performs very well for denoising of images, while preserving edges. However, smaller details, such as texture, are destroyed if the parameter λ is too small. To overcome this, Y. Meyer [5] proposed new minimization problems, changing in (1) the L^2 norm of $(f-u)$ by other weaker norms, more appropriate to represent textured or oscillatory patterns. One of the texture spaces proposed is defined as follows [5]:

Definition. Let G be the Banach space consisting of all generalized functions $f(x,y)$ which can be written as

$$f(x,y) = \partial_x g_1(x,y) + \partial_y g_2(x,y), \; g_1, g_2 \in L^\infty(\Omega), \; (2)$$

induced by the norm $\|f\|_*$ defined as the lower bound of all $L^\infty(\Omega)$ norms of the functions $|\vec{g}|$, where $\vec{g} = (g_1, g_2)$, $|\vec{g}(x,y)| = \sqrt{g_1(x,y)^2 + g_2(x,y)^2}$, and the infimum is computed over all decompositions (2) of f.

The space G coincides with the space $W^{-1,\infty}(\Omega)$, the dual space to $W^{1,1}(\Omega)$ (we recall that $W^{1,1}(\Omega)$ is the set of functions $f \in L^1(\Omega)$, with $\nabla f \in L^1(\Omega)^2$).

Y. Meyer shows that, if the component $v := f - u$ represents texture or noise, then it is better modeled by the space G instead of the space $L^2(\Omega)$, and proposes the following image restoration model [5]:

$$\inf_u \Big\{E(u) = \int_\Omega |\nabla u| + \lambda \|f-u\|_*\Big\}. \qquad (3)$$

Note that this convex minimization model cannot be directly solved in practice, due to the form of the $*$-norm of $(f-u)$. We cannot express directly the associated Euler-Lagrange equation with respect to u.

In a recent work [9], the authors have proposed a first method to overcome this difficulty. They have proposed the following convex minimization problem, as an approximation of (3):

$$\inf_{u,g_1,g_2} \Big\{ G_p(u,g_1,g_2) = \int_\Omega |\nabla u| \tag{4}$$

$$+\lambda \int_\Omega |f-(u+\mathrm{div}\vec{g})|^2 dxdy \tag{5}$$

$$+\mu \Big[\int_\Omega \Big(\sqrt{g_1^2+g_2^2} \Big)^p dxdy \Big]^{\frac{1}{p}} \Big\}, \tag{6}$$

where $\lambda, \mu > 0$ are tuning parameters, and $p \to \infty$. As $\lambda \to \infty$ and $p \to \infty$, the first term insures that $u \in BV(\Omega)$, and the second and third terms insure that $\mathrm{div}\vec{g} \approx (f-u) \in G$. In [9], image decomposition results and applications to texture discrimination have been proposed. For more details, we refer the reader to [9]. Note that, by this model, we have $(f-u) \approx \mathrm{div}\vec{g} \in W^{-1,p}(\Omega)$, the dual space to the Sobolev space $W^{1,p'}(\Omega)$, with $\frac{1}{p}+\frac{1}{p'}=1$. The case $p=2$ corresponds to the space $H^{-1}(\Omega)$.

Here, we propose a different and simplified practical algorithm for (3). Moreover, this new algorithm is a decomposition of the form $f=u+v$, while the original method [9] (which started this line of research) led to an $f=u+v+w$ model, with w a residual made as small as possible by increasing λ. The new algorithm is simplified; the minimization is performed only with respect to one unknown, u. Also, as we will see, the new model corresponds to the case $v := f-u \in H^{-1}(\Omega)$, but in the sense $v := f-u = \triangle P$, for a unique $P \in H^1(\Omega)$, with $\int_\Omega P(x,y)dxdy = 0$, and $\frac{\partial P}{\partial n} = 0$ on $\partial\Omega$. We will use this definition for $P = \triangle^{-1}v$ in the following section.

Other related work is [2], [3], [4], [10], [11], [1]. We refer to Mumford-Gidas [6] for a stochastic model, in a related approach, for natural images.

2. DESCRIPTION OF THE PROPOSED MODEL

Assume that $f-u = \mathrm{div}\vec{g}$, with $\vec{g} \in L^\infty(\Omega)^2$. We can then assume the existence of a unique Hodge decomposition of $\vec{g}$ as: $\vec{g} = \nabla P + \vec{Q}$, where P is a single-valued function and $\vec{Q}$ is a divergence-free vector field. From here, we obtain that $f-u = \mathrm{div}\vec{g} = \triangle P$. Now, we express $P = \triangle^{-1}(f-u)$, and we propose the following new convex minimization problem, a simplified and modified version of (3):

$$\inf_u E(u) = \int_\Omega |\nabla u| + \lambda \int_\Omega |\nabla(\triangle^{-1})(f-u)|^2 dxdy. \tag{7}$$

This new problem is obtained from (3) by neglecting $\vec{Q}$ from the expression for $\vec{g}$, and by considering the $(L^2-norm)^2$ instead of the $L^\infty - norm$ for $|\vec{g}|$. The problem can therefore be written using the norm in $H^{-1}(\Omega)$, as defined by $|v|^2_{H^{-1}} = \int_\Omega |\nabla(\triangle^{-1}v)|^2 dxdy$:

$$\inf_u E(u) = \int_\Omega |\nabla u| + \lambda |f-u|^2_{H^{-1}}. \tag{8}$$

Formally minimizing (7), we obtain the Euler-Lagrange equation:

$$2\lambda\triangle^{-1}(f-u) = \mathrm{div}\Big(\frac{\nabla u}{|\nabla u|}\Big). \tag{9}$$

Instead of directly solving (9) with a non-local term, we apply the Laplacian to (9), to obtain:

$$2\lambda(u-f) = -\triangle\Big[\mathrm{div}\Big(\frac{\nabla u}{|\nabla u|}\Big)\Big], \tag{10}$$

which we shall solve by driving to steady state

$$u_t = -\frac{1}{2\lambda}\triangle\Big[\mathrm{div}\Big(\frac{\nabla u}{|\nabla u|}\Big)\Big] - (u-f), \;\; u(0,x,y) = f(x,y).$$

The associated boundary conditions on $\partial\Omega$ are $\frac{\partial u}{\partial n} = \frac{\partial K}{\partial n} = 0$, where K is the curvature of level lines of u, $K(x,y) = \mathrm{div}\Big(\frac{\nabla u}{|\nabla u|}\Big)$.

Remark. We can justify in a simple general framework, that we still decrease the initial energy. Assume that we solve:

$$\inf_u \int_\Omega F(u)dxdy.$$

Embedding the minimization in a dynamic scheme based on gradient descent, we obtain: $u_t = -F_u$. We then replace this last equation by: $u_t = \triangle F_u$. We show that the initial energy is still decreasing under the new flow. Indeed, we have:

$$\begin{aligned} \frac{d}{dt}\int_\Omega F(u)dxdy &= \int_\Omega F_u u_t dxdy = \int_\Omega F_u \triangle F_u dxdy \\ &= \int_\Omega \mathrm{div}(F_u \nabla F_u)dxdy - \int_\Omega |\nabla F_u|^2 dxdy \\ &= \int_{\partial\Omega} F_u \frac{\partial}{\partial\vec{n}}(F_u)dS - \int_\Omega |\nabla F_u|^2 dxdy. \end{aligned}$$

This is a descent direction ($\frac{d}{dt}\int_\Omega F(u)dxdy < 0$) if: (a) $F_u = 0$ or $\frac{\partial}{\partial\vec{n}}(F_u) = 0$ on $\partial\Omega$. and (b) ∇F_u is not identically zero if F_u is not. These conditions are true in our framework.

We believe that it is remarkable that a TV minimization model leads to a fourth order Euler-Lagrange partial differential equation. Moreover, edges are kept in the u component, as we shall see in the numerical examples for image decomposition and denoising, presented in the next section.

3. NUMERICAL RESULTS FOR IMAGE DECOMPOSITION, IMAGE RESTORATION, AND TEXTURE MODELING

Figure 1 top corresponds to a real image where there is a high presence of textures combined with non textured parts. The u and v components are displayed in Figure 1 middle and bottom. The new model separates very well the textured details shown in v, from non-textured regions, kept in u. Also, details like the eyes, are very well represented in the u component of the new model. In other words, the model performs very well in keeping the main contours in the u component, instead of the v component (it performs extremely well in separating the main larger features from the textured features). Figure 2 shows the performance of the new model for the denoising problem. We show a zoom of the woman image, before and after corrupting it with white Gaussian noise of standard deviation 10, and the denoised result, which is remarkable. Finally, we end the paper with a decomposition result on an image with an object with fractal boundary (corresponding to the "Sierpinsky pentagon"), using the model. The result of the decomposition is remarkable, as shown in Figure 3: the cartoon part is well represented in the component u, while the oscillatory fractal-like boundaries are kept in the v component.

4. REFERENCES

[1] S. Casadei, S. Mitter, P. Perona, "Boundary detection in piecewise homogeneous textured images", LNCS, Vol. 588, 174-183, 1992.

[2] F. Malgouyres, "Mathematical analysis of a model which combines total variation and wavelet for image restoration", Journal of information processes, 2(1): 1-10, 2002.

[3] F. Malgouyres, "Combining total variation and wavelet packet approaches for image deblurring", Proceedings, IEEE Workshop on Variational and Level Set Methods in Computer Vision, 57-64, 2001.

[4] F. Malgouyres, "Minimizing the total variation under a general convex constraint for image restoration", to appear in IEEE T IP.

[5] Y. Meyer, "Oscillating Patterns in Image Processing and Nonlinear Evolution Equations", University Lecture Series, 22, AMS 2002.

[6] D. Mumford, B. Gidas, "Stochastic models for generic images", Quarterly of Applied Mathematics, 59(1): 85-111, 2001.

Fig. 1. Top: original image f. Decomposition into u (middle) and v (bottom) using our new method.

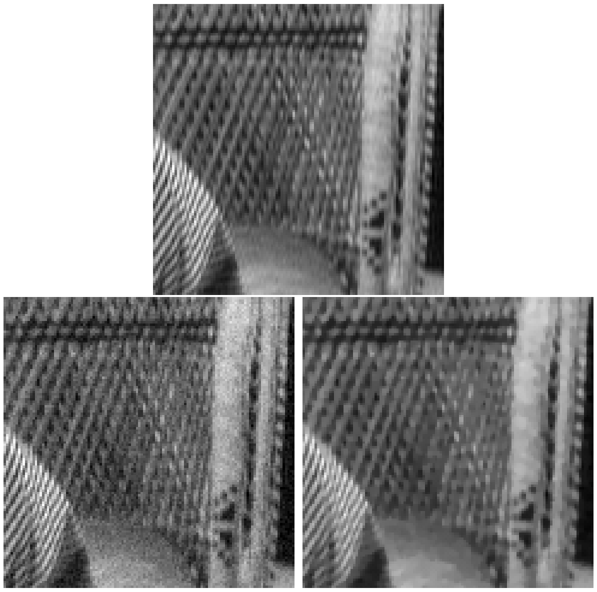

Fig. 2. Denoising of a textured image: original (top), noisy (bottom left), denoised result u (bottom right).

[7] L. Rudin, S. Osher, "Total Variation based image restoration with free local constraints", Proceedings of the IEEE ICIP 1994, I, 31-35.

[8] L. Rudin, S. Osher, E. Fatemi, "Nonlinear total variation based noise removal algorithms", Physica D, 60, 259-268, 1992.

[9] L. Vese, S. Osher, "Modeling Textures with Total Variation Minimization and Oscillating Patterns in Image Processing", UCLA CAM Report 02-19, May 2002, to appear, JSC.

[10] S.C. Zhu, Y.N. Wu, D. Mumford, "Filters, random fields and maximum entropy (FRAME): Towards a unified theory for texture modeling", IJCV 27(2): 107-126, 1998.

[11] S.C. Zhu, Y.N. Wu, D. Mumford, "Minimax entropy principle and its application to texture modeling", Neural Computation, 9(8): 1627-1660, 1997.

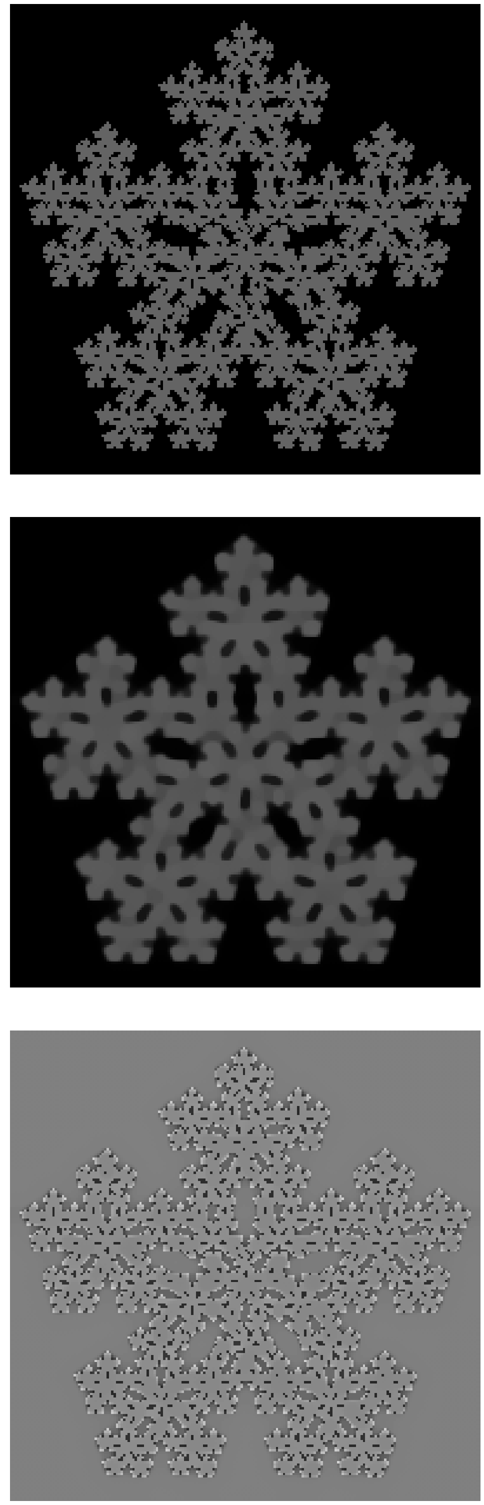

Fig. 3. Decomposition of an image with an object with fractal boundary (the "Sierpinsky pentagon"), using the new model. Top: f; middle: u; bottom: v.