Appl Math Optim 44:131–161 (2001)
DOI: 10.1007/s00245-001-0017-7

Applied Mathematics & Optimization

A Study in the BV Space of a Denoising–Deblurring Variational Problem*

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Communicated by R. Temam

Abstract. In this paper we study, in the framework of functions of bounded variation, a general variational problem arising in image recovery, introduced in [3]. We prove the existence and the uniqueness of a solution using lower semicontinuity results for convex functionals of measures. We also give a new and fine characterization of the subdifferential of the functional, together with optimality conditions on the solution, using duality techniques of Temam for the theory of time-dependent minimal surfaces. We study the associated evolution equation in the context of nonlinear semigroup theory and we give an approximation result in continuous variables, using Γ-convergence. Finally, we discretize the problems by finite differences schemes and we present several numerical results for signal and image reconstruction.

Key Words. Variational methods, Elliptic/parabolic PDEs, Functions of bounded variation, Convex functions of measures, Duality, Relaxation, Maximal monotone operators, Γ-Convergence, Finite differences scheme, Signal and image processing.

AMS Classification. 35, 49, 65.

* This work was done while the author was at the Laboratoire Jean-Alexandre Dieudonné, from the University of Nice - Sophia Antipolis, France.

1. Introduction

In this paper we study, in the space of functions of bounded variation, a variational model of image reconstruction introduced in [3], which now becomes more and more classical in the context of image analysis.

The general problem is to reconstruct a piecewise-smooth original image u from an observed and degraded initial image u_0.

Let u_0, u be two real functions defined on a bounded and open subset Ω of $\mathbb{R}^N$ (generally, Ω is a rectangle in $\mathbb{R}^2$). We assume here that u_0 is the result of a transformation or degradation, applied to the original image u, of the form

$$u_0 = Ku + \eta,$$

where K is a linear operator (for instance, the blur) and η is a random noise.

The problem is to find u, knowing u_0. To do this, we assume some knowledges on K (and/or on η) and we add some a priori constraints on the solution.

The model presented in [3] for image reconstruction allows us to search the image-function u among the minimizers of the following functional:

$$F_\alpha(u) = \int_\Omega (Ku - u_0)^2\,dx + \alpha \int_\Omega \varphi(|Du|)\,dx. \tag{1}$$

Here, $\alpha \geq 0$ is a weight parameter and $\varphi\colon \mathbb{R} \to \mathbb{R}^+$ is an even function. The a priori constraint on the solution is represented by the regularizing term $\varphi(|Du|)$.

The Euler–Lagrange equation associated to the minimization problem can be formally written as

$$2K^*Ku - \alpha\,\mathrm{div}\left(\frac{\varphi'(|Du|)}{|Du|}Du\right) = 2K^*u_0, \tag{2}$$

where K^* denotes the adjoint operator of K. If $\alpha = 0$, the equation becomes

$$2K^*Ku = 2K^*u_0.$$

Unfortunately, this is an ill-posed problem, because K^*K is not always invertible and the problem is often unstable. Then we choose $\alpha > 0$ to regularize the problem. This is also necessary to remove the noise.

As in [28], [11], or [3], it is clear that, to denoise an image by preserving its edges, we need to work with functions φ with at most a linear growth at infinity. To ensure the existence and the uniqueness of a solution u, we need in addition to assume that φ is a convex function, nondecreasing on $\mathbb{R}^+$ (sometimes φ has to be strictly convex). Then φ will be with "linear growth" and we will search the solution u in the space $BV(\Omega)$ of functions of bounded variation, well adapted to model images.

In order to diffuse the image in regions where variations of gray levels are weak (where $|Du| \ll \varepsilon$, with $\varepsilon > 0$ a threshold parameter) and to preserve the contours of these regions (where $|Du| \gg \varepsilon$), we have many possible choices for φ in this class of functions, for instance,

$$\varphi_1(z) = \begin{cases} \dfrac{1}{2\varepsilon}z^2 & \text{if} \quad z \leq \varepsilon, \\ z - \dfrac{\varepsilon}{2} & \text{if} \quad z > \varepsilon. \end{cases}$$

Indeed, for this function, in a neighborhood of a point $x \in \Omega$ where $|Du(x)| < \varepsilon$, (2) formally becomes

$$\frac{2}{\alpha}(K^*Ku - K^*u_0) = \frac{1}{\varepsilon} \triangle u, \tag{3}$$

which is a diffusion equation, with strong regularizing properties in all directions, which will remove the noise.

On a contour, where $|Du(x)| > \varepsilon$, (2) locally becomes

$$\frac{2}{\alpha}(K^*Ku - K^*u_0) = \operatorname{div}\left(\frac{Du}{|Du|}\right) = \frac{1}{|Du|}u_{\xi\xi},$$

where ξ is the unit orthogonal vector to Du and $u_{\xi\xi}$ denotes the second-order derivative of u in the ξ-direction. We note that $\operatorname{div}(Du(x)/|Du(x)|)$ represents the curvature of the level curve of u passing by x (the edge). In this case the diffusion will be weak, because $1/|Du|$ is small and this will be only in the ξ-direction, i.e., in the parallel direction to the contour. In this way, the edges will be preserved.

We can also use, instead of φ_1, other functions φ with the same behavior but more regular: for example, $\varphi_2(z) = \sqrt{1+z^2} - 1$ (the function of minimal surfaces) or $\varphi_3(z) = \log \cosh z$.

For more details on the choice of the function φ, we refer the reader to [3].

In the context of image analysis, Rudin and Osher [28] have introduced Total Variation minimization (for $\varphi(z) = |z|$), and Chambolle and Lions [11] and Acart and Vogel [1] have carried out the theoretical study in this particular case. In [1] the authors have also considered the function of minimal surfaces φ_2, but only to approach and regularize the total variation.

In this paper we study the general problem in the convex case, in the space of functions of bounded variation. We give in addition a characterization of the subdifferential of F. We also introduce the evolution equation associated to the minimization problem, using techniques from the theory of time-dependent minimal surfaces [17]. We show that, as the time tends to infinity, the solution of the evolution problem converges to the solution of the variational problem. We also approximate the BV solution by Sobolev functions, using the notion of Γ-convergence [14].

The outline of the paper is as follows. In Section 2 we review the basic properties of functions of bounded variation and of lower semicontinuous functionals of measures, and we give the assumptions on u_0, φ, and K. The existence and the uniqueness of the solution u of the minimization problem on the space $BV(\Omega)$ is presented in Section 3. In Section 4 we give a characterization of the subdifferential ∂F of F and therefore of the Euler–Lagrange equation associated to the minimization problem, written in $BV(\Omega)$, while in Section 5 we study the associated evolution problem, using the theory of maximal monotone operators. In Section 6 we approximate by Γ-convergence the problem in continuous variables. In Section 7, we present finite differences schemes for both the Euler–Lagrange and evolution equations, and, finally in Section 8 we show numerical results for signal and image reconstruction.

2. Notations, Assumptions, and Preliminary Results

Let Ω be an open, bounded, and connected subset of $\mathbb{R}^N$, with Lipschitz boundary Γ. We use standard notations for the Sobolev and Lebesgue spaces $W^{1,p}(\Omega)$ and $L^p(\Omega)$. For the theoretical study of the problem, we consider $\alpha = 1$ for simplicity, and the functional F_α will be denoted by F.

To ensure the existence and the uniqueness of a minimizer for (1) in $BV(\Omega)$, we make the following assumptions on φ and K:

H1. $\varphi\colon \mathbb{R} \to \mathbb{R}^+$ is an even and convex function, nondecreasing in $\mathbb{R}^+$, such that:
  (i) $\varphi(0) = 0$ (without loss of generality).
  (ii) There exist $c > 0$ and $b \geq 0$ such that $cz - b \leq \varphi(z) \leq cz + b$, $\forall z \in \mathbb{R}^+$.
H2. $K\colon L^p(\Omega) \to L^2(\Omega)$ is a linear and continuous operator, where $p = N/(N-1)$ if $N \geq 2$ and $p = 2$ if $N = 1$.
H3. $K\chi_\Omega \neq 0$.
H4. K is injective or φ is strictly convex.

Remark 2.1. Since $\varphi\colon \mathbb{R} \to \mathbb{R}^+$ is convex, then it is continuous. Moreover, its asymptote (recession) function φ^∞ exists (see, for instance, [21]) and it is finite (from H1(ii)):

$$\varphi^\infty(z) := \lim_{t\to\infty} \frac{\varphi(tz)}{t} \in [0; +\infty).$$

In fact, $c = \lim_{t\to\infty}(\varphi(t)/t)$ and $\varphi^\infty(z) = cz \cdot sign\, z$.

Remark 2.2. Thanks to H1(ii), the functional $j(u) := \int_\Omega \varphi(|Du|)\,dx$ is well-defined and finite on the space $W^{1,1}(\Omega)$. However, as is well known, $W^{1,1}(\Omega)$ is a nonreflexive Banach space and then the minimization problem (1) may not have the solution in this space. For these reasons, we work with functions of bounded variation and we use the notions of convex function of measures and relaxed functionals on measures to obtain the existence of a minimum. Moreover, the space of BV-functions is the proper class for many basic image processing tasks, because it allows discontinuities along curves or edges, while $W^{1,1}$-functions may not.

Example 2.3. For $E \subset \Omega$ with C^2 boundary, we consider the characteristic function χ_E, defined by

$$\chi_E(x) = \begin{cases} 1 & \text{if} \quad x \in \Omega, \\ 0 & \text{if} \quad x \in \Omega\backslash E. \end{cases}$$

Then $\chi_E \in BV(\Omega)$, because $TV(\chi_E) := \int_\Omega |D\chi_E| = \mathcal{H}^{N-1}(\partial E) < \infty$, but $\chi_E \notin W^{1,1}(\Omega)$, according, for instance, to Evans and Gariepy [19, Theorem 2 (characterization of Sobolev functions), Section 4.9.2]. In particular, the boundary of E, ∂E, could represent an edge in an image. We note that $\mathcal{H}^{N-1}(\partial E)$ is called the *perimeter* of E in Ω [20].

Remark 2.4. Examples of linear and continuous operators K from $L^p(\Omega)$ into $L^2(\Omega)$ include the identity operator ($K = I$) if $N = 1, 2$ and convolutions with a positive kernel. In image analysis, for $K = k * u$, the kernel k must satisfy $k(x) \geq 0$, $k(x) \to 0$ rapidly as $|x| \to \infty$, and $\int_{\mathbb{R}^N} k(x) = 1$. Generally, k is the heat kernel or a function which satisfies in addition the following properties: $k(x) = k(|x|)$, $k(|x|) = 0$ if $|x| \geq 1$ and $k \in C^\infty(\mathbb{R}^N)$ (see, for instance, [26]). In these particular cases, k belongs to $L^2(\Omega)$, and then, for $u \in L^p(\Omega)$, $Ku := k * u$ is well-defined, linear, and continuous from $L^p(\Omega)$ into $L^2(\Omega)$, even if $N > 2$. Assumption H3 means that K does not annihilate constant functions. This will guarantee the BV-coerciveness of the functional and it is always true for the convolution operator.

We now introduce the basic notations and preliminary results on the space $BV(\Omega)$, and we recall the notion of lower semicontinuity of functionals defined on this space.

We denote by $\mathcal{L}_N$ (or sometimes by dx) the Lebesgue N-dimensional measure in $\mathbb{R}^N$ and by $\mathcal{H}^\alpha$ the α-dimensional Hausdorff measure. We also set $|E| = \mathcal{L}_N(E)$, the Lebesgue measure of a measurable set $E \subset \mathbb{R}^N$. We use the notation $\mathcal{B}(\Omega)$ for the family of the Borel subsets of Ω. If $x, y \in \mathbb{R}^N$, then $x \cdot y$ will denote their scalar product.

Given a vector-valued measure $\mu\colon \mathcal{B}(\Omega) \to \mathbb{R}^M$, we use the notation $|\mu|$ for its total variation. We recall that

$$|\mu|(A) = \sup\left\{\sum_{j=1}^{M}\int_\Omega v_j\, d\mu_j\colon\ v = (v_1, \ldots, v_M) \in C_0(A; \mathbb{R}^M),\ \|v\|_\infty \leq 1\right\},$$

where $C_0(A; \mathbb{R}^M)$ denotes the closure, in the sup norm, of continuous functions with compact support in A. We denote by $\mathcal{M}(\Omega)$ the set of all signed measures on Ω with bounded total variation.

The usual *weak* $*$ topology on $\mathcal{M}(\Omega)$ is defined as the weakest topology on $\mathcal{M}(\Omega)$ for which the maps $\mu \to \int_\Omega \psi\, d\mu$ are continuous for every continuous function ψ vanishing on $\partial\Omega$.

We say that $u \in L^1(\Omega)$ is a function of bounded variation ($u \in BV(\Omega)$) if its distributional derivative $Du = (D_1 u, \ldots, D_N u)$ belongs to $\mathcal{M}(\Omega)$. For a general exposition of the theory of functions of bounded variation, we refer, for instance, to [34].

The space $BV(\Omega)$ endowed with the norm

$$\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + |Du|(\Omega)$$

is a Banach space.

The product topology of the strong topology of $L^1(\Omega)$ for u and of the $weak*$ topology of measures for Du will be called the *weak* $*$ topology of BV, and will be denoted by BV-$w*$. We recall that every bounded sequence in $BV(\Omega)$ admits a subsequence converging in BV-$w*$. This sequence is also relatively compact in $L^p(\Omega)$ for $1 \leq p < N/(N-1)$ and $N \geq 1$, and relatively weakly compact in $L^p(\Omega)$ for $p = N/(N-1)$ and $N \geq 2$ [20], [1].

We also have an extension to BV-functions of the Poincaré–Wirtinger inequality [9], [1]: for $u \in BV(\Omega)$, let

$$\bar{u} := \frac{1}{|\Omega|}\int_\Omega u(x)\, dx.$$

Then there exists $M > 0$ such that

$$\|u - \bar{u}\|_{L^p(\Omega)} \le M|Du|(\Omega),$$

for every $p < \infty$ if $N = 1$ and for $p = N/(N-1)$ if $N > 1$. Then, for $N = 1$, we can take $p = 2$. We deduce that if $u \in BV(\Omega)$, then $u \in L^p(\Omega)$ ($BV(\Omega)$ is continuously embedded in $L^p(\Omega)$).

For any function $u \in L^1(\Omega)$, we denote by S_u the complement of the Lebesgue set of u, i.e., $x \notin S_u$ if and only if there exists $\tilde{u}(x) \in \mathbb{R}$ such that

$$\lim_{\rho \to 0^+} \rho^{-N} \int_{B_\rho(x)} |u(y) - \tilde{u}(x)|\, dy = 0.$$

The limit $\tilde{u}(x)$ denotes the *approximate limit* of u at x and $\tilde{u}$ is a Borel function equal to u almost everywhere. The set S_u is of zero Lebesgue measure.

If $u \in BV(\Omega)$, then u is differentiable almost everywhere on $\Omega \backslash S_u$ and ∇u coincides with the Radon–Nikodym derivative of Du with respect to $\mathcal{L}_N$. Moreover, the Hausdorff dimension of S_u is at most $(N-1)$ and for $\mathcal{H}^{N-1}$-a.e. $x \in S_u$ it is possible to find unique $u^+(x), u^-(x) \in \mathbb{R}$, with $u^+(x) > u^-(x)$ and $\nu \in S^{n-1}$, such that

$$\lim_{\rho \to 0^+} \rho^{-N} \int_{B_\rho^\nu(x)} |u(y) - u^+(x)|\, dy = \lim_{\rho \to 0^+} \rho^{-N} \int_{B_\rho^{-\nu}(x)} |u(y) - u^-(x)|\, dy = 0,$$

where $B_\rho^\nu(x) = \{y \in B_\rho(x)\colon (y-x)\cdot\nu > 0\}$ and $B_\rho^{-\nu}(x) = \{y \in B_\rho(x)\colon (y-x)\cdot\nu < 0\}$ (we assume that the normal ν "points toward the larger value" of u; we have denoted by $B_\rho(x)$ the ball centered in x of radius ρ).

We have the Lebesgue decomposition

$$Du = \nabla u \cdot \mathcal{L}_N + D_s u,$$

where $\nabla u \in (L^1(\Omega))^N$ is the Radon–Nikodym derivative of Du and $D_s u$ is singular, with respect to $\mathcal{L}_N$. We also have the decomposition for $D_s u$:

$$D_s u = C_u + J_u,$$

where

$$J_u = (u^+ - u^-)\nu \cdot \mathcal{H}^{N-1}_{|S_u}$$

is the *Hausdorff part* or *jump part* and C_u is the *Cantor part* of Du. We recall that the measure C_u is singular with respect to $\mathcal{L}_N$ and it is "diffuse," i.e., $C_u(S) = 0$ for every set S of Hausdorff dimension $N-1$. Hence, we have, for every $B \in \mathcal{B}(\Omega)$, that

$$D_s u(B \backslash S_u) = C_u(B \backslash S_u) \quad \text{and} \quad D_s u(B \cap S_u) = J_u(B \cap S_u).$$

Finally, we can write Du and its total variation on Ω, $|Du|(\Omega)$, as

$$Du = \nabla u \cdot \mathcal{L}_N + C_u + (u^+ - u^-)\nu \cdot \mathcal{H}^{N-1}_{|S_u},$$
$$|Du|(\Omega) = \int_\Omega |\nabla u|\, dx + \int_{\Omega \backslash S_u} |Cu| + \int_{S_u} (u^+ - u^-)\, d\mathcal{H}^{N-1}$$

(we recall that $u^+ > u^-$).

It is then possible to define the convex function of measures $\varphi(|\cdot|)$ on $\mathcal{M}(\Omega)$, which is, for Du,

$$\varphi(|Du|) = \varphi(|\nabla u|)\cdot\mathcal{L}_N + \varphi^\infty(1)|D_s u|,$$

and the functional

$$J(u) = \varphi(|Du|)(\Omega) = \int_\Omega \varphi(|\nabla u|)\,dx + \varphi^\infty(1)\int_\Omega |D_s u|$$

(see [21], where it is proved that the functional $\varphi(|\cdot|)(\Omega)$ is weakly$*$ lower semicontinuous on $\mathcal{M}(\Omega)$, or [17]). It is also easy to see that $J(\cdot)$ is convex on $BV(\Omega)$ (for this, we use the fact that φ is convex and increasing on $\mathbb{R}_+$).

By the decomposition of $D_s u$, the properties of C_u, J_u, and the definition of the constant c, the functional J can be written as

$$J(u) = \int_\Omega \varphi(|\nabla u|)\,dx + c\int_{\Omega\setminus S_u} |C_u| + c\int_{S_u} (u^+ - u^-)\,d\mathcal{H}^{N-1}.$$

Now, the functional $J\colon BV(\Omega)\to[0,+\infty)$ is lower semicontinuous with respect to the BV-$w*$ topology and less than or equal to j, where j is defined by

$$j(u) = \begin{cases} \displaystyle\int_\Omega \varphi(|\nabla u|)\,dx & \text{if}\quad u\in W^{1,1}(\Omega),\\ +\infty & \text{if}\quad u\in BV(\Omega)\setminus W^{1,1}(\Omega).\end{cases}$$

We note that the functional j is not lower semicontinuous on $BV(\Omega)$ (or on $L^p(\Omega)$, $L^1(\Omega)$) with respect to BV-$w*$ (or the L^p, L^1 topologies, respectively). However, for each $u\in BV(\Omega)$, there exists (see p. 692 of [17]) a sequence $\{u_n\}_{n\ge 1}\in C^\infty(\Omega)\cap W^{1,1}(\Omega)$ such that $u_n\to u$ as $n\to\infty$ in BV-$w*$ and

$$J(u) = \varphi(|Du|)(\Omega) = \lim_{n\to\infty}\int_\Omega \varphi(|\nabla u_n|)\,dx = \lim_{n\to\infty} j(u_n).$$

In this way, we deduce that J is the relaxation of j on BV-$w*$, that is,

$$J(u) = \bar{j}(u) := \inf\left\{\liminf_{n\to\infty} j(u_n)\colon\ u_n\in BV(\Omega),\, u_n\to u \text{ in } BV{-}w^*\right\},$$

($\bar{j}$ is the greatest BV-w^* lower semicontinuous functional less than or equal to j).

For more general lower semicontinuity results for functionals defined on measures, we refer the reader to [5]–[7] and [4].

It is then natural to consider, instead of $j(u)$, $J(u)$ for the second term of $F(u)$ in (1) and we denote the new functional on $BV(\Omega)$ by $\hat{F}$ (this will be equal to F in $W^{1,1}(\Omega)$):

$$\hat{F}(u) = \int_\Omega |Ku - u_0|^2\,dx + \int_\Omega \varphi(|\nabla u|)\,dx + c\int_\Omega |D_s u|.$$

Remark 2.5. J is the lower semicontinuous envelope of j with respect to the L^p topology [5]–[7], [4], with $p = N/(N-1)$. Then, because K is linear and continuous from $L^p(\Omega)$ into $L^2(\Omega)$, if $u_0 \in L^2(\Omega)$, the functional

$$\hat{F}(u) = \int_\Omega |Ku - u_0|^2 \, dx + J(u)$$

is the lower semicontinuous envelope of F on $L^p(\Omega)$. In fact, if $u \in BV(\Omega)$, then there exists a sequence $u_n \in W^{1,1}(\Omega)$ such that $u_n \to u$ (as $n \to \infty$) in $L^p(\Omega)$, and

$$\hat{F}(u) = \lim_{n\to\infty} \hat{F}(u_n) = \lim_{n\to\infty} F(u_n).$$

3. The Minimization Problem

In this section we study the existence and the uniqueness of the solution of the minimization problem

$$\inf_u \left\{ \hat{F}(u) = \int_\Omega (Ku - u_0)^2 \, dx + \int_\Omega \varphi(|\nabla u|) \, dx + c \int_\Omega |D_s u| \right\}, \tag{4}$$

for $u \in BV(\Omega)$ (we recall that $BV(\Omega) \subset L^p(\Omega)$, with $p = 2$ if $N = 1$ and $p = N/(N-1)$ if $N \geq 2$), $Du = \nabla u \cdot \mathcal{L}_N + D_s u$, $Ku \in L^2(\Omega)$, and $u_0 \in L^2(\Omega)$.

To do this, we essentially follow Acart and Vogel [1] to show that

$$\varphi(|Du|)(\Omega) + \|Ku - u_0\|_{L^2(\Omega)}$$

is coercive in $BV(\Omega)$, and Chambolle and Lions [11] for passing to the limit in the minimizing sequences.

Proposition 3.1. *Let* $u_0 \in L^2(\Omega)$. *Under assumptions* H1–H4, *there exists a unique solution* $u \in BV(\Omega)$ *of* (4), *satisfying* $Ku \in L^2(\Omega)$.

Proof. Step 1: *Existence.* In what follows, we denote by M a strictly positive constant, which can be different from line to line.

Let $\{u_n\}_{n\geq 1}$ be a minimizing sequence for (4). Then $u_n \in BV(\Omega)$ thanks to assumption H1(ii) and we have

$$|Du_n|(\Omega) = \int_\Omega |\nabla u_n| \, dx + |D_s u_n|(\Omega) \leq M, \qquad \forall n \geq 1,$$

where $Du_n = \nabla u_n \, dx + D_s u_n$ is the Lebesgue decomposition of Du_n.

Now, we prove that $|\int_\Omega u_n| \leq M$, $\forall n \geq 1$.

Let

$$w_n = \frac{\int_\Omega u_n}{|\Omega|} \chi_\Omega \quad \text{and} \quad v_n = u_n - w_n.$$

Then $\int_\Omega v_n = 0$ and $Dv_n = Du_n$. Hence, $|Dv_n|(\Omega) \le M$. Using the Poincaré–Wirtinger inequality, we obtain that

$$\|v_n\|_{L^p(\Omega)} \le M.$$

We also have

$$\begin{aligned} M &\ge \|Ku_n - u_0\|_2^2 = \|Kv_n + Kw_n - u_0\|_2^2 \\ &\ge (\|Kv_n - u_0\|_2 - \|Kw_n\|_2)^2 \\ &\ge \|Kw_n\|_2(\|Kw_n\|_2 - 2\|Kv_n - u_0\|_2) \\ &\ge \|Kw_n\|_2[\|Kw_n\|_2 - 2(\|K\| \cdot \|v_n\|_p + \|u_0\|_2)]. \end{aligned}$$

Let $x_n = \|Kw_n\|_2$ and $a_n = \|K\| \cdot \|v_n\|_p + \|u_0\|_2$. Then

$$x_n(x_n - 2a_n) \le M, \qquad \text{with} \quad 0 \le a_n \le \|K\| \cdot M + \|u_0\|_{L^2(\Omega)} = M', \quad \forall n \ge 1.$$

Hence, we obtain

$$0 \le x_n \le a_n + \sqrt{a_n^2 + M} \le M'',$$

which implies

$$\|Kw_n\|_2 = \left|\int_\Omega u_n\, dx\right| \cdot \frac{\|K\chi_\Omega\|_2}{|\Omega|} \le M'', \qquad \forall n \ge 1,$$

and thanks to assumption H3, we obtain that $|\int_\Omega u_n\, dx|$ is uniformly bounded.

Again, by the Poincaré–Wirtinger inequality, we have

$$\left\| u_n - \frac{\int_\Omega u_n}{|\Omega|} \right\|_{L^p(\Omega)} \le C|Du_n|(\Omega) \le C \cdot M,$$

with $p = 2$ if $N = 1$ and $p = N/(N-1)$ if $N \ge 2$. Finally, we obtain

$$\|u_n\|_{L^p(\Omega)} = \left\| u_n - \frac{\int_\Omega u_n}{|\Omega|} + \frac{\int_\Omega u_n}{|\Omega|} \right\|_{L^p(\Omega)} \le \left\| u_n - \frac{\int_\Omega u_n}{|\Omega|} \right\|_{L^p(\Omega)} + \left| \int_\Omega u_n \right| \le M'''.$$

Therefore, u_n is bounded in $L^p(\Omega)$ and, in particular, in $L^1(\Omega)$. Then u_n is also bounded in $BV(\Omega)$, and there is a subsequence, still denoted u_n, and $u \in BV(\Omega)$, such that $u_n \rightharpoonup u$ weakly in $L^p(\Omega)$ and in BV-$w*$, $Du_n \rightharpoonup Du$ weakly$*$ in $\mathcal{M}(\Omega)$. Moreover, Ku_n converges weakly to Ku in $L^2(\Omega)$, from assumption H2.

Finally, we have (from the above lower semicontinuity results in BV-$w*$)

$$\int_\Omega (Ku - u_0)^2\, dx \le \liminf_{n\to\infty} \int_\Omega (Ku_n - u_0)^2\, dx$$

and

$$\int_\Omega \varphi(|Du|) \le \liminf_{n\to\infty} \int_\Omega \varphi(|Du_n|),$$

that is to say

$$\hat{F}(u) \leq \liminf_{n\to\infty} \hat{F}(u_n)$$

and u is a minimum of $\hat{F}$.

Step 2: *Uniqueness*. Let $u, v \in BV(\Omega)$ be two solutions of the minimization problem (4).

We first show that $Ku = Kv$: if, on the contrary, $Ku \neq Kv$, then

$$\hat{F}(\tfrac{1}{2}u + \tfrac{1}{2}v) < \tfrac{1}{2}\hat{F}(u) + \tfrac{1}{2}\hat{F}(v) = \inf \hat{F},$$

because $\hat{F}$ is the sum of two convex functions with independent variables, Ku and Du, the first one being strictly convex. However, this inequality cannot be true if u and v are minimizers $\hat{F}$. Then $Ku = Kv$.

If K is injective, we will have $u = v$. Otherwise, if K is not injective, but φ is strictly convex, then $Du = Dv$, which implies that $u = v + C$ and $K \cdot C = 0$. Therefore, from assumption H3, we obtain that $C = 0$, i.e., $u = v$. □

4. Characterization of Solutions

In this section we characterize the solution of the minimization problem by computing the subdifferential of $\hat{F}(u)$. We use the techniques of Temam for the problem of minimal surfaces [17] and duality results from [18].

We assume assumptions H1–H4 and that $u_0 \in L^2(\Omega)$.

We first extend $\hat{F}$ to $L^p(\Omega)$: for $u \in L^p(\Omega)\backslash BV(\Omega)$, $\hat{F}(u) = +\infty$ and for $u \in BV(\Omega)$, with $Du = \nabla u\, dx + C_u + J_u$, $\hat{F}(u)$ is given by

$$\hat{F}(u) = \int_\Omega (Ku - u_0)^2\, dx + \int_\Omega \varphi(|\nabla u|)\, dx + c\int_{\Omega\backslash S_u} |C_u| + c\int_{S_u} |J_u|.$$

The definition of the subdifferential $\partial\hat{F}$ at u is the following (see [18]): let $u \in L^p(\Omega)$ and $\xi \in L^{p'}(\Omega)$. Then

$$\xi \in \partial\hat{F}(u) \qquad \text{iff} \qquad \left\{ \begin{aligned} &\hat{F}(u) \in \mathbb{R} \text{ and} \\ &\hat{F}(u) - \int_\Omega \xi u \leq \hat{F}(v) - \int_\Omega \xi v, \forall v \in L^p(\Omega) \end{aligned} \right\}.$$

We have that $\hat{F}(u) = \inf_{v\in L^p(\Omega)} \hat{F}(v)$ if and only if $0 \in \partial\hat{F}(u)$ and for this reason it is natural to provide a characterization of $\partial\hat{F}$.

The subdifferential of $\hat{F}$: Let $u \in BV(\Omega) \subset L^p(\Omega)$ and $\xi \in L^{p'}(\Omega)$. We say that $\xi \in \partial\hat{F}(u)$ if u achieves the minimum on $BV(\Omega)$ of the following variational problem:

$$(\mathcal{P}_1) \qquad \inf_{v\in BV(\Omega)} \left\{ \hat{F}(v) - \int_\Omega \xi v\, dx \right\}.$$

By Remark 2.5, we can replace in $(\mathcal{P}_1)$ the infimum on $BV(\Omega)$ by the infimum on $W^{1,1}(\Omega)$ ($W^{1,1}(\Omega) \subset BV(\Omega) \subset L^p(\Omega)$). Using the following property: if $v \in W^{1,1}(\Omega)$, then $D_s v = 0$, the problem becomes

$$(\mathcal{P}_2) \qquad \inf_{v \in W^{1,1}(\Omega)} \left\{ \int_\Omega (Kv - u_0)^2\, dx + \int_\Omega \varphi(|\nabla v|)\, dx - \int_\Omega \xi v\, dx \right\}.$$

Now, problems $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ have the same infimum, which belongs to $\mathbb{R}$, because we have assumed that $\xi \in \partial \hat{F}(u)$, that is, u is a solution of $(\mathcal{P}_1)$.

We now write $(\mathcal{P}_2^*)$, the dual of $(\mathcal{P}_2)$, in the sense of Ekeland and Temam [18].

We first recall the definition of the Legendre transform (or polar) of a function: let V and V^* be two vector spaces in duality by a bilinear pairing denoted by $\langle \cdot, \cdot \rangle$. Let $\Phi\colon V \to \bar{\mathbb{R}}$ be a function. Then the Legendre transform $\Phi^*\colon V^* \to \bar{\mathbb{R}}$ of Φ is defined by

$$\Phi^*(u^*) = \sup_{u \in V} \{\langle u, u^* \rangle - \Phi(u)\}.$$

Let $\mathcal{F}\colon W^{1,1}(\Omega) \to \mathbb{R}$, $\mathcal{G}\colon L^2(\Omega) \times L^1(\Omega)^N \to \mathbb{R}$, $\mathcal{G}_1\colon L^2(\Omega) \to \mathbb{R}$, and $\mathcal{G}_2\colon L^1(\Omega)^N \to \mathbb{R}$, such that

$$\mathcal{F}(v) = -\int_\Omega v\xi\, dx = -\langle v, \xi \rangle_{L^p \times L^{p'}},$$

$$\mathcal{G}_1(w_0) = \int_\Omega (w_0 - u_0)^2\, dx, \qquad \mathcal{G}_2(\bar{w}) = \int_\Omega \varphi(|\bar{w}|)\, dx,$$

$$\mathcal{G}(w) = \mathcal{G}_1(w_0) + \mathcal{G}_2(\bar{w}),$$

with $w = (w_0, \bar{w}) = (w_0, w_1, \ldots, w_N) \in L^2(\Omega) \times L^1(\Omega)^N$.

Then $(\mathcal{P}_2^*)$ is given by

$$(\mathcal{P}_2^*) \qquad \sup_{p^* \in L^2(\Omega) \times L^\infty(\Omega)^N} \{-\mathcal{F}^*(\Lambda^* p^*) - \mathcal{G}^*(-p^*)\},$$

where the operator $\Lambda\colon W^{1,1}(\Omega) \to L^2(\Omega) \times L^1(\Omega)^N$ is defined by

$$\Lambda v = (Kv, D_1 v, D_2 v, \ldots, D_N v)$$

and Λ^* is the adjoint.

We compute $\mathcal{F}^*$ and $\mathcal{G}^*$ using the definition of the Legendre transform: if $V = W^{1,1}(\Omega)$ with V^* the dual, then

$$\mathcal{F}^*(\Lambda^* p^*) = \sup_{v \in W^{1,1}(\Omega)} \langle \Lambda^* p^* + \xi, v \rangle_{V \times V^*} = \begin{cases} 0 & \text{if} \quad \Lambda^* v^* + \xi = 0 \quad \text{on } V, \\ +\infty & \text{elsewhere}. \end{cases}$$

It is easy to see that

$$\mathcal{G}^*(p^*) = \mathcal{G}_1^*(p_0^*) + \mathcal{G}_2^*(\bar{p}^*),$$

where $p^* = (p_0^*, \bar{p}^*) = (p_0^*, p_1^*, \ldots, p_N^*)$.

We have that

$$\mathcal{G}_1^*(p_0^*) = \int_\Omega \left(\frac{(p_0^*)^2}{4} + p_0^* u_0 \right) dx.$$

Since $\varphi\colon \mathbb{R} \to \mathbb{R}^+$ is convex, lower semicontinuous, and even, we also have [18]

$$\mathcal{G}_2^*(\bar{p}^*) = \int_\Omega \varphi^*(|\bar{p}^*|)\, dx,$$

if $|\bar{p}^*(\cdot)| \in Dom(\varphi^*)$.

In this way, we can also write $(\mathcal{P}_2^*)$ in the following form:

$$(\mathcal{P}_2^*) \qquad \sup_{p^* \in \mathcal{K}} \left\{ -\int_\Omega \left(\frac{(p_0^*)^2}{4} - p_0^* u_0 \right) dx - \int_\Omega \varphi^*(|\bar{p}^*|)\, dx \right\},$$

where

$$\mathcal{K} = \{p^* \in L^2(\Omega) \times L^\infty(\Omega)^N\colon |\bar{p}^*(x)| \in Dom(\varphi^*),\ \Lambda^* p^* + \xi = 0 \text{ in } \mathcal{D}'(\Omega)\}.$$

We can simply see from assumption H1(ii) that if $m \in \mathbb{R}$, then $m \in Dom(\varphi^*)$ if and only if $|m| \leq c = \varphi^\infty(1)$ (see also [17]).

From $\Lambda^* p^* + \xi = 0$ in $\mathcal{D}'(\Omega)$, we get

$$\begin{aligned}\langle \Lambda^* p^*, w\rangle + \langle \xi, w\rangle &= \langle p^*, \Lambda w\rangle + \langle w, \xi\rangle \\ &= \langle p_0^*, Kw\rangle + \langle \bar{p}^*, Dw\rangle + \langle w, \xi\rangle = 0, \qquad \forall w \in \mathcal{D}(\Omega).\end{aligned}$$

Then we have

$$K^* p_0^* - \operatorname{div} \bar{p}^* + \xi = 0 \qquad \text{in } \mathcal{D}'(\Omega).$$

For p^* satisfying this relation, we obtain that $\operatorname{div} \bar{p}^* \in L^{p'}(\Omega)$, and then we can define (by a theorem of Lions and Magenes [24]) the trace of $\bar{p}^* \cdot \nu$ on $\Gamma = \partial\Omega$, where ν represents the unit normal to Γ, and integrating by parts, we get, for $v \in W^{1,1}(\Omega)$

$$\begin{aligned}\int_\Gamma \bar{p}^* \cdot \nu v\, d\Gamma &= \int_\Omega \sum_{i=1}^N (D_i \bar{p}_i^* v)\, dx + \int_\Omega \sum_{i=1}^N (\bar{p}_i^* D_i v)\, dx \\ &= \langle K^* p_0^*, v\rangle + \langle v, \xi\rangle - \langle p_0^*, Kv\rangle - \langle v, \xi\rangle = 0.\end{aligned}$$

In this way, we deduce, for $p^* \in \mathcal{K}$, that $\bar{p}^* \cdot \nu = 0$ $d\Gamma$-a.e. on Γ.

Finally, we rewrite $\mathcal{K}$ in the following way:

$$\begin{aligned}\mathcal{K} = \{&p^* \in L^2(\Omega) \times L^\infty(\Omega)^N: \\ &|\bar{p}^*(x)| \leq c,\ K^* p_0^* - \operatorname{div} \bar{p}^* + \xi = 0 \text{ in } \mathcal{D}'(\Omega),\ \bar{p}^* \cdot \nu = 0 \text{ on } \Gamma\}.\end{aligned}$$

We now apply the duality Theorem III.4.1 from [18], since the functional in $(\mathcal{P}_2)$ is convex, continuous with respect to Λv in $L^2(\Omega) \times L^1(\Omega)^N$, and $\inf \mathcal{P}_2$ is finite. Then $\inf(\mathcal{P}_2) = \sup(\mathcal{P}_2^*) \in \mathbb{R}$ and $(\mathcal{P}_2^*)$ has a solution $M \in \mathcal{K}$ [18]. This solution is unique if φ^* is strictly convex, which is equivalent to saying that $\varphi \in C^1(\mathbb{R})$, according to a result of Rockafellar [27].

Now we write that u is a solution of $(\mathcal{P}_1)$, that M is a solution of $(\mathcal{P}_2^*)$, and that $\inf(\mathcal{P}_1) = \inf(\mathcal{P}_2) = \sup(\mathcal{P}_2^*)$ (the extremality relations):

$$\begin{aligned}&\int_\Omega (Ku - u_0)^2\, dx + \int_\Omega \varphi(|Du|) - \int_\Omega \xi u\, dx \\ &\quad = -\int_\Omega \left(\frac{M_0^2}{4} - M_0 u_0 \right) dx - \int_\Omega \varphi^*(|\bar{M}|)\, dx,\end{aligned}$$

where $M \in L^2(\Omega) \times L^\infty(\Omega)^N$, $|\bar{M}(x)| \leq c$, $K^* M_0 - \operatorname{div} \bar{M} + \xi = 0$ in $\mathcal{D}'(\Omega)$, and $\bar{M} \cdot \nu = 0$ $d\Gamma$-a.e. on Γ.

Following Demengel and Temam [17], we can associate to u and $\bar{M}$ a bounded unsigned measure denoted $Du \cdot \bar{M}$ which is defined, as a distribution on Ω, by

$$\langle Du \cdot \bar{M}, \psi \rangle = -\int_\Omega u(\operatorname{div} \bar{M})\psi \, dx - \int_\Omega \bar{M} \cdot (\nabla\psi) u \, dx, \qquad \forall \psi \in C_0^\infty(\Omega)$$

(see also [30] and [22]).

By the generalized Green's formula (see also [30], [22], and [31])

$$\int_\Omega Du \cdot \bar{M} = -\int_\Omega u \cdot \operatorname{div} \bar{M} + \int_\Gamma u(\bar{M} \cdot \nu) \, d\Gamma,$$

since $\bar{M} \cdot \nu = 0$ $d\Gamma$-a.e., we get

$$\int_\Omega (Ku - u_0)^2 \, dx + \int_\Omega \varphi(|Du|) + \int_\Omega M_0 Ku + \int_\Omega \left(\frac{M_0^2}{4} - M_0 u_0 \right) dx$$
$$+ \int_\Omega Du \cdot \bar{M} + \int_\Omega \varphi^*(|\bar{M}|) \, dx = 0.$$

Using the decomposition $Du = \nabla u \, dx + C_u + (u^+ - u^-)\nu \, d\mathcal{H}^{N-1}|_{S_u}$, where $C_u(S_u) = 0$, we finally have

$$\int_\Omega (Ku - u_0)^2 \, dx + \int_\Omega M_0 Ku \, dx + \int_\Omega \left(\frac{M_0^2}{4} - M_0 u_0 \right) dx$$
$$+ \int_\Omega (\varphi(|\nabla u|) + \nabla u \cdot \bar{M} + \varphi^*(|\bar{M}|)) \, dx$$
$$+ \int_{\Omega \setminus S_u} (c|C_u| + \bar{M} C_u) + \int_{S_u} (u^+ - u^-)(c + \bar{M} \cdot \nu) \, d\mathcal{H}^{N-1} = 0.$$

Now, we have the following:

1°. $(Ku - u_0)^2 - (-M_0)u_0 + ((-M_0)^2/4 + (-M_0)u_0) \geq 0$, by the definition of $\mathcal{G}_1^*$ and for dx-a.e. $x \in \Omega$.
2°. $\varphi(|\nabla u(x)|) + \bar{M}(x) \cdot \nabla u(x) + \varphi^*(|\bar{M}(x)|) \geq \varphi(|\nabla u(x)|) - |\bar{M}(x)| \cdot |\nabla u(x)| + \varphi^*(|\bar{M}(x)|) \geq 0$, by the definition of φ^* and for dx-a.e. $x \in \Omega$, where ∇u is defined.
3°. $C_u << |C_u|$ and there exists $h \in L^1(|C_u|)^N$ such that $|h| = 1$ and $C_u = h|C_u|$ (the Radon–Nikodym theorem). We then obtain $c|C_u| + \bar{M} \cdot C_u = (C + \bar{M} \cdot h)|C_u| \geq 0$, because $|\bar{M}| \leq c$.
4°. When u^+ and u^- are defined, we have $u^+ - u^- \geq 0$ and $c + \bar{M} \cdot \nu \geq 0$.

We can now give a characterization of $\xi \in \partial \hat{F}(u)$:

Proposition 4.1. *Let $\xi \in L^{p'}(\Omega)$ and $u \in BV(\Omega)$, with $Du = \nabla u \, dx + C_u + J_u$ the decomposition of Du. Then $\xi \in \partial \hat{F}(u)$ if and only if there exists $M \colon \Omega \to \mathbb{R}^{N+1}$ with $M(x) = (M_0(x), \bar{M}(x)) \in \mathbb{R} \times \mathbb{R}^N$, $|\bar{M}(\cdot)| \leq c$, and $M_0 \in L^2(\Omega)$, such that*

$$\varphi(|\nabla u|) + \nabla u \cdot \bar{M} + \varphi^*(|\bar{M}|) = 0 \qquad dx\text{-a.e.} \quad x \in \Omega. \tag{5}$$

If

$$\mathcal{H} = \{x \in \Omega \backslash S_u\colon\ c + \bar{M}(x) \cdot h(x) = 0, C_u = h|C_u|, h \in L^1(|C_u|)^N, |h| = 1\},$$

then

$$\operatorname{supp}(|C_u|) \subset \mathcal{H}, \tag{6}$$
$$c + \bar{M} \cdot \nu = 0, \quad |\bar{M}| = c \qquad d\mathcal{H}^{N-1}\text{-a.e.} \quad x \in S_u, \tag{7}$$
$$K^* M_0 - \operatorname{div} \bar{M} + \xi = 0 \qquad \text{in } \mathcal{D}'(\Omega), \tag{8}$$
$$-M_0 = 2(Ku - u_0) \qquad dx\text{–a.e. in } \Omega, \tag{9}$$
$$\bar{M} \cdot \nu = 0 \qquad d\Gamma\text{-a.e. on } \Gamma. \tag{10}$$

If in addition φ *is differentiable, then we can compute* $\bar{M}$ *as*

$$\bar{M}(x) = -\frac{\varphi'(|\nabla u(x)|)}{|\nabla u(x)|} \nabla u(x), \qquad dx\text{-a.e.} \quad x \in \Omega, \quad \text{if} \quad |\nabla u(x)| \neq 0, \tag{11}$$

and $\bar{M}(x) = (0, \dots, 0)$ *if* $|\nabla u(x)| = 0$.

Finally, we have a characterization for the solution u *of* $\{\inf_{v \in L^p(\Omega)} \hat{F}(v)\}$, *taking* $\xi = 0$ *and writing that* $0 \in \partial \hat{F}(u)$.

Proof. The direct implication has just been proved. Conversely, if such an M exists, it is easy to check that M is a solution of $(\mathcal{P}_2^*)$ and u is a solution of $(\mathcal{P}_1)$, which amounts to saying that $\xi \in \partial \hat{F}(u)$.

Now, if φ is differentiable, we only show how we obtain the expression of M, the other results follow from before.

Let $x \in \Omega$ such that (5) is true at x and we denote by $\bar{M}_i(x)$ (and by $\nabla_i u(x)$), $i = 1, \dots, N$, the components of $\bar{M}(x)$ ($\nabla u(x)$, respectively). We have the following:

$$\varphi^*(|-\bar{M}(x)|) = -\bar{M}(x) \cdot \nabla u(x) - \varphi(|\nabla u(x)|) = \sup_{T \in \mathbb{R}^N} \{-\bar{M} \cdot T - \varphi(|T|)\}.$$

Let x be a Lebesgue point for $|Du|$. If $|\nabla u(x)| \neq 0$, then for $T = \nabla u(x)$ we have that

$$\nabla_T(-\bar{M} \cdot T - \varphi(|T|)) = (0, \dots, 0),$$

which implies that

$$\bar{M}_i(x) = -\frac{\varphi'(|\nabla u(x)|)}{|\nabla u(x)|} \nabla_i u(x).$$

We also deduce from 2° and (5) that

$$\varphi(|\nabla u(x)|) - |\bar{M}(x)| \cdot |\nabla u(x)| + \varphi^*(|\bar{M}(x)|) = 0.$$

Then

$$\varphi^*(|\bar{M}(x)|) = |\bar{M}(x)| \cdot |\nabla u(x)| - \varphi(|\nabla u(x)|) = \sup_{t\in\mathbb{R}}\{|\bar{M}(x)| \cdot t - \varphi(t)\},$$

which implies that $|\bar{M}(x)| = \varphi'(t)$ for $t = |\nabla u(x)|$. Hence, if $|\nabla u(x)| = 0$, then $\bar{M}(x) = (0, \dots, 0)$, using $\varphi'(0) = 0$. □

Remark 4.2. Unfortunately, the functional is not lower semicontinuous on $SBV(\Omega)$, the space of special functions of bounded variation, introduced by De Giorgi and Ambrosio [16], defined by

$$SBV(\Omega) = \{u \in BV(\Omega)\colon\ Du = \nabla u\, dx + J_u \Leftrightarrow C_u = 0\}.$$

This fact is proved in [4]. Therefore, we cannot say a priori that the solution belongs to $SBV(\Omega)$. For instance, the Mumford–Shah functional for image segmentation (see [25] and [15])

$$MS(u) = \frac{1}{2}\int_\Omega |u - u_0|^2\, dx + \int_\Omega |\nabla u|^2\, dx + \int_{S_u} d\mathcal{H}^{N-1} \tag{12}$$

is convex *and* lower semicontinuous on $SBV(\Omega)$. Maybe the subspace $SBV(\Omega)$ is more convenient than $BV(\Omega)$ to model the reconstructed images.

Remark 4.3. The dual problem $(\mathcal{P}_2^*)$ with the solution $M = (M_0, \bar{M})$ can offer a new method to compute the solution u numerically, at least for the case $K = I$. Indeed, if we solve the following constrained minimization problem with the solution $\bar{M}$,

$$\inf_{\bar{M}\colon\ |\bar{M}(x)|\le c} \left\{\int_\Omega \left(\frac{(\operatorname{div}\bar{M})^2}{4} - (\operatorname{div}\bar{M})u_0\right) dx + \int_\Omega \varphi^*(|\bar{M}|)\, dx\right\},$$

then we can compute u by the relations $M_0 = \operatorname{div}\bar{M}$, $2(u - u_0) = -M_0$, from Proposition 4.1. We have remarked that φ^* is strictly convex if and only if $\varphi \in C^1(\mathbb{R})$, and this is true for the potentials φ_1, φ_2, and φ_3 defined in the Introduction. For instance, for $m \in [0, 1]$, we have $\varphi_1^*(m) = m^2/2$ and $\varphi_2^*(m) = 1 - \sqrt{1 - m^2}$. For these two functions, the constant c is equal to one. This problem could be solved by a quasi-Newton method with constraints, but we do not analyze this possible approach here.

5. The Evolution Problem

In this section we study the evolution equation associated with the problem $0 \in \partial\hat{F}(u)$, in the particular cases of one and two dimensions, i.e., $N = 1$ and $N = 2$. Then $BV(\Omega)$ is continuously embedded in the Hilbert space $L^2(\Omega)$, this being necessary in order to apply general results on maximal monotone operators and evolution equations on Hilbert spaces [8]. The function φ satisfies the same assumptions as in the previous sections. For the moment, we only assume that $K\colon\ L^2(\Omega) \to L^2(\Omega)$ is linear and continuous.

We can associate the following evolution problem to the minimization of the functional F (given in (1)) if, for instance, $\varphi \in C^1(\mathbb{R})$:

$$(Ev_1) \quad \begin{cases} 0 = \dfrac{\partial u}{\partial t} + \mathcal{A}u & \text{in} \quad]0, \infty[\times \Omega, \\ u(0, x) = u_0(x) & \text{for} \quad x \in \Omega, \\ \dfrac{\varphi'(|Du|)}{|Du|} Du \cdot \nu = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $u\colon [0, \infty) \times \Omega \to \mathbb{R}$ is the unknown function and $\mathcal{A}$ is the operator:

$$\mathcal{A}u = 2K^*(Ku - u_0) - \operatorname{div}\left(\frac{\varphi'(|\nabla u|)}{|\nabla u|} \nabla u\right).$$

To study such an evolution problem, we apply nonlinear semigroup theory and the notion of a maximal monotone operator [8].

Unfortunately, the operator $\mathcal{A}$ is not maximal monotone because it is the subdifferential of the functional F, $F\colon L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}$,

$$F(u) = \begin{cases} \displaystyle\int_\Omega (Ku - u_0)^2\, dx + \int_\Omega \varphi(|\nabla u|)\, dx & \text{if} \quad u \in W^{1,1}(\Omega), \\ +\infty & \text{if} \quad u \in L^2(\Omega) \setminus W^{1,1}(\Omega), \end{cases}$$

which is not lower semicontinuous on $L^2(\Omega)$. To overcome this difficulty, as in the previous sections, we consider the relaxed functional $\hat{F}$ of F on $L^2(\Omega)$:

$$\hat{F}(u) = \int_\Omega (Ku - u_0)^2\, dx + \int_\Omega \varphi(|\nabla u|)\, dx + c|D_s u|(\Omega) \qquad \text{if} \quad u \in BV(\Omega),$$

with $Du = \nabla u\, dx + D_s u$, and $\hat{F}(u) = +\infty$ if $u \in L^2(\Omega) \setminus BV(\Omega)$. Then we associate to $\hat{F}$ the following evolution problem on $L^2(\Omega)$:

$$(Ev_2) \quad \begin{cases} 0 \in \dfrac{\partial u}{\partial t} + \partial \hat{F}(u) & \text{on} \quad]0, \infty[\times \Omega, \\ u(0, x) = u_0(x) & \text{for} \quad x \in \Omega. \end{cases}$$

It is easy to establish the following theorem from the above relaxation results and a general result of an evolution equation governed by a maximal monotone operator.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded, and connected subset of $\mathbb{R}^N$ ($N = 1, 2$) with Lipschitz boundary $\Gamma = \partial\Omega$. Let $u_0 \in Dom(\partial\hat{F})$. Then there exists a unique function $u(t)\colon [0, +\infty[\to L^2(\Omega)$ such that*

$$u(t) \in Dom(\partial\hat{F}), \quad \forall t > 0, \qquad \frac{\partial u}{\partial t} \in L^\infty((0, +\infty); L^2(\Omega)), \tag{13}$$

$$-\frac{\partial u}{\partial t} \in \partial\hat{F}(u(t)), \qquad a.e. \quad t \in]0, +\infty[, \quad u(0) = u_0. \tag{14}$$

If $\hat{u}$ is a solution of (13)–(14), *with $\hat{u}_0$ instead of u_0, then*

$$\|u(t) - \hat{u}(t)\|_{L^2(\Omega)} \le \|u_0 - \hat{u}_0\|_{L^2(\Omega)}, \qquad \forall t \ge 0. \tag{15}$$

Let $Du(\cdot,t) = \nabla u(\cdot,t)\,dx + D_s u(\cdot,t)$ *be the Lebesgue decomposition of* $Du(\cdot,t)$. *Then, for almost every* $t > 0$, *there exists* $M(t,\cdot) \in L^2(\Omega) \times L^\infty(\Omega)^N$, $M = (M_0, \bar{M}) = (M_0, \ldots, M_N)$ *satisfying* (5)–(7), (9), (10), *and, instead of* (8),

$$-\frac{du}{dt} + K^* M_0 - \operatorname{div} \bar{M} = 0 \qquad \textit{in } \mathcal{D}'(\Omega).$$

If, in addition, φ *is differentiable, then* $M(t,x)$ *is given by* (11).

Proof. The functional $\hat{F}$ is clearly convex, proper, and lower semicontinuous in $L^2(\Omega)$, from Remark 2.5. Then $\partial\hat{F}$ is maximal monotone and (13) and (14) follow from nonlinear semigroup theory [8, Theorem 3.1]. The other conditions follow immediately from (14) and the characterization of $\partial\hat{F}$. □

Remark 5.2. For each $t > 0$, the map $u_0 \to u(t)$ is a contraction from $Dom(\partial\hat{F})$ into $Dom(\partial\hat{F})$. We denote by $S(t)$ its unique extension to a continuous nonexpansive semigroup on $\overline{Dom(\partial\hat{F})} = Dom\,\hat{F} = BV(\Omega)$ (see, for instance, [8] and [33]). If $u_0 \in BV(\Omega)$, then $u(t) = S(t)u_0$ is called the generalized solution of (Ev_2). Moreover, $S(t)u_0 \in Dom(\partial\hat{F})$ for all $t > 0$, i.e., the operator $\partial\hat{F}$ has a regularizing effect.

Behavior of solutions as $t \to +\infty$: Let φ and K satisfy assumptions H1–H4 and $u_0 \in Dom(\partial\hat{F})$. Then the problem (Ev_2) has a unique solution $u(t)$: $[0;+\infty[\to L^2(\Omega)$, which satisfies (13)–(15) and we also know that $\hat{F}$: $L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}$ has a unique minimum $\bar{u}$ on $BV(\Omega)$.

We now prove, as in [23], that $u(t)$ converges strongly in $L^1(\Omega)$ and weakly in $L^2(\Omega)$ to $\bar{u}$ as $t \to \infty$.

First, we recall a result of Bruck [10] which proves the weak convergence in $L^2(\Omega)$ to $\bar{u}$.

Proposition 5.3 [10]. *Let* H *be a Hilbert space and let* $\mathcal{A}$ *be the subdifferential* ∂F *of a proper lower semicontinuous function* F: $H \to\ ]-\infty,+\infty]$ *which assumes a minimum in* H.

If u: $[0,\infty[\to H$ *is absolutely continuous and satisfies*

$$u(t) \in Dom(\mathcal{A}), \qquad \forall t \geq 0,$$

$$0 \in \frac{\partial u}{\partial t} + \mathcal{A}u \qquad a.e.,$$

$$\left\| \frac{\partial u}{\partial t} \right\|_H \in L^\infty(0,\infty),$$

then $u(t)$ *has a weak limit* $\bar{u}$ *in* H *as* $t \to \infty$ *and* $\bar{u}$ *belongs to* $\mathcal{A}^{-1}(0)$.

Theorem 5.4. *Let* $u_0 \in Dom(\partial\hat{F})$. *Then the solution* u *of* (Ev_2) *converges as* $t \to \infty$ *to the minimum* $\bar{u}$ *of* $\hat{F}$ *in the following sense*:

$$u(t) \to \bar{u} \textit{ strongly in } L^1(\Omega) \textit{ and weakly in } L^2(\Omega).$$

Proof. The existence of a weak limit $\bar{u}$ in $L^2(\Omega)$ is a consequence of the existence result from Section 3, Theorem 5.1, and Proposition 5.3. It remains to show the strong convergence in $L^1(\Omega)$.

We prove that $\hat{F}(u(t))$ is uniformly bounded and therefore $u(t)$ will be uniformly bounded in $BV(\Omega)$.

From

$$-\frac{\partial u}{\partial t} \in \partial \hat{F}(u(t)),$$

by the definition of the subdifferential, we have

$$\hat{F}(u(t)) + \int_\Omega \frac{\partial u}{\partial t} u(t)\, dx \leq \hat{F}(v) + \int_\Omega \frac{\partial u}{\partial t} v\, dx, \qquad \forall v \in L^2(\Omega).$$

Let $v = \bar{u}$. Then

$$\hat{F}(u(t)) \leq \hat{F}(\bar{u}) + \int_\Omega \frac{\partial u}{\partial t}(\bar{u} - u(t))\, dx \leq \hat{F}(\bar{u}) + \left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)} \cdot \|\bar{u} - u(t)\|_{L^2(\Omega)}.$$

Now, $\hat{F}(\bar{u}) < \infty$ because $\bar{u} \in Dom(\hat{F}) = BV(\Omega)$, $\|\partial u/\partial t\|_{L^2(\Omega)}$ is uniformly bounded from (13), like $\|\bar{u} - u(t)\|_{L^2(\Omega)}$, from the weak convergence.

Finally, as in Section 3, there is a subsequence $u(t_n)$ which converges in $L^1(\Omega)$ to a limit, which must be $\bar{u}$. Moreover, all the sequence $u(t)$ converges strongly in $L^1(\Omega)$ to $\bar{u}$ (for instance by contradiction). □

Remark 5.5. Unfortunately, we have obtained the strong convergence of $u(t)$ to $\bar{u}$ only in $L^1(\Omega)$, and not in $L^2(\Omega)$, since $Dom(\hat{F}) = BV(\Omega)$ is only continuously embedded in $L^2(\Omega)$. Generally, we could obtain the strong convergence in $L^2(\Omega)$ under the following assumption on $\hat{F}$: for each $C \geq 0$, the set $\{u \in L^2(\Omega)\colon\ \hat{F}(u) + \|u\|^2_{L^2(\Omega)} \leq C\}$ is strongly compact (see [8]), but this is not true in our case.

6. Approximation by Γ-Convergence

In order to solve the minimization problem (4) numerically, we first need to regularize it and to work on a more regular space than $BV(\Omega)$, because we do not know how to approximate directly in the energy the term

$$\int_\Omega |D_s u| = \int_{\Omega\setminus S_u} |C_u| + \int_{S_u} |u^+ - u^-|\, d\mathcal{H}^{N-1},$$

for $u \in BV(\Omega)$. Therefore, it is necessary to approach in some sense the functional $\hat{F}$ by a sequence $(F_\varepsilon)_{\varepsilon>0}$ of quadratic functionals, finite, lower semicontinuous, and well-defined on a subspace of $W^{1,p}(\Omega)$ (we recall that $p = 2$ if $N = 1$ and $p = N/(N-1)$ if $N \geq 2$), and where the functions have the singular part of the gradient equal to zero.

There are many possibilities to construct the sequence $(F_\varepsilon)_{\varepsilon>0}$, and we consider here two cases. The most classical approximation and regularization is obtained by defining $\varphi_{1\varepsilon}\colon \mathbb{R}^+ \to \mathbb{R}^+$, $\varphi_{1\varepsilon}(z) = \varphi(z) + \varepsilon z^2$.

We can also approach and regularize the function φ, which is assumed to be continuously differentiable on $]0, +\infty[$, in the following manner: let $\varphi_{2\varepsilon}\colon \mathbb{R}^+ \to \mathbb{R}^+$ be ($\varepsilon > 0$)

$$\varphi_{2\varepsilon}(z) = \begin{cases} \dfrac{\varphi'(\varepsilon)}{2\varepsilon} z^2 + \varphi(\varepsilon) - \dfrac{\varepsilon\varphi'(\varepsilon)}{2}, & \text{if} \quad z \le \varepsilon, \\ \varphi(z), & \text{if} \quad \varepsilon \le z \le \frac{1}{\varepsilon}, \\ \dfrac{\varepsilon\varphi'(1/\varepsilon)}{2} z^2 + \varphi\left(\dfrac{1}{\varepsilon}\right) - \dfrac{\varphi'(1/\varepsilon)}{2\varepsilon}, & \text{if} \quad z \ge \frac{1}{\varepsilon}. \end{cases} \tag{16}$$

By the following assumption,

$$]0, \infty[\ \ni z \mapsto \frac{\varphi'(z)}{z} \qquad \text{is continuously decreasing,} \tag{17}$$

we have that $\varphi_{2\varepsilon}(z) \ge \varphi(z)$, for all $z \ge 0$ (this is of course true for the previous approximation $\varphi_{1\varepsilon}$ of φ).

Now, choosing one of these two sequences $(\varphi_{i\varepsilon})_{\varepsilon>0}$, we define the sequence $(F_{i\varepsilon})_{\varepsilon>0}$, $i = 1, 2$, by

$$F_{i\varepsilon}(u) = \begin{cases} \displaystyle\int_\Omega |Ku - u_0|^2\, dx + \int_\Omega \varphi_{i\varepsilon}(|\nabla u|)\, dx, & \text{if} \quad u \in W^{1,1}(\Omega), \\ & \qquad \nabla u \in L^2(\Omega), \\ +\infty, & \text{elsewhere}. \end{cases}$$

We also define

$$\bar{F}(u) = \begin{cases} F(u), & \text{if} \quad u \in W^{1,1}(\Omega), \quad \nabla u \in L^2(\Omega), \\ +\infty, & \text{elsewhere} \end{cases}$$

($\bar{F}$ is the restriction of F from Section 2 to functions $u \in W^{1,1}(\Omega)$, with $\nabla u \in L^2(\Omega)$).

Sometimes, we use the notation $(F_\varepsilon)_{\varepsilon>0}$ instead of $(F_{i\varepsilon})_{\varepsilon>0}$, F_ε being one of these two approximations.

From now on, we assume assumptions H1–H4 from Section 2. For the results concerning the sequence $(F_{2\varepsilon})_{\varepsilon>0}$, we need in addition to assume that $\varphi \in C^1(0, +\infty)$ and (17).

Proposition 6.1. *For every $\varepsilon > 0$, the functional F_ε has a unique minimum $u^\varepsilon \in W^{1,1}(\Omega)$, with $\nabla u^\varepsilon \in L^2(\Omega)$.*

Proof. Let u_n be a minimizing sequence for F_ε. Then $u_n \in W^{1,1}(\Omega)$, with $\nabla u_n \in L^2(\Omega)$, and there exists a constant $M > 0$ such that

$$\int_\Omega |Ku_n - u_0|^2\, dx \le M, \qquad \|\nabla u_n\|_{L^2(\Omega)} \le M,$$

from the construction of φ_ε. Then we prove, as in the existence result from Section 3, that

$$\|u_n\|_{L^p(\Omega)} \leq M.$$

Then there is $u \in W^{1,1}(\Omega)$, with $\nabla u \in L^2(\Omega)$, and a subsequence of u_n, still denoted u_n, such that

$$u_n \rightharpoonup u \quad \text{weakly in } L^p(\Omega), \qquad \nabla u_n \rightharpoonup \nabla u \quad \text{weakly in } L^2(\Omega).$$

Since φ_ε is convex and continuous, and $K\colon\ L^p(\Omega) \to L^2(\Omega)$ is linear and continuous, we obtain that

$$F_\varepsilon(u) \leq \liminf_{n\to\infty} F_\varepsilon(u_n),$$

i.e., u is a minimum of F_ε, denoted u^ε. The uniqueness is deduced as in Section 3. □

Now, to show that $(u^\varepsilon)_{\varepsilon>0}$ converges to the unique minimum of $\hat{F}$, we use the notion of Γ-*convergence* and its relation with the pointwise convergence, presented by Dal Maso in [14].

Let X be a topological space. The set of all open neighborhoods of x in X will be denoted by $\mathcal{N}(x)$. Let (F_h) be a sequence of functions from X into $\bar{\mathbb{R}}$.

Definition 6.2. The Γ-*lower limit* and the Γ-*upper limit* of the sequence (F_h) are the functions from X into $\bar{\mathbb{R}}$ defined by

$$\left(\Gamma\text{-}\liminf_{h\to\infty} F_h\right)(x) = \sup_{U\in\mathcal{N}(x)} \liminf_{h\to\infty} \inf_{y\in U} F_h(y),$$
$$\left(\Gamma\text{-}\limsup_{h\to\infty} F_h\right)(x) = \sup_{U\in\mathcal{N}(x)} \limsup_{h\to\infty} \inf_{y\in U} F_h(y).$$

If there exists a function $F\colon\ X \to \bar{\mathbb{R}}$ such that

$$\Gamma\text{-}\liminf_{h\to\infty} F_h = \Gamma\text{-}\limsup_{h\to\infty} F_h = F,$$

then we write $F = \Gamma\text{-}\lim_{h\to\infty} F_h$ and we say that the sequence (F_h) Γ-*converges* to F (in X) or that F is the Γ-*limit* of (F_h) (in X).

We also use the following two results from [14]:

Proposition 6.3. *If (F_h) is a decreasing sequence converging to F pointwise, then (F_h) Γ-converges to the lower semicontinuous envelope of F in X, denoted by $sc^- F$.*

Corollary 6.4. *Suppose that (F_h) is equi-coercive and Γ-converges to a function F, with a unique minimum point x_0 in X. Let (x_h) be a sequence in X such that x_h is a minimum for F_h in X for every $h \in \mathbb{N}$. Then (x_h) converges to x_0 in X and $(F_h(x_h))$ converges to $F(x_0)$.*

Proposition 6.5. *The sequence* $(u^{\varepsilon})_{\varepsilon>0}$ *from Proposition* 6.1 *converges in* $L^1(\Omega)$ *to the unique minimum* u *of* $\hat{F}$ *and* $F_{\varepsilon}(u^{\varepsilon})$ *converges to* $\hat{F}(u)$.

Proof. In our case, for $X = L^1(\Omega)$, we have that $F_{\varepsilon}(u) \searrow \bar{F}(u)$ as $\varepsilon \searrow 0$, for every $u \in W^{1,1}(\Omega)$, with $\nabla u \in L^2(\Omega)$. Of course, the sequence (F_{ε}) is equi-coercive in $L^1(\Omega)$, since $F_{\varepsilon} \geq \hat{F}$ for all $\varepsilon > 0$ and $\hat{F}$ is coercive in $L^1(\Omega)$. To apply the above results, we need to check that $\hat{F} = sc^{-}\bar{F}$ in $L^1(\Omega)$. We consider two steps.

Step 1: $\hat{F}$ *is lower semicontinuous in* $L^1(\Omega)$ *with respect to the* L^1*-topology.* It is easy to verify this: let $u, u_n \in L^1(\Omega)$, such that $u_n \to u$ in $L^1(\Omega)$, as $n \to \infty$ and $\liminf_{n\to\infty} \hat{F}(u_n) < +\infty$. Then, as $\hat{F}(u_n)$ is bounded (or for a subsequence), we deduce that $u_n \in BV(\Omega)$ with $\|u_n\|_{BV(\Omega)}$ uniformly bounded. Then $u \in BV(\Omega)$, $u_n \rightharpoonup u$ in $L^p(\Omega)$ and $u_n \rightharpoonup u$ in BV-$w*$, as $n \to \infty$. Finally, we have

$$\hat{F}(u) \leq \liminf_{n\to\infty} \hat{F}(u_n),$$

i.e., step 1 is proved.

Step 2: $\hat{F}$ *is the lower semicontinuous envelope of* $\bar{F}$ *in* $L^1(\Omega)$, *with respect to the* L^1*-topology.* From step 1, it suffices to show that, for $u \in BV(\Omega)$, there exists a sequence $u_n \in W^{1,1}(\Omega)$, with $\nabla u_n \in L^2(\Omega)$, such that

$$u_n \to u \quad \text{in } L^1(\Omega) \text{ as } \quad n \to \infty \qquad \text{and} \qquad \hat{F}(u) = \liminf_{n\to\infty} \bar{F}(u_n).$$

Let $u \in BV(\Omega)$. From Remark 2.5 we have that $\hat{F}$ is the lower semicontinuous envelope in $L^p(\Omega)$ of its restriction $\hat{F}|_{W^{1,1}(\Omega)} = F$. Then there is a sequence $u_n \in W^{1,1}(\Omega)$, such that $u_n \to u$ in $L^p(\Omega)$ and

$$\hat{F}(u) = \lim_{n\to\infty} \hat{F}(u_n) = \lim_{n\to\infty} F(u_n).$$

Now, for each $u_n \in W^{1,1}(\Omega)$, there is a sequence $(u_n^k)_{k\in\mathbb{N}} \in W^{1,1}(\Omega) \cap C^{\infty}(\bar{\Omega})$, such that $u_n^k \to u_n$ in $W^{1,1}(\Omega)$, as $k \to \infty$ (see [19]). In particular, we have that $\nabla u_n^k \in L^2(\Omega)$ and $u_n^k \to u_n$ in $L^p(\Omega)$, as $k \to \infty$, by the Sobolev embedding. Then, since the map $u \mapsto \int_{\Omega} \varphi(|Du|)\, dx$ is a convex and continuous function from $W^{1,1}(\Omega)$ into $\mathbb{R}$ (see, for instance, [17]), and K is linear and continuous from $L^p(\Omega)$ into $L^2(\Omega)$, we deduce in addition that

$$\hat{F}(u_n) = F(u_n) = \lim_{k\to\infty} \bar{F}(u_n^k).$$

Then, by a double approximation argument, we deduce that, for $u \in BV(\Omega)$, there exists $u^n \in W^{1,1}(\Omega)$, with $\nabla u^n \in L^2(\Omega)$, such that $u^n \to u$ in $L^p(\Omega)$ (and, in particular, in $L^1(\Omega)$), and

$$\hat{F}(u) = \lim_{n\to\infty} \bar{F}(u^n),$$

i.e., step 2 is proved.

In this way we obtain that the sequence $(u^\varepsilon)_{\varepsilon>0}$ converges in $L^1(\Omega)$ to u, the unique minimum of $\tilde{\tilde{F}}$ (equation (4)). □

Remark 6.6. To compute u^ε numerically, with $\varepsilon > 0$ small enough, we can use the associated Euler–Lagrange equation, having the solution u^ε, the minimum of F_ε, which is now Gâteaux-differentiable at each point. However, unfortunately, the problem is still nonlinear. To overcome this difficulty, we will construct a sequence of functions $(u_n^\varepsilon)_{n\in\mathbb{N}}$, which will converge to u^ε, and u_n^ε will be the solution of a linear equation. For instance (for $K = I$), if we consider the second regularization $\varphi_{2\varepsilon}$ (16) of φ and denote $\varphi_{2\varepsilon}$ by Φ for simplicity, and the values

$$L = \lim_{z\to\infty} \frac{\Phi'(z)}{2z}, \qquad M = \lim_{z\to 0^+} \frac{\Phi'(z)}{2z},$$

then we can show that there exists a strictly convex and decreasing function Ψ defined on $[M, L]$ such that

$$\Phi(z) = \inf_{M\le w\le L} (wz^2 + \Psi(w)).$$

The minimum will be reached for $w = \Phi'(z)/2z$. Then we let

$$E(u, b) = \int_\Omega |Ku - u_0|^2 + \int_\Omega b|Du|^2 + \int_\Omega \Psi(b),$$

and we obtain the following algorithm: start from any u^1 and b^1 and let

$$u^{n+1} = \arg\min_{u\in H^1(\Omega)} E(u, b^n),$$

$$b^{n+1} = \arg\min_{M\le b\le L} E(u^{n+1}, b) = M \vee \frac{\Phi'(|Du^{n+1}|)}{2|Du^{n+1}|} \wedge L,$$

where we have used the notations $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$. Therefore, u^{n+1} will be characterized by

$$u^{n+1} - \operatorname{div}(b^n Du^{n+1}) = u_0.$$

The discrete version of this algorithm is introduced in [3] for the Euler–Lagrange equation associated to the minimization problem (this algorithm will be used here), and in [12] and [13] for the minimized energy. In those papers, stability and convergence results are presented. Also, in continuous variables, in [11] the authors have proved the convergence of the algorithm for the total variation minimization.

Remark 6.7. In this section we have presented the results in the general N-dimensional case. In practice, for signal and image reconstruction, we will have $N = 1$ or $N = 2$. Then $p = 2$ and the regularizing sequence $(u^\varepsilon)_{\varepsilon>0}$ will belong to $H^1(\Omega)$.

7. The Numerical Approximation of the Problem

In this section we recall the version of the previous algorithm introduced in [3], to approach by finite differences schemes the associated Euler–Lagrange equation written in conservative form:

$$K^*Ku - \alpha \operatorname{div}\left(\frac{\varphi'(|Du|)}{|Du|}Du\right) = K^*u_0. \tag{18}$$

To discretize the divergence operator, a method of Rudin et al. [29] for the total variation minimization is used. We also adapt the algorithm to the associated evolution equation.

Remark 7.1. For numerical reasons, we need to compute u^ε, the continuous approximation of the BV solution u, with $\varepsilon > 0$ small enough, defined in Section 6 as a minimizer of $F_{i\varepsilon}$, the approximation of $\hat{F}$. If we use the approximation $F_{2\varepsilon}$ of $\hat{F}$, then it is not necessary to consider, in (16), the case $z \geq 1/\varepsilon$, since, in practice, for discrete images, the gradients are always bounded. Moreover, if the function φ is regular and "quadratic" at the origin (like, for instance, φ_1, φ_2, and φ_3 from the Introduction), we will not consider the case $z \leq \varepsilon$. In the description of the algorithm, we assume that $\varphi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is of class C^1, $\varphi'(0) = 0$, with $z \mapsto \varphi'(z)/z$ strictly positive, continuous, and decreasing in $[0, +\infty[$. Therefore, we discretize directly (18).

Remark 7.2. We need to specify boundary conditions on $\Gamma = \partial\Omega$ associated to (18). From Section 4 (see (10)), the natural condition is

$$\frac{\varphi'(|Du|)}{|Du|}Du \cdot n = 0, \qquad \partial\Gamma\text{-a.e. on } \Gamma,$$

where n is the unit normal to Γ. However, because $\varphi'(z)/z$ is strictly positive in $[0, +\infty[$, and the discrete gradients are bounded in the norm, we get the following classical boundary condition:

$$\frac{\partial u}{\partial n} = 0 \qquad \text{on} \quad \Gamma = \partial\Omega.$$

We approach the solution u by a sequence $(u^n)_{n\geq 0}$, with $u^0 = u_0$, such that u^{n+1} is the solution of the following linear problem:

$$K^*Ku^{n+1} - \alpha \operatorname{div}\left(\frac{\varphi'(|Du^n|)}{|Du^n|}Du^{n+1}\right) = K^*u_0. \tag{19}$$

Let ψ be the function defined by

$$\psi\colon \mathbb{R}^+ \to \mathbb{R}^+, \qquad \psi(z) = \begin{cases} \dfrac{\varphi'(z)}{z} & \text{if} \quad z > 0, \\ \lim\limits_{z\to 0} \dfrac{\varphi'(z)}{z} & \text{if} \quad z = 0. \end{cases}$$

Then, for each type of potential, the function ψ is strictly positive and bounded on $\mathbb{R}^+$.

Now we move to the precise description of the algorithm in dimension one.

7.1. *The One-Dimensional Case*

Let for the moment $K = I$. In this case, (18) becomes

$$u - \alpha \frac{\partial}{\partial x}(\psi(|u_x|)u_x) = u_0. \tag{20}$$

Assume that $\Omega =]0,1[$. Let $M \in \mathbb{N}^*$, $x_i = ih$, where $h = 1/M$ and $0 \le i \le M$, be the discrete points. Let u_0: $[0,1] \to \mathbb{R}$ be given and let u: $]0,1[\to \mathbb{R}$ be a solution to the problem (20). We define the discrete approximations u_h of u and $u_{0,h}$ of u_0 by

$$u_h(x_i) = u_i \approx u(x_i), \quad \text{for} \quad 0 < i < M,$$
$$u_{0,h}(x_i) = u_{0,i} \approx u_0(x_i), \quad \text{for} \quad 0 \le i \le M,$$

with the following discrete boundary conditions (corresponding to Neumann boundary conditions):

$$u_h(x_0) := u_h(x_1), \qquad u_h(x_M) := u_h(x_{M-1}). \tag{21}$$

We may assume that the initial discrete signal, the data, satisfies the following property:

$$\text{there exist } m_2 \ge m_1 \ge 0 \text{ such that } m_1 \le u_{0,i} \le m_2, \qquad \text{for} \quad 0 \le i \le M,$$

which will be used to establish the so-called L^∞*-stability* for the solution. We also recall the usual notations for finite differences in dimension one. Let

$$\triangle_+ u_i := u_{i+1} - u_i, \quad \triangle_- u_i := u_i - u_{i-1}, \qquad \text{for} \quad 0 < i < M.$$

The numerical approximation of (20) will be

$$u_i - \frac{\alpha}{h}\triangle_-\left[\psi\left(\left|\frac{\triangle_+ u_i}{h}\right|\right)\left(\frac{\triangle_+ u_i}{h}\right)\right] = u_{0,i}, \qquad 0 < i < M, \tag{22}$$

with the boundary conditions (21).

Since the problem is still nonlinear, as we have mentioned, we approach the numerical solution u_h by a sequence $(u_h^n)_{n\ge 0}$, which is obtained by a fixed point algorithm (see also [2]) as follows (sometimes we write u, u_0, u^n instead of $u_h, u_{0,h}, u_h^n$):

1. u^0 is arbitrarily given, such that $m_1 \le u_i^0 \le m_2$ (for instance, $u^0 = u_0$).
2. If u^n is calculated, then we compute u^{n+1} as the solution to the discrete linear problem:

$$u_i^{n+1} - \frac{\alpha}{h}\triangle_-\left[\psi\left(\left|\frac{\triangle_+ u_i^n}{h}\right|\right)\left(\frac{\triangle_+ u_i^{n+1}}{h}\right)\right] = u_{0,i}, \qquad 0 < i < M, \tag{23}$$

with the discrete boundary conditions for u^{n+1}.

In fact, (23) is an approximation of (19). Now, we multiply (23) by h^2/α, and we define $c_1(u_i^n),\ c_2(u_i^n),\ C_1(u_i^n),\ C_2(u_i^n)$, and $C(u_i^n)$ by

$$c_1(u_i^n) = c_1 := \psi\left(\left|\frac{u_{i+1}^n - u_i^n}{h}\right|\right), \qquad c_2(u_i^n) = c_2 := \psi\left(\left|\frac{u_i^n - u_{i-1}^n}{h}\right|\right), \tag{24}$$

$$C_i = \frac{c_i}{(h^2/\alpha) + c_1 + c_2}, \qquad C = \frac{(h^2/\alpha)}{(h^2/\alpha) + c_1 + c_2}. \tag{25}$$

We remark that $c_i, C_i, C > 0$, for $i = 1, 2$, and $C_1 + C_2 + C = 1$ (we note that these coefficients depend on (u_i^n)). All these properties on the coefficients will guarantee the L^∞-*stability* of the scheme. With these notations, (23) becomes

$$u_i^{n+1} = C_1(u_i^n)u_{i+1}^{n+1} + C_2(u_i^n)u_{i-1}^{n+1} + C(u_i^n)u_{0,i}. \tag{26}$$

For the **evolution equation**:

$$\frac{\partial u(t)}{\partial t} - \alpha\frac{\partial}{\partial x}(\psi(|u_x|)u_x) = u_0, \qquad u(0) = u_0,$$

in $]0, T[\, \times\, \Omega$, we use the same approximation for the divergence term, and we have the choice between an explicit or implicit scheme. Let $\triangle t > 0$, $n \in \mathbb{N}$, and define $u_h(n \triangle t, x_i) = u_i^n \approx u(n \triangle t, x_i)$, with $u_h^0 = u_{0,h}$.

The explicit scheme is, for $1 \le i \le M - 1$,

$$\frac{u_i^{n+1} - u_i^n}{\triangle t} + u_i^n - \frac{\alpha}{h^2}[c_1(u_i^n)(u_{i+1}^n - u_i^n) + c_2(u_i^n)(u_i^n - u_{i-1}^n)] = u_{0,i}$$

or

$$u_i^{n+1} = \left[1 - \frac{\alpha \triangle t}{h^2}(c_1 + c_2) - \triangle t\right]u_i^n + \frac{\alpha \triangle t}{h^2}c_1 u_{i+1}^n + \frac{\alpha \triangle t}{h^2}c_2 u_{i-1}^n + \triangle t u_{0,i}.$$

We have that

$$\{m_1 \le u_i^n \le m_2,\ \text{for } 0 \le i \le M\} \Rightarrow \{m_1 \le u_i^{n+1} \le m_2,\ \text{for } 0 \le i \le M\},$$

under the following stability condition:

$$1 - \triangle t\left(\frac{2\alpha}{h^2}\sup_{[0,+\infty[}\psi - 1\right) \ge 0.$$

The implicit scheme will be, for $0 < i < M$,

$$\frac{u_i^{n+1} - u_i^n}{\triangle t} + u_i^{n+1} - \frac{\alpha}{h^2}[c_1(u_i^n)(u_{i+1}^{n+1} - u_i^{n+1}) + c_2(u_i^n)(u_i^{n+1} - u_{i-1}^{n+1})] = u_{0,i},$$

which can be written in the form (26).

7.2. *The Two-Dimensional Case*

In this subsection we describe the extension of the previous approximation to the two-dimensional problem (following [3]), which is, for $N = 2$, with $Du = (u_x, u_y)$,

$$u - \alpha\frac{\partial}{\partial x}(\psi(|Du|)u_x) - \alpha\frac{\partial}{\partial y}(\psi(|Du|)u_y) = u_0, \tag{27}$$

in $\Omega \subset \mathbb{R}^2$ and with $\partial u/\partial n = 0$ on $\Gamma = \partial\Omega$. We follow [3].

Assume $\Omega =]0, 1[\, \times\,]0, 1[$, $h > 0$, and let $x_i = ih, y_j = jh, h = 1/M$, for $0 \le i, j \le M$, be the discrete points. As in the one-dimensional case, we recall the following usual notations:

1°. $u_h(x_i, y_j) = u_{ij} \approx u(x_i, y_j)$, $u_{0,h}(x_i, y_j) = u_{0,ij} \approx u_0(x_i, y_j)$.
2°. $m(a, b) = \text{minmod}(a, b) = ((\text{sign}\, a + \text{sign}\, b)/2) \min(|a|, |b|)$.
3°. $\triangle^x_\mp u_{ij} = \mp(u_{i\mp1,j} - u_{ij})$ and $\triangle^y_\mp u_{ij} = \mp(u_{i,j\mp1} - u_{ij})$.

So, $(u_{0,ij})_{i,j=0,M}$ is the initial discrete image, the data, such that $m_1 \le u_{0,ij} \le m_2$, where $m_2 \ge m_1 \ge 0$. We approach the numerical solution $(u_{ij})_{i,j=0,M}$ by a sequence $(u^n_{ij})_{i,j=0,M}$ for $n \to \infty$, which is obtained as follows:

1. u^0 is arbitrarily given, such that $m_1 \le u^0_{ij} \le m_2$ (we can take $u^0 = u_0$).
2. If u^n is calculated, then we compute u^{n+1} as the solution of the linear discrete problem:

$$u^{n+1}_{ij} - \frac{\alpha}{h} \triangle^x_- \left[\psi \left(\left(\left(\frac{\triangle^x_+ u^n_{ij}}{h} \right)^2 + \left(m \left(\frac{\triangle^y_+ u^n_{ij}}{h}, \frac{\triangle^y_- u^n_{ij}}{h} \right) \right)^2 \right)^{1/2} \right) \left(\frac{\triangle^x_+ u^{n+1}_{ij}}{h} \right) \right]$$
$$- \frac{\alpha}{h} \triangle^y_- \left[\psi \left(\left(\left(\frac{\triangle^y_+ u^n_{ij}}{h} \right)^2 + \left(m \left(\frac{\triangle^x_+ u^n_{ij}}{h}, \frac{\triangle^x_- u^n_{ij}}{h} \right) \right)^2 \right)^{1/2} \right) \left(\frac{\triangle^y_+ u^{n+1}_{ij}}{h} \right) \right]$$
$$= u_{0,ij}, \tag{28}$$

for $i, j = 1, \ldots, M-1$, and with the boundary conditions

$$u^{n+1}_{0,j} = u^{n+1}_{1,j}, \qquad u^{n+1}_{Mj} = u^{n+1}_{M-1,j}, \qquad u^{n+1}_{i0} = u^{n+1}_{i1}, \qquad u^{n+1}_{i,M} = u^{n+1}_{i,M-1}.$$

We use here the minmod function, in order to reduce the oscillations and to get the correct values of derivatives in the case of local maxima and minima. We observe that, to approach, for instance, the term $(\partial/\partial x)(\psi(|(u_x, u_y)|)u_x)$, we do not use the minmod function for u_x, since we wish to obtain a five point finite differences scheme.

We multiply (28) by (h^2/α) and then we denote by $c_1(u^n_{ij})$, $c_2(u^n_{ij})$, $c_3(u^n_{ij})$, and $c_4(u^n_{ij})$, the coefficients of $u^{n+1}_{i+1,j}$, $u^{n+1}_{i-1,j}$, $u^{n+1}_{i,j+1}$, and $u^{n+1}_{i,j-1}$, respectively. We remark, for $i = 1, \ldots, 4$, that $c_i > 0$, since the function ψ is strictly positive. Now, for u^n_{ij}, let $C_i(u^n_{ij})$ and $C(u^n_{ij})$ be defined by ($i = 1, \ldots, 4$)

$$C_i = \frac{c_i}{(h^2/\alpha) + c_1 + c_2 + c_3 + c_4}, \qquad C = \frac{(h^2/\alpha)}{(h^2/\alpha) + c_1 + c_2 + c_3 + c_4}.$$

Then we have that $C_i, C > 0$ and $C_1 + C_2 + C_3 + C_4 + C = 1$ (we recall that these coefficients depend on u^n_{ij}).

Hence, we write (28) as

$$u^{n+1}_{ij} = C_1(u^n_{ij})u^{n+1}_{i+1j} + C_2(u^n_{ij})u^{n+1}_{i-1j} + C_3(u^n_{ij})u^{n+1}_{ij+1}$$
$$+ C_4(u^n_{ij})u^{n+1}_{ij-1} + C(u^n_{ij})u_{0,ij}. \tag{29}$$

We do not describe the approximation for the two-dimensional evolution problem, this being similar to the one-dimensional case. Also, in order to verify the L^∞-stability of these schemes, the existence and uniqueness of u^{n+1} for a fixed u^n, and the convergence, we refer the reader to [3] and [32].

At the end of this subsection we briefly consider the case $K \neq I$ (see [3]). In many cases the degradation operator K, the blur, is a convolution type integral operator.

In the numerical approximations, $(K_{mn})_{m,n=0,d}$ is a symmetric matrix with

$$\sum_{m,n=1}^{d} K_{mn} = 1$$

and an approximation of Ku can be

$$Ku_{ij} = \sum_{m,n=1}^{d} K_{mn} u_{i+d/2-m,\, j+d/2-n}.$$

Since K is symmetric, then $K^* = K$ and $K^*Ku = KKu$ is approximated by

$$KKu_{ij} = \sum_{m,n=1}^{d} \sum_{r,t=1}^{d} K_{mn} K_{rt} u_{i+d-r-m,\, j+d-t-n}.$$

Then we use the same approximation of the divergence term and the same iterative algorithm, with a slight modification. The equation, in this case, is approximated by (using the same notations c_i as before)

$$\begin{aligned}(h^2/\alpha) KKu_{ij}^{n+1} &+ (c_1(u_{ij}^n) + c_2(u_{ij}^n) + c_3(u_{ij}^n) + c_4(u_{ij}^n))u_{ij}^{n+1} \\ &= c_1(u_{ij}^n)u_{i+1,j}^{n+1} + c_2(u_{ij}^n)u_{i-1,j}^{n+1} + c_3(u_{ij}^n)u_{i,j+1}^{n+1} + c_4(u_{ij}^n)u_{i,j-1}^{n+1} + (h^2/\alpha)Ku_{0,ij}.\end{aligned}$$

8. Experimental Results

8.1. *Reconstruction of Noisy Signals*

In this subsection we present experimental results to reconstruct two noisy signals, using the potential $\varphi(z) = \sqrt{\varepsilon + z^2}$ (the function of minimal surfaces with a parameter $\varepsilon > 0$) and the discretization of the stationary equation. The first signal is piecewise-constant, while the second is piecewise-linear (see Figure 1).

A priori, there is no optimal choice for the parameters. Then, for each case, we first tested the algorithm for different values of the parameters and we show here our best results. We remark that for the piecewise-constant case, the parameter ε is smaller than for the piecewise-linear case (we will find the same behavior for images). In this case, with a very small ε, the problem is close to the Total Variation minimization [29].

8.2. *Reconstruction of Noisy and Blurred Images*

In this last subsection we present several results on synthetic and real images, degraded by noise and blur, using the same potential $\varphi(z) = \sqrt{\varepsilon + z^2}$. We show the SNR (the mean signal to noise ratio between the original and the noisy image or the result, after normalization) each time. The SNR is smaller for a noisy image, and larger for the

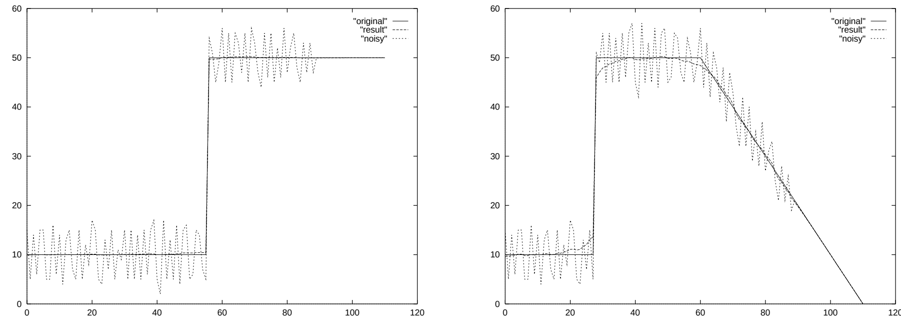

Figure 1. (Left) The piecewise-constant signal: original, noisy, and result, superposed ($h = 1$, $\alpha = 7$, $\varepsilon = 0.001$). (Right) The piecewise-linear signal: original, noisy, and result, superposed ($h = 1$, $\alpha = 20$, $\varepsilon = 1$).

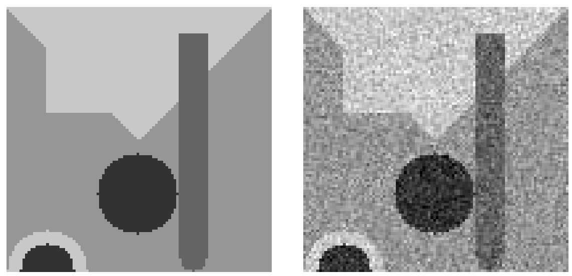

Figure 2. Synthetic picture: original and noisy (SNR = 7.38 dB).

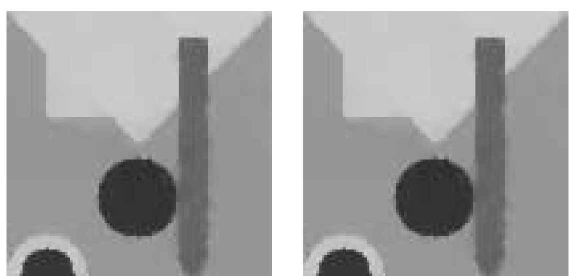

Figure 3. Results with the synthetic picture. (Left) The stationary case (SNR = 18.54 dB, $h = 1$, $\alpha = 20$, $\varepsilon = 0.001$). (Right) The associated evolution case (SNR = 18.54 dB).

reconstructed image. Therefore, the choice of parameters is made in order to increase the initial SNR.

The first three images have been degraded with an additive Gaussian noise.

We begin with a synthetic picture. The corresponding SNR is 7.38 dB (see Figure 2).

In Figure 3 we present the results on the synthetic image, using for the first (left) the stationary equation, and for the second (right) the evolution equation. We see that in the evolution case we obtain the same result (with the same SNR), and this agrees with the theoretical result on the evolution problem.

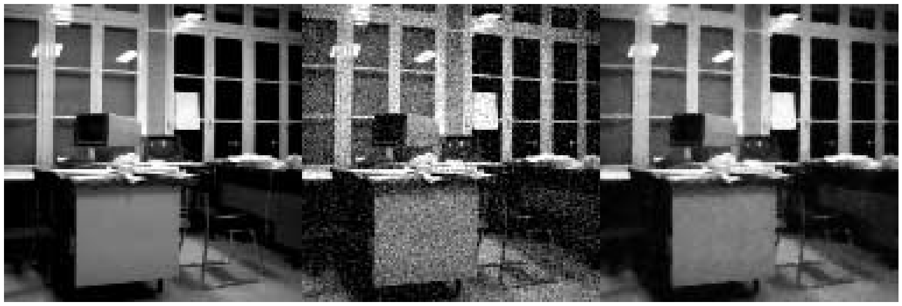

Figure 4. From left to right: original, noisy (SNR = 11.56 dB), and result (SNR = 18.23 dB, $\alpha = 16$, $h = 1$, $\varepsilon = 1$).

Figure 5. From left to right: original, noisy (SNR = 11.00 dB), and result (SNR = 18.30 dB, $\alpha = 15$, $h = 1$, $\varepsilon = 1$).

Figure 6. From left to right: original, noisy (SNR = 3.68 dB), and result (SNR = 16.63 dB, $h = 0.09$, $\alpha = 2800$, $\varepsilon = 1$).

We continue with results for two real pictures, representing an office and a lady, in the stationary case (see Figures 4 and 5).

For the last two results (Figures 6 and 7), we test a uniform impulsive noise (strong "salt and pepper" noise).

For the lady image, Figure 6, it was necessary to consider a large α (the regularizing parameter), but very few iterations.

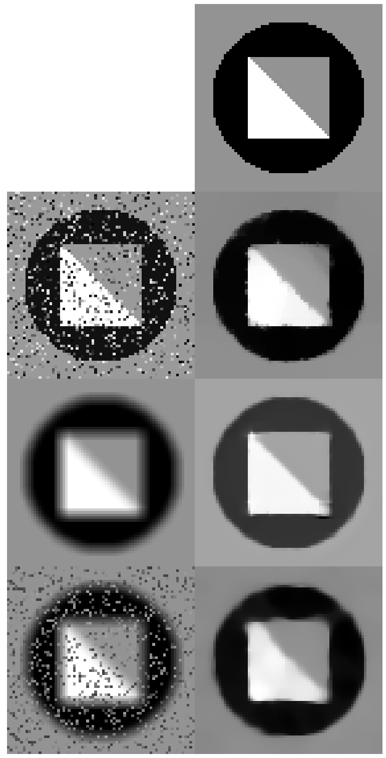

Figure 7. Results on another synthetic picture in the stationary case. (Left) The initial data, (right) the result, on each line. From top to bottom: original and results, for denoising, deblurring, and denoising–deblurring.

We end this section with Figure 7, where we show results with a synthetic image degraded by both noise and blur.

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Accepted 5 *March* 2001. *Online publication* 20 *June* 2001.