“Convex Formulation of Continuous Multi-label Problems”

by Pock, Schoenemann, Graber, Bischof, Cremers (2008)

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A General Problem

Let $G = (\mathcal{V}, \mathcal{E})$ be a graph with nodes $x \in \mathcal{V}$ and edges $(x, y) \in \mathcal{E}$, and let L be a node label set, and consider

$$\min_{u:\mathcal{V}\to L} E(u) = \sum_{(x,y)\in\mathcal{E}} R\big(u(x) - u(y)\big) + \sum_{x\in\mathcal{V}} \rho\big(u(x), x\big) \tag{1}$$

where R is convex and ρ is arbitrary. Applications include segmentation, stereo estimation, and denoising.

Ishikawa showed that *even though E is nonconvex*, a global minimizer of (1) can be found with graph cuts.

Applications

$$\min_{u:\mathcal{V}\to L} \sum_{(x,y)\in\mathcal{E}} R\big(u(x)-u(y)\big) + \sum_{x\in\mathcal{V}} \rho\big(u(x),x\big)$$

- *Multiphase segmentation:* $c_{\ell_1}, c_{\ell_2}, \ldots$ are given segment intensities

  $$\rho(u,x) = |I(x) - c_u|^2 \qquad u \text{ is the segment label}$$

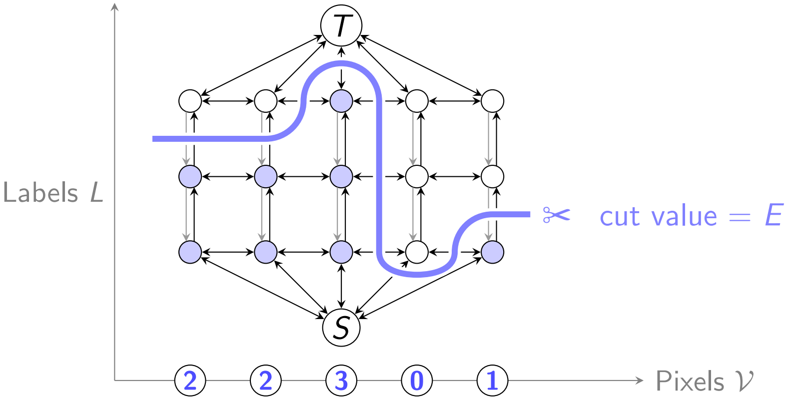
- *Stereo estimation:* I_L and I_R are a stereo pair of left and right images

  $$\rho(u,x) = \left|I_L(x) - I_R\left(x + \binom{u}{0}\right)\right| \qquad u \text{ is the displacement}$$

- *Multiplicative noise removal:* $f = n \cdot u_{\text{exact}}$ with $n \sim \mathit{Gamma}$

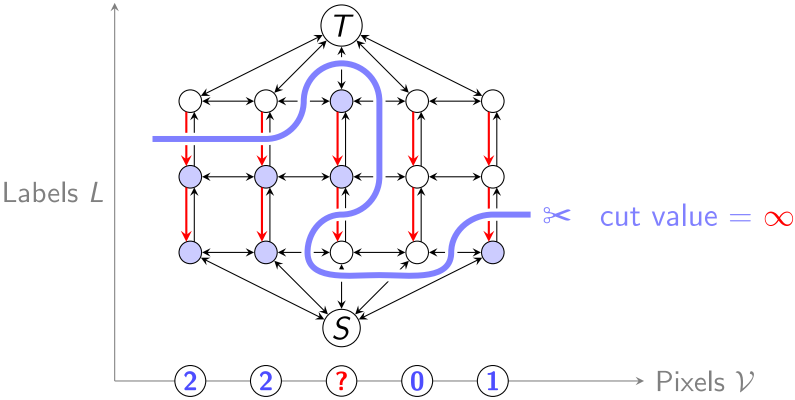
  $$\rho(u,x) = \log u + \frac{f(x)}{u} \qquad u \text{ is the denoised pixel value}$$

Ishikawa's Method



Each pixel corresponds to a column of nodes. The horizontal edges encode R and the vertical edges encode ρ.

Ishikawa's Method



To prevent cutting any column more than once, the red edges are given infinite weight.

Ishikawa's Method

Problems

- Memory: many edges, all represented explicitly
- Parallelization: currently no fast parallel algorithm for graph cuts
- Metrification (grid bias) artifacts

Continuous Problem

Pock et al. consider the variational problem

$$\min_{u:\Omega\to\Gamma} E(u) = \int_\Omega |\nabla u(x)|\, dx + \int_\Omega \rho\big(u(x), x\big)\, dx \tag{2}$$

where $\Gamma = [\gamma_{\text{min}}, \gamma_{\text{max}}]$ and $\rho\big(u(x), x\big)$ is any nonnegative function.

The authors show how to obtain a global minimizer of this nonconvex problem by reinterpreting Ishikawa's method.

Functional Lifting

Define the super level sets of u

$$\varphi(x,\gamma) = \mathbb{1}_{\{u>\gamma\}}(x) = \begin{cases} 1 & \text{if } u(x) > \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Then u is recovered from φ as

$$u(x) = \gamma_{\text{min}} + \int_{\gamma_{\text{min}}}^{\gamma_{\text{max}}} \varphi(x,\gamma)\, d\gamma.$$

For notational convenience, let $\Sigma = \Omega \times \Gamma$ and

$$D' = \{\varphi : \Sigma \to \{0,1\} \mid \varphi(x,\gamma_{\text{min}}) = 1, \varphi(x,\gamma_{\text{max}}) = 0\}$$

Functional Lifting

Theorem

The minimization for u (2) is equivalent to

$$\min_{\varphi \in D'} \int_\Sigma |\nabla \varphi(x,\gamma)| + \rho(x,\gamma)\, |\partial_\gamma \varphi(x,\gamma)| \; d\Sigma \tag{3}$$

Proof: By the co-area formula, we have for the TV term

$$\begin{aligned} \int_\Omega |\nabla u(x)| \; dx &= \int_\Gamma \text{perimeter}(\mathbb{1}_{\{u>\gamma\}})\, d\gamma \\ &= \int_\Gamma \int_\Omega |\nabla \varphi| \; dx\, d\gamma. \end{aligned}$$

Functional Lifting

Observe that

$$|\partial_\gamma \varphi(x,\gamma)| = \delta(u(x)-\gamma).$$

So for the fidelity term,

$$\begin{aligned}
&\int_\Omega \rho\big(u(x),x\big)\,dx \\
&= \int_\Omega \int_\Gamma \rho(\gamma,x)\delta\big(u(x)-\gamma\big)\,d\gamma\,dx \\
&= \int_\Omega \int_\Gamma \rho(\gamma,x)\,|\partial_\gamma \varphi(x,\gamma)|\;d\gamma\,dx.
\end{aligned}$$

Functional Lifting

Thus we have that the problem in u

$$\min_{u:\Omega\to\Gamma} E(u) = \int_\Omega |\nabla u(x)|\; dx + \int_\Omega \rho\big(u(x), x\big)\, dx$$

is equivalent to the lifted problem in φ

$$\min_{\varphi\in D'} E(\varphi) = \int_\Sigma |\nabla\varphi(x,\gamma)| + \rho(x,\gamma)\,|\partial_\gamma\varphi(x,\gamma)|\; d\Sigma.$$

Convex Relaxation

Still, at this point, the lifted problem

$$\min_{\varphi \in D'} E(\varphi) = \int_\Sigma |\nabla \varphi| + \rho \, |\partial_\gamma \varphi| \; d\Sigma$$

is nonconvex because D' is nonconvex:

$$D' = \big\{ \varphi : \Sigma \to \{0, 1\} \mid \varphi(x, \gamma_{\text{min}}) = 1, \varphi(x, \gamma_{\text{max}}) = 0 \big\}.$$

Convex Relaxation

To make the problem convex, define the relaxed set

$$D = \{\varphi : \Sigma \to [0,1] \mid \varphi(x,\gamma_{\text{min}}) = 1, \varphi(x,\gamma_{\text{max}}) = 0\}.$$

Then the problem

$$\min_{\varphi \in D} E(\varphi) = \int_\Sigma |\nabla\varphi| + \rho\,|\partial_\gamma\varphi| \; d\Sigma \tag{4}$$

is convex.

We can find a minimizer $\varphi^* \in D$ of (4) and then threshold it, $\mathbb{1}_{\{\varphi^* \geq \mu\}} \in D'$.

Convex Relaxation

Theorem

Let $\varphi^* \in D$ be a minimizer of the relaxed problem (4). Then for a.e. $\mu \in [0, 1]$, the thresholded solution

$$\mathbb{1}_{\{\varphi^* \geq \mu\}} \in D'$$

is a minimizer of the unrelaxed problem (3).

Proof: Again using the co-area formula.

Convex Relaxation

(Proof by contradiction) By the co-area formula,

$$\begin{aligned} E(\varphi) &= \int_\Sigma |\nabla \varphi(x,\gamma)| + \rho(x,\gamma)\,|\partial_\gamma \varphi(x,\gamma)|\; d\Sigma \\ &= \int_0^1 \int_\Sigma \left|\nabla \mathbb{1}_{\{\varphi \geq \mu\}}\right| + \rho(x,\gamma)\left|\partial_\gamma \mathbb{1}_{\{\varphi \geq \mu\}}\right|\; d\Sigma\, d\mu \\ &= \int_0^1 E(\mathbb{1}_{\{\varphi \geq \mu\}})\, d\mu. \end{aligned}$$

Suppose there exists $\varphi' \in D'$ such that $E(\varphi') < E(\mathbb{1}_{\{\varphi^* \geq \mu\}})$ for all μ in a measurable subset of $[0,1]$ of nonzero measure, then

$$E(\varphi') = \int_0^1 E(\varphi')\, d\mu < \int_0^1 E(\mathbb{1}_{\{\varphi^* \geq \mu\}})\, d\mu = E(\varphi^*).$$

But this contradicts that φ^* is a minimizer of (4).

Convex Relaxation

So, we can find a minimizer φ^* of the relaxed convex problem

$$\min_{\varphi \in D} \int_\Sigma |\nabla \varphi| + \rho \, |\partial_\gamma \varphi| \ d\Sigma,$$

then thresholding it $\mathbb{1}_{\{\varphi^* \geq \mu\}}$ gives a minimizer of the unrelaxed problem

$$\min_{\varphi \in D'} \int_\Sigma |\nabla \varphi| + \rho \, |\partial_\gamma \varphi| \ d\Sigma.$$

Convex Relaxation

Then a solution u^* is recovered by

$$u^* = \gamma_{\text{min}} + \int_{\gamma_{\text{min}}}^{\gamma_{\text{max}}} \mathbb{1}_{\{\varphi^* \geq \mu\}} \, d\gamma,$$

and it is a minimizer of the original problem

$$\min_{u:\Omega \to \Gamma} \int_\Omega |\nabla u(x)| \; dx + \int_\Omega \rho\big(u(x), x\big) \, dx.$$

Minimization Algorithm

Now that we have established a convex formulation of the problem, we wish to solve it. To find the minimizer of

$$\min_{\varphi \in D} E(\varphi) = \int_{\Sigma} |\nabla \varphi| + \rho \, |\partial_\gamma \varphi| \; d\Sigma,$$

one could attempt to solve the associated Euler-Lagrange equations

$$-\operatorname{div}\left(\frac{\nabla \varphi}{|\nabla \varphi|}\right) - \partial_\gamma \left(\rho \frac{\partial_\gamma \varphi}{|\partial_\gamma \varphi|}\right) = 0, \quad \text{s.t.} \quad \varphi \in D.$$

But this is hard because of the singularities as $|\nabla \varphi|$ or $|\partial_\gamma \varphi| \to 0$.

Minimization Algorithm

Instead, write $E(\varphi)$ in a dual formulation: observe that

$$|\nabla\varphi| + \rho\,|\partial_\gamma\varphi| = \max_p\{p\cdot\nabla_3\varphi\}, \quad \text{s.t.} \quad \sqrt{p_1^2+p_2^2}\le 1,\ |p_3|\le\rho,$$

where $p = (p_1, p_2, p_3)$ is the dual variable and $\nabla_3 := (\partial_{x_1}, \partial_{x_2}, \partial_\gamma)^T$. This leads to the primal-dual formulation

$$\min_{\varphi\in D}\left\{\max_{p\in C}\int_\Sigma p\cdot\nabla_3\varphi\, d\Sigma\right\}, \tag{5}$$

where $C = \big\{p : \Sigma\to\mathbb{R}^3 \,|\, \sqrt{p_1(x,\gamma)^2 + p_2(x,\gamma)^2}\le 1,$
$|p_3(x,\gamma)|\le\rho(\gamma,x)\big\}.$

Minimization Algorithm

The authors solve $\min_{\varphi \in D} \left\{ \max_{p \in C} \int_\Sigma p \cdot \nabla_3 \varphi \, d\Sigma \right\}$ with a primal-dual proximal point method:

Primal Step: Solve for φ^{k+1} as the minimizer of

$$\min_{\varphi \in D} \int_\Sigma p^k \cdot \nabla_3 \varphi + \tfrac{1}{2\tau_p} \int_\Sigma (\varphi - \varphi^k)^2$$

$$\implies \quad \varphi^{k+1} = \mathcal{P}_D(\varphi^k + \tau_c \operatorname{div}_3 p^k)$$

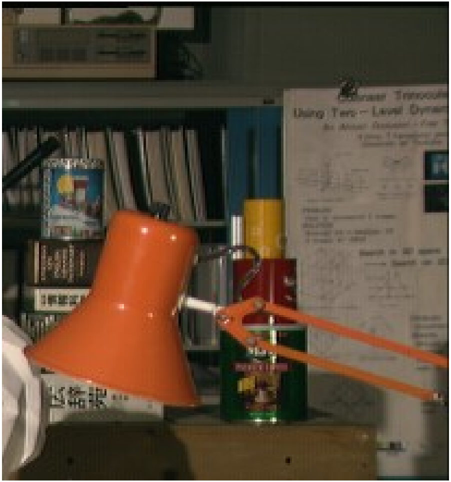
Dual Step: Solve for p^{k+1} as the maximizer of

$$\max_{p \in C} \int_\Sigma p \cdot \nabla_3 \varphi^{k+1} - \tfrac{1}{2\tau_d} \int_\Sigma (p - p^k)^2$$

$$\implies \quad p^{k+1} = \mathcal{P}_C(p^k + \tau_d \nabla_3 \varphi^{k+1})$$

Numerical Results

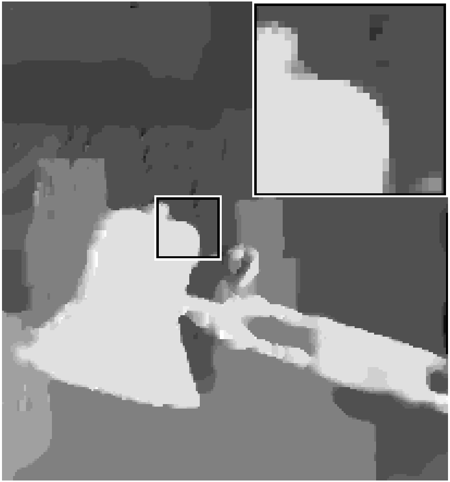
Pock et al. compare their method with Ishikawa's for color stereo estimation. The left image is



Numerical Results

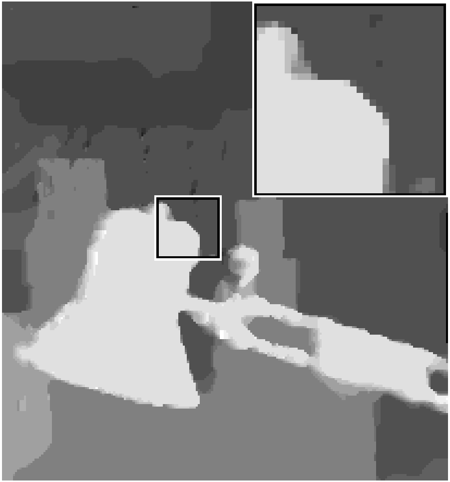
Ishikawa 4-neighbor

Pock et al.

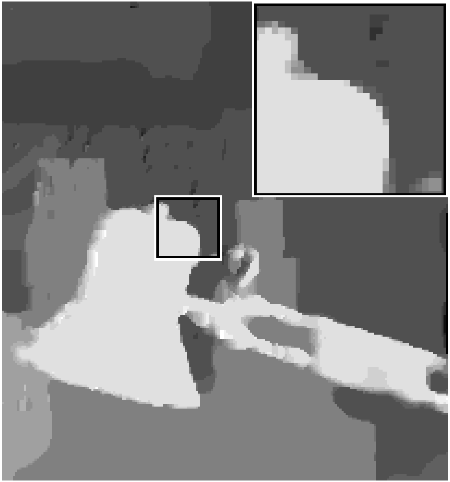


Numerical Results

Ishikawa 8-neighbor

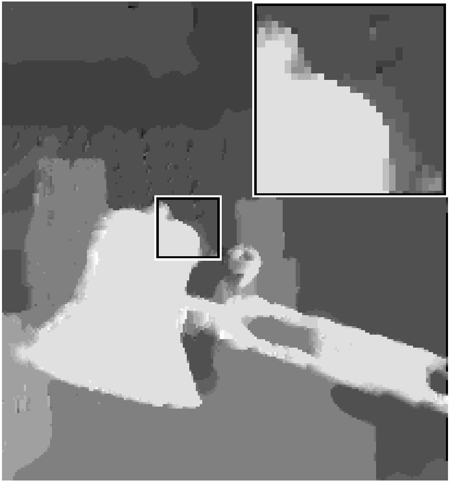


Pock et al.

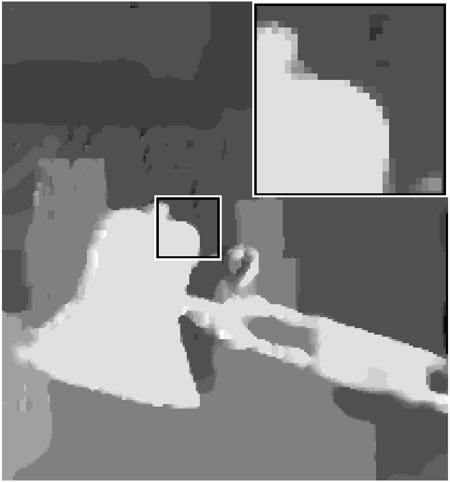


Numerical Results

Ishikawa 16-neighbor



Pock et al.



Numerical Results

Method	Error (%)	Runtime (s)	Memory (MB)
Ishikawa 4-neighbor	2.90	2.9	450
Ishikawa 8-neighbor	2.63	4.9	630
Ishikawa 16-neighbor	2.71	14.9	1500
Pock et al., CPU	2.57	25	54
Pock et al., GPU	**2.57**	**0.75**	**54**

The authors tested both CPU and a GPU implementations of their method (on a fancy NVidia GeForce GTX 280). Ishikawa's method is only on CPU as there is currently no parallel algorithm for graph cuts.

Summary

- Pock et al. consider nonconvex problems of the form

$$\min_{u:\Omega\to\Gamma} \int_\Omega |\nabla u(x)| \; dx + \int_\Omega \rho\big(u(x), x\big) \, dx.$$

- Functional lifting is used to obtain a convex formulation

$$\min_{\varphi\in D} \int_\Sigma |\nabla\varphi| + \rho \, |\partial_\gamma \varphi| \; d\Sigma.$$

- The convex formulation is solved by a proximal primal-dual method on

$$\min_{\varphi\in D} \left\{ \max_{p\in C} \int_\Sigma p \cdot \nabla_3 \varphi \, d\Sigma \right\}.$$

The Paper

- T. Pock, T. Schoenemann, G. Grabe, H. Bischof, D. Cremers, "A Convex Formulation of Continuous Multi-label Problems," *Proc. ECCV*, 2008.

Related Works

- E. Brown, T.F. Chan, X. Bresson, "Convex Formulation and Exact Global Solutions for Multi-phase Piecewise Constant Mumford-Shah Image Segmentation," *UCLA CAM Report* 09-66, 2009.
- T. Goldstein, X. Bresson, S. Osher, "Global Minimization of Markov Random Fields with Applications to Optical Flow," *UCLA CAM Report* 09-77, 2009.
- H. Ishikawa, "Exact optimization for Markov random fields with convex priors," *IEEE Trans. PAMI* 25(10), pp. 1333–1336, 2003.

Related Work on Mumford-Shah*

The Mumford-Shah functional is

$$E(u) = \lambda \int_\Omega (f - u)^2 \, dx + \int_{\Omega \setminus S_u} |\nabla u|^2 \, dx + \nu \mathcal{H}^1(S_u)$$

Pock et al. use a result from Bouchitte, Alberti, and Dal Maso:

$$E(u) = \sup_{p \in K} \int_{\Omega \times \mathbb{R}} p \cdot D\mathbb{1}_{\{u > \gamma\}}$$

where K is the set of vectorfields $p = (p_1, p_2, p_3)$ satisfying

- $p_1, p_2, p_3 \in C_0(\Omega \times \mathbb{R})$
- $p_3(x, \gamma) \geq \frac{1}{4}\left[p_1(x,\gamma)^2 + p_2(x,\gamma)^2\right] - \lambda\big(\gamma - f(x)\big)^2$
- $\left|\int_{\gamma_1}^{\gamma_2} \binom{p_1(x,\mu)}{p_2(x,\mu)} \, d\mu\right| \leq \nu$ for all $x \in \Omega, \gamma_1, \gamma_2 \in \mathbb{R}$

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*T. Pock, D. Cremers, H. Bischof, A. Chambolle, "An Algorithm for Minimizing the Mumford-Shah Functional," *ICCV*, 2009.