Math 285J, Assignment 5:

I. Consider the binary segmentation model based on the minimal variance in a level set formulation:

$$\inf_{c_1,c_2,\phi} F(c_1,c_2,\phi)$$

$$= \lambda_1 \int_\Omega (c_1-f)^2 H(\phi)dxdy + \lambda_2 \int_\Omega (c_2-f)^2(1-H(\phi))dxdy + \mu \int_\Omega |\nabla H(\phi)|dxdy,$$

where $f : \Omega \to I\!R$ is the given image data to be segmented, $\lambda_1, \lambda_2, \mu > 0$ are tuning parameters (usually, we may take $\lambda_1 = \lambda_2 = 1$), and H is the 1D Heaviside function.

• For fixed level set function ϕ, find explicit expressions for c_1, c_2 minimizers of the functional F.

• For fixed c_1, c_2, give the time-dependent Euler-Lagrange equation in ϕ, associated with the minimization. Thus $\phi = \phi(t,x,y)$ solves a PDE. Simplify this equation if possible by using cancellations (you may need to assume that the Heaviside function H has been substituted by a smooth approximation).

• Show that $F(c_1(t), c_2(t), \phi(t,\cdot,\cdot))$ is decreasing in time if c_1, c_2 and ϕ satisfy the obtained equations.

• Implement the time-dependent pde in ϕ together with the expressions for c_1 and c_2 and segment the image given on the class webpage. You need to start with an initial guess for the level set function (for example the signed distance function to a circle). Visualize the evolving curve (zero level line of ϕ) at intermediate steps and at steady state, and the binary segmented image given by $u = c_1H(\phi) + c_2(1 - H(\phi))$. Use a special approximation to the Heaviside function and to its derivative δ, given by

$$H_\epsilon(z) = \frac{1}{2}\Big(1 + \frac{2}{\pi} \arctan\Big(\frac{z}{\epsilon}\Big)\Big), \quad \delta_\epsilon(z) = H'_\epsilon(z).$$

Usually you can take $\epsilon = \triangle x = \triangle y = 1$. You can keep $\lambda_1 = \lambda_2 = 1$ and vary only μ to get the desired result. For more details, see the manuscript "Active contours without edges" from the class webpage.

II. Using the notations from the lecture, find a geometric time-dependent gradient descent evolution that minimizes the robust alignment term

$$E(C) := \int_0^1 \Big|\Big\langle V, \vec{N}\Big\rangle\Big| |C_q| dq,$$

where $V = (u, v)$ is a given vector field and $\vec{N} = \frac{(-\frac{\partial C_2}{\partial q}, \frac{\partial C_1}{\partial q})}{|C_q|}$ is the unit normal. Particularize this to the case $V = \nabla I$.