[1] Affine invariance: Let \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) be an arbitrary matrix, such that \( ad - bc > 0 \), and let \( X = (x, y) \). Check that, if \( u \) satisfies
\[
\frac{\partial u}{\partial t} = |\nabla u| \text{curv}(u)^{1/3},
\]
then \( v(X) = u(AX) \) satisfies
\[
\frac{\partial v}{\partial t} = c(A)|\nabla v| \text{curv}(v)^{1/3}.
\]
What is \( c(A) \)?


Consider given image data \( f \in L^\infty(\Omega) \subset L^2(\Omega) \). We wish to solve in practice the following minimization problem (where \( \varepsilon \to 0^+ \) is a small parameter)

\[
\inf_{u,v \in H^1(\Omega)} G^\varepsilon_{AT}(u,v) = \int_\Omega \left[ \varepsilon |\nabla v|^2 + \alpha (v^2 + \alpha \varepsilon) |\nabla u|^2 + \frac{(v - 1)^2}{4\varepsilon} + \beta |u - f|^2 \right] dx dy,
\]

where \( \alpha \varepsilon \) is any non negative infinitesimal faster than \( \varepsilon \), and \( \alpha, \beta \) are positive parameters. The unknown \( u = u_\varepsilon \) will be an optimal piecewise-smooth approximation of the data \( f \), while the unknown \( v = v_\varepsilon \) will be an edge function: \( 0 \leq v \leq 1 \), \( v \approx 0 \) near edges, and \( v \approx 1 \) inside homogenous regions.

(i) Give the Euler-Lagrange equations in \( u \) and \( v \), associated with the minimization (using alternating minimization), together with the corresponding boundary conditions.

(ii) Discretize the obtained system and implement it for an image \( f \). Visualize the energy decrease versus iterations, and the final \( u \) and \( v \) at steady state.