Assignment #3

[1] Given $f \in L^2(\Omega)$, consider the minimization problem

$$
\inf_{u \in H^1(\Omega)} F(u) = \int_{\Omega} \lambda |f - Ku|^2 dx dy + \sqrt{\int_{\Omega} |\nabla u|^2 dx dy},
$$

where $\lambda > 0$ and $K$ is linear and continuous operator from $L^2(\Omega)$ to $L^2(\Omega)$, with adjoint $K^*$, such that $K1 = 1$.

Formulate and show a similar characterization of minimizers (as done in class for the BV ROF model). Define the dual star norm $\| \cdot \|_*$ necessary in this formulation and mention to what known norm this corresponds.

[2] Let $u : \mathbb{R}^2 \to \mathbb{R}$. The upper level set of an image function $u$ at level $\lambda \in \mathbb{R}$ is the set $\chi_\lambda(u) = \{x \in \mathbb{R}^2 : u(x) \geq \lambda\}$. Show that $u$ can be retrieved by the reconstruction formula

$$
u(x) = \sup \{\lambda : x \in \chi_\lambda(u)\}.
$$

[3] Let $u, v : \mathbb{R}^2 \to \mathbb{R}$. Assume that two image functions $u$ and $v$ have the same level sets, that is for all $\lambda \in \mathbb{R}$, there is $\mu \in \mathbb{R}$ such that $\chi_\lambda(u) = \chi_\mu(v)$. Let us define $g$ by $g(\lambda) = \sup \{\mu : \chi_\lambda(u) = \chi_\mu(v)\}$. Then $g$ is nondecreasing and $v = g \circ u$. (show first that $g$ is nondecreasing, then show that $v \geq g \circ u$ and that $g \circ u \geq v$).

[4] Implement the projection algorithm by A. Chambolle, introduced in the paper posted on the class web-page, for ROF total variation minimization (implement equation (9) from the paper, and obtain the denoised output image $u$ using (7)). Compare with your implementation from the previous assignment.