

285J, L. Vese

Assignment 2:

[1] Implement a numerical scheme for the Euler-Lagrange equation of the ROF model in the presence of known blur:

$$\inf_u \int_{\Omega} \sqrt{\epsilon^2 + |\nabla u|^2} dx dy + \frac{\lambda}{2} \int_{\Omega} |k * u - f|^2 dx dy,$$

where you can choose a Gaussian blur or a uniform blur for k (of mean 1), with small support (between 3×3 and 9×9). If you compute the convolution in the spatial domain, then you can use the commands “conv2” or “imfilter” in matlab. You can also evaluate the convolutions in the frequency domain. Define a blurry data f (with a small amount of noise). Output the root-mean-square-error between u and u_{orig} , and plot the numerical energy versus iterations. Choose a value λ that gives better results.

[2] Assume that $\phi(z)$ is even and differentiable, increasing on $[0, \infty)$ and positive. We know that the time-dependent Euler-Lagrange equation obtained by minimizing

$$\inf_{u \in W^{1,1}(\Omega)} F(u) = \int_{\Omega} \phi(|\nabla u|) dx dy + \frac{\lambda}{2} \int_{\Omega} |f - u|^2 dx dy$$

is formally given by

$$(1) \quad \frac{\partial u}{\partial t} = -\lambda(u - f) + \operatorname{div} \left(\frac{\phi'(|\nabla u|)}{|\nabla u|} \nabla u \right).$$

(i) Express the above differential operator in (1) as $(\cdot)_{\vec{N}\vec{N}} + (\cdot)_{\vec{T}\vec{T}}$, where $\vec{N} = \frac{\nabla u}{|\nabla u|}$, and \vec{T} is normalized and orthogonal to \vec{N} .

(ii) Assume $\lambda = 0$ in (1). Under what conditions is the obtained PDE (weakly) parabolic? (in other words, under what conditions on ϕ , does the quasi-linear 2nd order operator $\operatorname{div} \left(\frac{\phi'(|\nabla u|)}{|\nabla u|} \nabla u \right) = \sum_{i,j=1}^2 a_{ij} u_{x_i x_j}$ satisfy the weakly elliptic property $\sum_{i,j=1}^2 a_{ij} \xi_i \xi_j \geq 0$ for all $\xi_1, \xi_2 \in \mathbb{R}$?)

[3] Let u be sufficiently smooth and satisfy

$$\frac{\partial u}{\partial t} = |\nabla u| G(\operatorname{curv} u),$$

where $\operatorname{curv} u = \operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right)$ is the curvature operator, and G is a function such that $kG(k) \geq 0$. Show that this flow decreases the total variation of u in time.

[4] Let $g \in C^1(\mathbb{R})$ be a function, with $g' > 0$. Let $v = g(u)$. If u satisfies

$$\frac{\partial u}{\partial t} = |\nabla u| G(\operatorname{curv}(u)),$$

so does v (this is called contrast invariance or geometric invariance).