Review: Definitions and properties of functions of bounded variation

Let Ω be an open subset of $\mathbb{R}^N$.

Definition: A function $u \in L^1(\Omega)$ whose partial derivatives in the sense of distributions are measures with finite total variation in Ω is called a *function of bounded variation.* The vector space of functions of bounded variation in Ω is denoted by $BV(\Omega)$. Thus $u \in BV(\Omega)$ if and only if $u \in L^1(\Omega)$ and there are Radon measures μ_1, ..., μ_N with finite total mass in Ω such that

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} dx = -\int_\Omega \varphi d\mu_i \quad \forall \varphi \in C_c^1(\Omega), \quad i = 1, ..., N.$$

If $u \in BV(\Omega)$, the total variation of the measure Du is

$$\|Du\| = \sup\Big\{ \int_\Omega u \mathrm{div}\phi dx : \ \phi \in C_c^1(\Omega, \mathbb{R}^n), \ |\phi(x)| \leq 1 \text{ for } x \in \Omega. \Big\} < \infty.$$

The space $BV(\Omega)$, endowed with the norm $\|u\|_{BV} = \|u\|_{L^1} + \|Du\|$, is a Banach space. We also use $\int_\Omega |Du|$ to denote the total variation $\|Du\|(\Omega)$.

Example: Assume $u \in W^{1,1}(\Omega)$. Then for any $\phi \in C_c^1(\Omega, \mathbb{R}^N)$, with $|\phi| \leq 1$, we have

$$\int_\Omega u \mathrm{div}\phi dx = -\int_\Omega \nabla u \cdot \phi dx \leq \int_\Omega |\nabla u| dx < \infty.$$

Thus $u \in BV(\Omega)$ and $\int_\Omega |Du| = \int_\Omega |\nabla u| dx$.

Properties:

• (lower semi-continuity of the total variation) Suppose $u_n \in BV(\Omega)$, $n = 1, 2, ...$ and that $u_n \to u$ in $L^1_{loc}(\Omega)$. Then

$$\int_\Omega |Du| \leq \liminf_{n\to\infty} \int_\Omega |Du_n|.$$

• (approximation by smooth functions) Assume that $u \in BV(\Omega)$. There is a sequence of functions $u_n \in BV(\Omega) \cap C^\infty(\Omega)$ such that
(i) $u_n \to u$ in $L^1(\Omega)$ and
(ii) $\int_\Omega |Du_n| \to \int_\Omega |Du|$ as $n \to \infty$.
Moreover, if $u \in BV(\Omega) \cap L^q(\Omega)$, $q < \infty$, we can find $u_n \in L^q(\Omega)$, $u_n \to u$ in $L^q(\Omega)$.

Definition: Let $u_n, u \in BV(\Omega)$. We say that u_n weakly* converges to u in $BV(\Omega)$ if $u_n \to u$ in $L^1_{loc}(\Omega)$ and Du_n weakly* converges to Du as measures in Ω.

• Let $u_n, u \in BV(\Omega)$. Then $u_n \to u$ weakly* in $BV(\Omega)$ if and only if $\{u_n\}$ is bounded in $BV(\Omega)$ and converges to u in $L^1_{loc}(\Omega)$.

• (compactness) Let $\Omega \subset \mathbb{R}^N$ be open, bounded, with $\partial\Omega$ Lipschitz. Assume $u_n \in BV(\Omega)$ satisfying $\|u_n\|_{BV(\Omega)} \leq M < \infty$ for all $n \geq 1$. Then there is a subsequence u_{n_j} and a function $u \in BV(\Omega)$ such that $u_{n_j} \to u$ in $L^1(\Omega)$.

Isoperimetric inequalities

• (Sobolev inequality) There is a constant $C > 0$ such that

$$\|u\|_{L^{N/N-1}(\mathbb{R}^N)} \leq C \int_{\mathbb{R}^N} |Du|$$

for all $u \in BV(\mathbb{R}^N)$.

Notation: If $u \in L^1(\Omega)$, the mean value of u in Ω is $u_\Omega = \frac{1}{|\Omega|}\int_\Omega u(x)dx$.

• (Poincaré inequality) Let Ω be open, bounded, connected, with $\partial\Omega$ Lipschitz. Then

$$\int_\Omega |u - u_\Omega| dx \leq C \int_\Omega |Du| \quad \forall u \in BV(\Omega)$$

for some constant C depending only on Ω.

Moreover, there is a constant C depending only on Ω such that

$$\|u - u_\Omega\|_{L^p(\Omega)} \leq C \int_\Omega |Du| \quad \forall u \in BV(\Omega),\ 1 \leq p \leq \frac{N}{N-1}.$$

If $u \in L^1_{loc}(\mathbb{R}^2)$, then its total variation $\int_\Omega |Du|$ can still be defined (finite or infinite).

• (another version of Poincaré inequality in $\mathbb{R}^2$) For any $u \in L^2(\mathbb{R}^2)$ (subset of $L^1_{loc}(\mathbb{R}^2)$), the following inequality holds:

$$\|u\|_{L^2(\mathbb{R}^2)} \leq C \int_{\mathbb{R}^2} |Du|$$

for some constant C independet of u.

• *Fatou's Lemma:* If f_n is a sequence of non-negative measurable functions in Ω, then $\int_\Omega \liminf_{n\to\infty} f_n(x)dx \leq \liminf_{n\to\infty} \int_\Omega f_n(x)dx$.

• *Lebesgue's dominated convergence theorem:* Let f_n be a sequence of measurable functions in Ω. Assume that $|f_n(x)| \leq g(x)$, for some integrable function g, and that f_n converges pointwise to a limit f. Then $\int_\Omega f(x)dx = \lim_{n\to\infty} \int_\Omega f_n(x)dx$.